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ON THE ORDER OF MONOID COHYPERSUBSTITUTIONS OF SOME TYPE

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Abstract

A mapping σ which assigns to every n-ary cooperation symbol f_i an n_i-ary coterm of type $\tau = (n_i)_{i \in I}$ is said to be a cohypersubstitution of type τ. The concepts of cohypersubstitutions were introduced in [4]. Every cohypersubstitution σ of type τ induces a mapping $\hat{\sigma}$ on the set of all coterms of type τ. The set of all cohypersubstitutions of type τ under the binary operation $\hat{\circ}$ which is defined by $\sigma_1 \hat{\circ} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp(\tau)$ forms a monoid which is called the monoid of cohypersubstitution of type τ. In [7], it was shown that the order of a cohypersubstitution of type $\tau = (2)$ is 1, 2 or infinite. In this paper, we characterize orders of $Cohyp(\tau)$, where $\tau = (3)$.

1. Introduction

Let A be a non-empty set and n be a positive integer. The nth copower

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$A^{\sqcup n}$ of A is the union of n disjoint copies of A; formally, we define $A^{\sqcup n}$ as the Cartesian product $A^{\sqcup n} := \underline{n} \times A$, where $\underline{n} := \{1, ..., n\}$. An element (i, a) in this copower corresponds to the element a in the ith copy of A, for $1 \leq i \leq n$. A co-operation on A is a mapping $f^A : A \to A^{\sqcup n}$ for some $n \geq 1$; the natural number n is called the *arity* of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A : A \to \underline{n}$ and $f_2^A : A \to A$; the first mapping gives the labelling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A that any element is mapped to.

We shall denote by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$ the set of all n-ary co-operations defined on A, and by $cO_A := \bigcup_{n \geq 1} cO_A^{(n)}$ the set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A; (f_i^A)_{i \in I})$, where f_i^A is an n_i-ary cooperation defined on A, and $\tau = (n_i)_{i \in I}$ for $n_i \geq 1$ is called the *type of the coalgebra*. Coalgebras were studied by Drbohlav [5]. In [2], the following superposition of cooperations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, ..., g_{n-1}^A \in cO_A^{(k)}$, then the k-ary co-operation $f^A[g_0^A, ..., g_{n-1}^A] : A \to A^{\sqcup k}$ is defined by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. The co-operation $f^A[g_0^A, ..., g_{n-1}^A]$ is called the *superposition* of f^A and $g_0^A, ..., g_{n-1}^A$. It will also be denoted by

$$comp_k^n(f^A, g_0^A, ..., g_{n-1}^A).$$

The *injection co-operations* $\iota_i^{n, A} : A \to A^{\sqcup k}$ are special cooperations which are defined for each $0 \leq i \leq n-1$ by $\iota_i^{n,A} : A \to A^{\sqcup k}$ with $a \mapsto (i, a)$ for all $a \in A$. Then we get a multi-based algebra

$$(({}_C O_A^{(n)})_{n\geq 1} (comp_k^n)_{k,n\geq 1}, (\iota_i^{n,A})_{0\leq i\leq n-1}),$$

called the *clone of co-operations* on A. In [2], it is mentioned that this algebra is a clone, i.e., it satisfies the three clone axioms ($C1$), ($C2$) and ($C3$). In [3], Denecke and Saengsura gave a full proof of this fact. In [3], the following coterms of type $\tau = (n_i)_{i\in I}$ were introduced. Let $(f_i)_{i\in I}$ be an indexed set of co-operation symbols such that for each $i \in I$, f_i has arity n_i. Let $\bigcup\{e_j^n \mid n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n-1\}$ be a set of symbols which is disjoint from the set $\{f_i \mid i \in I\}$ such that for each $0 \leq j \leq n-1$, e_j^n has arity n. Then coterms of type τ are defined as follows:

(i) For every $i \in I$, the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$, the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $n \geq 1$ and $0 \leq j \leq n-1$.

Let $cT_\tau^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_\tau := \bigcup_{n\geq 1} cT_\tau^{(n)}$ be the set of all (finitary) coterms of type τ.

The superposition of coterms was introduced in [4] as follows: the operation $S_m^n : cT_\tau^{(n)} \times (cT_\tau^{(m)})^n \to cT_\tau^{(m)}$ is defined by induction on the complexity of coterm definition as follows:

(i) $S_m^n(e_i^n, t_0, \ldots, t_{n-1}) := t_i$ for $0 \leq i \leq n-1$.

(ii) $S_{n_i}^{n_i}(f_i, e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}) := f_i$ for an n_i-ary co-operation symbol f_i.

(iii) $S_m^{n_j}(g_j, t_1, \ldots, t_{n_j}) := g_j[t_1, \ldots, t_{n_j}]$ if g_j is an n_j-ary co-operation symbol.

(iv) $S_m^n(f_i[s_1, \ldots, s_{n_i}], t_1, \ldots, t_n) := f_i[S_m^n(s_1, t_1, \ldots, t_n), \ldots, S_m^n(s_{n_i}, t_1, \ldots, t_n)]$, where f_i is an n_i-ary co-operation symbol, $s_1, \ldots, s_{n_i}$ are n-ary coterms of type τ and $t_1, \ldots, t_n$ are m-ary coterms of type τ.

