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ON THE HUREWICZ HOMOMORPHISMS IN SHAPE THEORY

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Abstract

In this note, under suitable conditions, it can be seen that the homotopy group $\pi_k(\lim \mathfrak{X}, *)$ is isomorphic to the integral homology group $H_k(\lim \mathfrak{X}; \mathbb{Z})$ of the inverse limit of a k-connected inverse system $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ of pointed topological spaces X_λ and pointed preserving continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda'$ over a directed set Λ.

1. Introduction

In algebraic topology, it appears very often that a certain cohomology expression can be described as a derived functor $\lim^n(-)$, $n \geq 0$ defined by

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Roos [13] and Növeling [12] independently and simultaneously. The first derived limit is an important algebraic tool in the computation of phantom maps in rational homotopy theory. McGibbon [10] wrote a nice paper on the derived limits and phantom maps. McGibbon and Steiner [11] introduced some questions about the first derived limits of the inverse limits and phantom maps. Using the Növeling-Roos cohomology (derived limit in this paper), Watanabe [14] gave an elementary and concrete proof of the properties of derived functors on two categories. We can also find some results on the set of phantom maps in the sense of the extension product of inverse systems [2] and the same n-type structures of CW-spaces (see [3] and [4]).

In 1985, Lisica and Mardešić [5] defined the strong homology groups which are the generalizations of the Steenrod homology groups. Mardešić proved that the strong homology group does not have compact supports and that there exists a paracompact space whose nth derived limit is not trivial (see [6] and [7]). Mardešić and Prasolov [8] constructed a structure theorem which shows a lot of information about the derived limits and strong homology groups of some inverse systems. Actually, strong homology theory deeply involved in the derived limits of various inverse systems.

Let $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of pointed topological spaces X_λ and pointed preserving continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda'$ over a directed set Λ and let $\mathbb{Z}$ be the set of all integers. In this paper, we show that if $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ is a k-connected inverse system of pointed topological spaces and pointed preserving weak fibrations, inducing epimorphic chain maps, over the directed set, then the homotopy group $\pi_k(\lim \mathfrak{X}, *)$ of the inverse limit is isomorphic to the integral homology group $H_k(\lim \mathfrak{X}; \mathbb{Z})$.

2. Preliminaries

Let $(X, *)$ be a pointed n-connected space, i.e., $\pi_k(X, *) = 0$ for $0 \leq k \leq n$. We note that if $n \geq 1$, then $H_k(X; \mathbb{Z}) = 0$, for $1 \leq k \leq n$,

$h_{n+1} : \pi_{n+1}(X, *) \to H_{n+1}(X; \mathbb{Z})$ is an isomorphism and h_{n+2} is an epimorphism.

Let $HTop_*$ be the homotopy category of pointed topological spaces and let $pro\text{-}HTop_*$ be the pro-homotopy category of pointed inverse systems.

Definition 2.1. An object $(\mathfrak{X}, *) \in pro\text{-}HTop_*$ is *n-connected* if $\pi_k(\mathfrak{X}, *) = 0$ for $0 \le k \le n$.

Let $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ be an inverse system of abelian groups A_λ and group homomorphisms $a_{\lambda\lambda'} : A_{\lambda'} \to A_\lambda$, $\lambda \le \lambda'$ over the directed set Λ. Let Λ^n, $n \ge 0$ be the set of all increasing sequences $\overline{\lambda} = (\lambda_0, \lambda_1, ..., \lambda_n)$, $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$, $\lambda_i \in \Lambda$. The sequence $\overline{\lambda}_j = (\lambda_0, \lambda_1, ..., \lambda_{j-1}, \lambda_{j+1}, ..., \lambda_n) \in \Lambda^{n-1}$ is obtained from $\overline{\lambda}$ by deleting the *j*th factor λ_j, $0 \le j \le n$.

We define *n*-cochain groups $C^n(\mathfrak{A})$ of $\mathfrak{A}$ by

$$C^n(\mathfrak{A}) = \prod_{\overline{\lambda} \in \Lambda^n} A_{\overline{\lambda}}, \quad n \ge 0,$$

where $A_{\overline{\lambda}} = A_{\lambda_0}$.

