Far East Journal of Mathematical Sciences (FJMS)

Published Online: October 2015
http://dx.doi.org/10.17654/FJMSNov2015_665_669
Volume 98, Number 5, 2015, Pages 665-669 **ISSN: 0972-0871**

AN APPLICATION OF MONOTONE CONVERGENCE THEOREM IN PDEs AND FOURIER ANALYSIS

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Abstract

Using the monotone convergence theorem, we examine the interchangeability of infinite series and integral in inducing process of Fourier series and PDEs.

1. Introduction

To begin with, let us look into the concept of the integral transform. Normally, solving an ODE can be reduced to an algebraic problem, and the integral transform is a typical operational method that has an advantage in inputs theories. If $f(t)$ is a function defined for all $t \geq 0,$ then its integral transform is the integral of $f(t)$ times $K(s, t)$ from $t = 0$ to $\infty,$ where $K(s, t)$ is a kernel. That is

$$F(s) = \int_0^\infty K(s, t) f(t) dt.$$

Received: May 25, 2015; Revised: July 28, 2015; Accepted: August 14, 2015
2010 Mathematics Subject Classification: 35A22, 65R10.
Keywords and phrases: monotone convergence theorem, Lebesgue dominated convergence theorem.
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Communicated by K. K. Azad

The position shifting of integral and infinite series, or integral and limit frequently appears in many contexts in Fourier analysis and PDEs, and almost all teaching materials assume the interchange integration and taking the limit, or integration and infinite series. By the reason, we would like to clarify the validity of interchangeability of integral and infinite series, and integral and limit symbol.

Let us see two examples from the book [4] of Kreyszig. One is in the inducing process of Fourier series (p. 483), where he expressed that

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) dx = \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} cosnxdx + b_n \int_{-\pi}^{\pi} sinnxdx \right)$$

if termwise integration is allowed. The other is found in the solution of wave equation with respect to the semi-infinite string (p. 595).

$$\lim_{x \to \infty} \int_0^{\infty} e^{-st} w(x, t) dt = \int_0^{\infty} e^{-st} \lim_{x \to \infty} w(x, t) dt$$

under the condition that the integration and taking the limit can be interchanged. If so, then let us check this point by using monotone convergence theorem in measure theory.

2. An Application of Monotone Convergence Theorem in PDEs and Fourier Analysis

In this section, we will check the validity with respect to interchangeability of integration and taking limit. Similarly, the integral and infinite series do as well. The used tool is Lebesgue dominated convergence theorem, a kind of variation of monotone convergence theorem.

Lemma 2.1 (Monotone convergence theorem (MCT) [5])**.** *Let* (X, M, μ) *be a measure space and* $\{f_n\}$ *be a* μ-*a.e. nondecreasing sequence of nonnegative finite* μ-*a.e. measurable functions defined on X. Then* $\lim_{n\to\infty} f_n(x) = f(x)$ *exists* μ-*a.e. on* X, $f(x)$ *is nonnegative and measurable and*

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

The above lemma is a kind of μ-a.e. version of Torchinsky and while Bartle's representation is simpler than this.

Lemma 2.2 (Monotone convergence theorem (MCT) [1])**.** *Let* M^+ *be the collection of all nonnegative measurable functions and let* μ *be a measure. If* (f_n) *is a monotone increasing sequence of functions in* M^+ *which converges to f, then*

$$\int f d\mu = \lim \int f_n d\mu.$$

It is easily caught that the conclusion of MCT is

$$\int \lim f_n d\mu = \lim \int f_n d\mu.$$

MCT can be rewritten as Lebesgue dominated convergence theorem.

Lemma 2.3 (Lebesgue dominated convergence theorem (LDCT) [3, 5])**.** *Let* (X, M, μ) *be a measure space and suppose* $\{f_n\}$ *is a sequence of extended real-valued measurable functions defined on X such that*

(a) $\lim_{n\to\infty} f_n(x) = f(x)$ *exists* μ-*a.e.*

(b) *There is an integrable function g so that for each* n, $|f_n| \leq g$ μ-*a.e.* *Then f is integrable and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Let us put $f_n = g_1 + \cdots + g_n$ and apply the MCT. Then we obtain the following lemma.

Lemma 2.4 (Beppo Livi's theorem [1])**.** *Let* (g_n) *be a sequence in* M^+. *Then*

$$\int \sum_{n=1}^{\infty} g_n d\mu = \sum_{n=1}^{\infty} \int g_n d\mu.$$

It is clear that MCT gives a validity to Beppo Livi's theorem. That is, $\int \sum_{n=1}^{\infty} g_n d\mu = \int \lim \sum_{n=1}^{m} g_n d\mu = \lim \sum_{n=1}^{m} \int g_n d\mu = \sum_{n=1}^{\infty} \int g_n d\mu.$ By using these lemmas, we can make a suitable form in applied mathematics as following:

Theorem 2.5. *We can interchange places of integral and infinite series in the inducing process of Fourier coefficients without any condition. In other word, the equality*

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) dx = \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} cosnxdx + b_n \int_{-\pi}^{\pi} sinnxdx \right)$$

has a validity.

Proof. We note that $a_k coskx \leq 2a_k$ and $b_k sinkx \leq 2b_k$, and both measurable. Let us put

$$f_k = \sum_{k=1}^{n} (a_k coskx + b_k sinkx)$$

and put $c_k = (a_k coskx + b_k sinkx)$. Then f_k less than an integrable function $2\sum_{k=1}^{n} (a_k + b_k)$ holds at almost every k, and

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) dx \tag{$*$}$$

can be written as $\int_{-\pi}^{\pi} \lim_{n\to\infty} f_k dx.$ Hence by LDCT, the equation $(*)$ can be written as

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f_k dx = \lim_{n\to\infty} \int_{-\pi}^{\pi} \sum_{k=1}^{n} c_k dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} c_k dx.$$

Having checked the next theorem in [2], we would like to re-prove it more neatly than [2].

Theorem 2.6. *In the modeling of semi-infinite string, let* $w(x, t)$ *be the displacement of an elastic string. Then the equality*

$$\lim_{x\to\infty}\int_0^\infty e^{-st}w(x, t)dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x, t)dt$$

has a validity.

Proof. Since $\lim_{x\to\infty} w(x, t) = 0$ for $t \geq 0$ and $|w(x, t)| \leq 1$ for all t, by LDCT, we have

$$\lim_{x\to\infty}\int_0^\infty e^{-st}w(x, t)dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x, t)dt.$$

By the above reason, the interchange of $\int$ and $\lim\left(\text{or } \int \text{ and } \sum_{n=1}^{\infty}\right)$ has a validity in PDEs and Fourier analysis.

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