These operations give us a heterogeneous algebra

$$c\mathcal{T}_\tau := ((cT_\tau^{(n)})_{n \geq 1}(S_m^n)_{m,\, n \geq 1}, (e_j^n)_{1 \leq j \leq n}).$$

We shall show that it is a clone, i.e., it satisfies the clone axioms ($C1$), ($C2$) and ($C3$).

Theorem 1.1 [4]. *The heterogeneous algebra* $c\mathcal{T}_\tau$ *satisfies the following identities*:

($C1$) $\hat{S}_m^p(z, \hat{S}_m^n(y_1, x_1, \ldots, x_n), \ldots, \hat{S}_m^n(y_p, x_1, \ldots, x_n))$

$$\approx \hat{S}_m^n(\hat{S}_n^p(z, y_1, \ldots, y_p), x_1, \ldots, x_n) \ (m, n, p \in \mathbb{N}^+),$$

($C2$) $\hat{S}_m^n(e_i^n, x_1, \ldots, x_n) \approx x_i \ (m \in \mathbb{N}^+, 1 \leq i \leq n)$,

($C3$) $\hat{S}_n^n(y, e_1^n, \ldots, e_n^n) \approx y \ (n \in \mathbb{N}^+)$.

(*Here* $\hat{S}_m^n$, e_i^n *are operation symbols corresponding to the clone type*).

A *cohypersubstitution* of type τ was introduced in [4] as a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$ from the set of all cooperation symbols to the set of all coterms which preserves the arities. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}[e_j^n] := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}[f_i] := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}[f_i[t_1, ..., t_{n_i}]] := S_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$ for $t_1, ..., t_{n_i} \in cT_\tau^{(n)}$.

Let $Cohyp(\tau)$ be the set of all cohypersubstitutions of type τ. On the set $Cohyp(\tau)$ of all cohypersubstitutions of type τ, we may define a binary operation $\hat{\circ}$ by $\sigma_1 \hat{\circ} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of mappings. Let σ_{id} be the cohypersubstitution defined by $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then we have

Lemma 1.1 [4]. *For any two cohypersubstitutions* $\sigma_1, \sigma_2 \in Cohyp(\tau)$, *we have* $(\sigma_1 \hat{\circ} \sigma_2)^\wedge = \hat{\sigma}_1 \circ \sigma_2$. *The cohypersubstitution* σ_{id} *satisfies the equation* $\hat{\sigma}_{id}[t] = t$ *for all* $t \in cT_\tau$.

Theorem 1.2 [4]. $(Cohyp(\tau); \hat{\circ}\, \sigma_{id})$ *is a monoid.*

2. The Order of Cohypersubstitutions of Type $\tau = (3)$

Here we recall that an element *a* of a semigroup *S* is called an *idempotent* if $a^2 = a$. The order of *a* is the cardinal number of the set $\{a^n \mid n \in \mathbb{N}^*\}$ and denoted $|a|$. For any $\sigma \in Cohyp(\tau)$ and $\tau = (n)$, if $\sigma(f) = t$, we denote σ by σ_t. For any positive integer *n*, we call the symbol e_j^n the *injection symbol*, for all $0 \leq j \leq n-1$ and for each coterm *t*, let $E(t)$ be the set of all injection symbols which occur in *t*. In this section, we have to consider the order of elements of the semigroup $Cohyp(3)$. First of all, we start with the order of idempotent cohypersubstitutions. In [1], Boonchari and Saengsura had characterized all idempotents of $Cohyp(3)$ as the following proposition:

Proposition 2.1 [1]. *Let* $t \in CT_{(3)}$ *and* $t = f[t_1, t_2, t_3]$. *If* $e_i^3 \in E(t)$ *for some* $i \in \{0, 1, 2\}$, *then* σ_t *is an idempotent if and only if* $t_{i+1} = e_i^3$.

There follow the elements of $Cohyp(3)$ in Proposition 2.1 are of order 1. Next, we have to consider the order of cohypersubstitution $\sigma_t \in Cohyp(3)$, where $E(t) = \{e_0^3\}$.

Lemma 2.1. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_0^3\}$. *If* $s_1, s_2, s_3 \in CT_{(3)}$ *such that* $s_1 \neq e_0^3$, *then* $t[s_1, s_2, s_3] \neq t$.