Let $pr_{\overline{\lambda}} : C^n(\mathfrak{A}) \to A_{\overline{\lambda}}$ be a projection. If x is an element of $C^n(\mathfrak{A})$, then we denote the element $x_{\overline{\lambda}}$ of $A_{\overline{\lambda}}$ by

$$x_{\overline{\lambda}} = pr_{\overline{\lambda}}(x).$$

The coboundary operators $\delta^n : C^{n-1}(\mathfrak{A}) \to C^n(\mathfrak{A})$, $n \ge 1$ are defined by

$$(\delta^n x)_{\overline{\lambda}} = a_{\lambda_0\lambda_1}(x_{\overline{\lambda}_0}) + \sum_{j=1}^{n} (-1)^j x_{\overline{\lambda}_j},$$

where $x \in C^{n-1}(\mathfrak{A})$. For $n = 0$, we put $\delta^0 = 0 : 0 \to C^0(\mathfrak{A})$. Then we have a cochain complex

$$(C^*(\mathfrak{A}), \delta) : 0 \xrightarrow{\delta^0} C^0(\mathfrak{A}) \xrightarrow{\delta^1} C^1(\mathfrak{A})$$

$$\to \cdots \to C^{n-1}(\mathfrak{A}) \xrightarrow{\delta^n} C^n(\mathfrak{A}) \to \cdots.$$

The nth *derived limit* [12] $\lim^n \mathfrak{A}$ of $\mathfrak{A}$ is defined by

$$\lim^n \mathfrak{A} = \ker(\delta^{n+1})/\mathrm{im}(\delta^n).$$

There is another notion about the first derived limit of inverse systems. Let $\{G_n\} = (G_n, g_n^{n+1}, \mathbb{N})$ be an inverse tower of (possibly non-abelian) groups G_n and homomorphisms $g_n^{n+1} : G_{n+1} \to G_n$ indexed by the set of all nonnegative integers $\mathbb{N}$. We consider a left action of $\prod G_n$ on $\prod G_n$ by the formula

$$(\cdots, s_n, s_{n+1}, \cdots) \circ (\cdots, t_n, t_{n+1}, \cdots) = (\cdots, s_n t_n g_n^{n+1}(s_{n+1}^{-1}), \cdots).$$

We define the first derived functor, $\lim^1\{G_n\}$, of an inverse tower as the set of orbits of $\prod G_n$ under this action in the sense of Bousfield-Kan [1, p. 251]. We can also define the inverse limit, $\lim\{G_n\}$, of the inverse tower $\{G_n\}$ by using this action:

$$\lim\{G_n\} = \left\{g \in \prod G_n \,|\, g \circ * = *\right\}.$$

Moreover, the set $\lim^1\{G_n\} = \prod G_n/\sim$ can be viewed as the quotient set of the direct products $\prod G_n$ by an equivalence relation $\sim$ defined as follows: For $x = (\cdots, x_n, \cdots)$, $y = (\cdots, y_n, \cdots) \in \prod G_n$, one has $x \sim y$ if and only if there exists an element $s = (\cdots, s_n, \cdots) \in \prod G_n$ such that $y = s \circ x$.

Let $\mathfrak{D} = (D_\lambda, d_{\lambda\lambda'}, \Lambda)$ and $\mathfrak{E} = (E_\gamma, e_{\lambda\lambda'}, \Gamma)$ be inverse systems in any category $\mathfrak{C}$. We say that $s = \{\varphi, s_\gamma : \gamma \in \Gamma\} : \mathfrak{D} \to \mathfrak{E}$ is a *rigid system map* from $\mathfrak{D}$ to $\mathfrak{E}$ if $\varphi : \Gamma \to \Lambda$ is an increasing function, $s_\gamma : D_{\varphi(\gamma)} \to E_\gamma$,

$\gamma \in \Gamma$ is a morphism in the category $\mathfrak{E}$ and for any $\gamma \leq \gamma'$ in Γ, the following diagram:

$$\begin{tikzcd} D_{\varphi(\gamma)} \arrow[d, "s_\gamma"'] & D_{\varphi(\gamma')} \arrow[l, "d_{\varphi(\gamma)\varphi(\gamma')}"'] \arrow[d, "s_{\gamma'}"] \\ E_\gamma & E_{\gamma'} \arrow[l, "e_{\gamma\gamma'}"'] \end{tikzcd}$$

is commutative. We can make a category inv-$\mathfrak{E}$ of inverse systems in $\mathfrak{E}$ and rigid system maps. The rigid system map is called a *level system map* provided $\Gamma = \Lambda$ and φ is an identity map on Λ. It is easy to see that the category $\mathfrak{E}^{\Lambda}$ of the inverse systems and the level system maps is not full subcategory but subcategory of inv-$\mathfrak{E}$.

3. Hurewicz Homomorphisms

For a given pointed inverse system $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ of pointed topological spaces X_λ and pointed preserving continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda'$ over a directed set Λ, we obtain the following inverse systems:

(1) $\pi_k(\mathfrak{X}, *) = (\pi_k(X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ and

(2) $H_k(\mathfrak{X}, \mathbb{Z}) = (H_k(X_\lambda; \mathbb{Z}), p_{\lambda\lambda'}, \Lambda)$

in homotopy and homology, respectively, induced by the inverse system $(\mathfrak{X}, *)$.

The Hurewicz homomorphism $h_\lambda : \pi_k(X_\lambda, *) \to H_k(X_\lambda; \mathbb{Z})$, $\lambda \in \Lambda$ induces a morphism (level system map) $h : \pi_k(\mathfrak{X}, *) \to H_k(\mathfrak{X}; \mathbb{Z})$ in the category Gr^{Λ} of inverse systems of groups and level system maps over Λ.

Definition 3.1. A level system map $h : \pi_k(\mathfrak{X}, *) \to H_k(\mathfrak{X}; \mathbb{Z})$ in Gr^{Λ} is called the *Hurewicz level system map* of $(\mathfrak{X}, *)$.

Proposition 3.2 (Hurewicz isomorphism theorem)**.** *Let* $(\mathfrak{X}, *)$ *be a pointed k-connected inverse system. If* $k \geq 1$, *then we have the following facts*:

(1) $H_n(\mathfrak{X}, \mathbb{Z}) = 0,\ 1 \leq n \leq k$ *and*

(2) $h : \pi_{k+1}(\mathfrak{X}, *) \to H_{k+1}(\mathfrak{X}; \mathbb{Z})$ *is an isomorphism of inverse systems induced by* $(\mathfrak{X}, *)$.

Proof. See Theorem 2, Section 4.1 of the second chapter in [9]. □

A pointed preserving map $f : (X, *) \to (Y, *)$ is called a *pointed preserving weak fibration* provided *f* has the homotopy lifting property with respect to the collection of cubes $\{I_n\}_{n \geq 0}$.

Lemma 3.3. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be an inverse system of pointed topological spaces and pointed preserving weak fibrations. Then the sequence*

$$0 \to \lim{}^1 \pi_{k+1}(\mathfrak{X}, *) \to \pi_k(\lim \mathfrak{X}, *) \to \lim \pi_k(\mathfrak{X}, *) \to 0$$

is exact for any $k \geq 0$.

Proof. See Theorem 2, Section 4.1 of the second chapter in [9]. □

Lemma 3.4. *Let* $\mathfrak{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an inverse system of topological spaces and continuous maps inducing epimorphic chain maps* $p_{\lambda\lambda'\sharp}$: $C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z}),\ \lambda \leq \lambda'$. *Then the sequence*

$$0 \to \lim{}^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) \to H_k(\lim \mathfrak{X}; \mathbb{Z}) \to \lim H_k(\mathfrak{X}; \mathbb{Z}) \to 0$$

is also exact.

Proof. See Theorem 2 of [9]. □

Theorem 3.5. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be a* k*-connected inverse system of pointed topological spaces. If the bonding morphisms are weak fibrations inducing epimorphic chain maps* $p_{\lambda\lambda'\sharp} : C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z})$, $\lambda \le \lambda'$, *then*

$$\pi_k(\lim \mathfrak{X}, *) \cong H_k(\lim \mathfrak{X}; \mathbb{Z}).$$

Proof. Considering an exact sequence of derived limits of homotopy groups and the Hurewicz isomorphism theorem, by Lemma 3.3 and Lemma 3.4, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \lim^1 \pi_{k+1}(\mathfrak{X}, *) \arrow[r] \arrow[d] & \pi_k(\lim \mathfrak{X}, *) \arrow[r] \arrow[d] & \lim \pi_k(\mathfrak{X}, *) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) \arrow[r] & H_k(\lim \mathfrak{X}; \mathbb{Z}) \arrow[r] & \lim H_k(\mathfrak{X}; \mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Since the inverse system $(\mathfrak{X}, *)$ is k-connected, by Proposition 3.2, we have

$$\pi_{k+1}(\mathfrak{X}, *) \cong H_{k+1}(\mathfrak{X}; \mathbb{Z})$$

and

$$H_n(\mathfrak{X}; \mathbb{Z}) = 0$$

for $1 \le n \le k$. Therefore, we obtain

$$\begin{aligned}
\pi_k(\lim \mathfrak{X}, *) &\cong \lim^1 \pi_{k+1}(\mathfrak{X}, *) (\pi_k(\mathfrak{X}, *) \text{ is trivial}) \\
&\cong \lim^1 H_{k+1}(\mathfrak{X}; \mathbb{Z}) \text{ (Hurewicz isomorphism theorem)} \\
&\cong H_k(\lim \mathfrak{X}; \mathbb{Z})
\end{aligned}$$

which shows the proof. □

An inverse system $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ has the *Mittag-Leffler property* if every $\lambda \in \Lambda$ admits a $\lambda' \in \Lambda$, $\lambda' \ge \lambda$ such that

$$a_{\lambda\lambda'}(A_{\lambda'}) = a_{\lambda\lambda''}(A_{\lambda''})$$

for any $\lambda'' \geq \lambda'$.

Proposition 3.6. *If the inverse system* $\mathfrak{A} = (A_\lambda, a_{\lambda\lambda}, \Lambda)$ *has the Mittag-Leffler property*, *then*

$$\lim{}^1 \mathfrak{A} = 0.$$

Proof. See Theorem 10, Section 6.2 of the second chapter in [9]. □

Corollary 3.7. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be a k-connected inverse system of pointed topological spaces*. *If the bonding morphisms are weak fibrations inducing epimorphic chain maps* $p_{\lambda\lambda'\sharp} : C_\sharp(X_{\lambda'}; \mathbb{Z}) \to C_\sharp(X_\lambda; \mathbb{Z})$, $\lambda \leq \lambda'$ *and if* $\pi_{k+1}(\mathfrak{X}, *)$ *has the Mittag-Leffler property*, *then*

$$H_k(\lim \mathfrak{X}; \mathbb{Z}) = 0.$$

Proof. By Lemma 3.3, Theorem 3.5 and Proposition 3.6, we have

$$\begin{aligned} H_k(\lim \mathfrak{X}; \mathbb{Z}) &\cong \pi_k(\lim \mathfrak{X}, *) \\ &\cong \lim{}^1 \pi_{k+1}(\mathfrak{X}, *) \\ &= 0 \end{aligned}$$

as required. □

Let HPol and HPol_* be homotopy category and pointed homotopy category of polyhedra, respectively. Also, let $p : X \to \mathfrak{X}$ be an HPol-expansion [9]. The *Čech homology group* $\check{H}_k(X; A)$ of X with coefficients in an abelian group A is defined by

$$\check{H}_k(X; A) = \lim[H_k(\mathfrak{X}; A)],$$

where [ ] means the equivalence class of inverse systems.

Let $p : (X, *) \to (\mathfrak{X}, *)$ be an HPol_*-expansion. The *k*th shape group $\check{\pi}_k(X, *)$ is defined by

$$\check{\pi}_k(X, *) = \lim[\pi_k(\mathfrak{X}, *)].$$

Corollary 3.8. *Let* $(\mathfrak{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$ *be a k-connected inverse system of pointed topological spaces and* $p : (X, *) \to (\mathfrak{X}, *)$ *be an* HPol_*-*expansion of* $(X, *)$. *Then*

$$\check{\pi}_{k+1}(X, *) \cong \check{H}_{k+1}(X; \mathbb{Z}).$$

Proof. By the Hurewicz isomorphism theorem, we have

$$\begin{aligned}\check{\pi}_{k+1}(X, *) &\cong \lim[\pi_{k+1}(\mathfrak{X}, *)] \\ &\cong \lim[H_{k+1}(\mathfrak{X}; \mathbb{Z})] \\ &\cong \check{H}_{k+1}(X; \mathbb{Z})\end{aligned}$$

as required. □

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