Proof. We give a proof by induction on the complexity of the coterm t. If $t = e_0^3$, then

$$\begin{aligned} e_0^3[s_1, s_2, s_3] &= s_1, \\ &\neq e_0^3. \end{aligned}$$

If $t = f[t_1, t_2, t_3]$ and assume that $t_1[s_1, s_2, s_3] \neq t_1$ and $t_2[s_1, s_2, s_3] \neq t_2$ and $t_3[s_1, s_2, s_3] \neq t_3$, then

$$\begin{aligned} t[s_1, s_2, s_3] &= (f[t_1, t_2, t_3])[s_1, s_2, s_3] \\ &= f[t_1[s_1, s_2, s_3], t_2[s_1, s_2, s_3], t_3[s_1, s_2, s_3]] \\ &\neq f[t_1, t_2, t_3]. \end{aligned}$$

Therefore, $t[s_1, s_2, s_3] \neq t$. □

Lemma 2.2. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_0^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_1 \neq e_0^3$, *then* $\sigma_t^m \neq \sigma_t^n$ *for all* $m, n \in \mathbb{N}$.

Proof. Let $m, n \in \mathbb{N}$ such that $m > n$. Then there is $k \in \mathbb{N}$ such that $m = n + k$.

If $k = 1$, then

$$\sigma_t^m(f) = \sigma_t^{n+1}(f)$$

$$= \hat{\sigma}_t^n(f[t_1, t_2, t_3])$$

$$= \sigma_t^n(f)[\hat{\sigma}_t^n(t_1), \hat{\sigma}_t^n(t_2), \hat{\sigma}_t^n(t_3)].$$

Since $t_1 \neq e_0^3$, $\hat{\sigma}_t^n(t_1) \neq e_0^3$. By Lemma 2.1, we get that

$$\sigma_t^n(f)[\hat{\sigma}_t^n(t_1), \hat{\sigma}_t^n(t_2), \hat{\sigma}_t^n(t_3)] \neq \sigma_t^n(f).$$

This means that $\sigma_t^{n+1}(f) \neq \sigma_t^n(f)$. Assume that $k > 1$, let $\hat{\sigma}_t^{k-1}(f) = f[w_1, w_2, w_3]$ for some $w_1, w_2, w_3 \in CT_{(3)}$. Since $E(t) = \{e_0^3\}$ and $\hat{\sigma}_t\{e_0^3\} = e_0^3$, $E(\hat{\sigma}_t^{k-1}(f)) = \{e_0^3\}$. This implies that $E(w_1) = \{e_0^3\}$, $E(w_2) = \{e_0^3\}$ and $E(w_3) = \{e_0^3\}$. Since $t_1 \neq e_0^3$, $\sigma_t^{k-1}(t_1) \neq e_0^3$, so $w_1[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)] \neq e_0^3$. By Lemma 2.1, we get that

$$\begin{aligned}
\sigma_t^m(f) &= \sigma_t^{n+k}(f) \\
&= \hat{\sigma}_t^n(\sigma_t^k(f)) \\
&= \hat{\sigma}_t^n(\hat{\sigma}_t^{k-1}(f[t_1, t_2, t_3])) \\
&= \hat{\sigma}_t^n(\sigma_t^{k-1}(f)[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]) \\
&= \hat{\sigma}_t^n((f[w_1, w_2, w_3])[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]) \\
&= \sigma_t^n(f)[\hat{\sigma}_t^n(w_1[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]), \\
&\quad \hat{\sigma}_t^n(w_2[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]), \\
&\quad \hat{\sigma}_t^n(w_3[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)])] \\
&\neq \sigma_t^n(f).
\end{aligned}$$

Therefore, $\sigma_t^m \neq \sigma_t^n$ for all $m, n \in \mathbb{N}$. □

Theorem 2.1. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_0^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_1 \neq e_0^3$, *then* $|\sigma_t|$ *is infinite.*

Proof. Clearly, by Lemma 2.2, the cyclic subsemigroup of $Cohyp(3)$ generated by σ_t is infinite. □

If we use the same procedures as Lemma 2.1 and Lemma 2.2 for $\sigma_t \in Cohyp(3)$, where $E(t) = \{e_1^3\}$ and $E(t) = \{e_2^3\}$, we obtain the following results:

Theorem 2.2. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_1^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_2 \neq e_1^3$, *then* $|\sigma_t|$ *is infinite.*

Theorem 2.3. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_2^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_3 \neq e_2^3$, *then* $|\sigma_t|$ *is infinite.*

For any cohypersubstitution $\sigma_t \in Cohyp(3)$, where $t = f[t_1, t_2, t_3]$ and $t_1, t_2, t_3 \in \{e_0^3, e_1^3, e_2^3\}$, we have the following results of the order of cohypersubstitutions which are not idempotents:

Proposition 2.2. *Let* $\sigma_t \in Cohyp(3)$ *and* $t = f[t_1, t_2, t_3]$. *Then*

(i) *If* $t = f[e_0^3, e_0^3, e_1^3]$, *then* $|\sigma_t|$ *is two.*

(ii) *If* $t = f[e_1^3, e_0^3, e_0^3]$, *then* $|\sigma_t|$ *is two.*

(iii) *If* $t = f[e_1^3, e_0^3, e_1^3]$, *then* $|\sigma_t|$ *is two.*

(iv) *If* $t = f[e_1^3, e_1^3, e_0^3]$, *then* $|\sigma_t|$ *is two.*

Proof. (i) If $t = f[e_0^3, e_0^3, e_1^3]$, then

$$\begin{aligned}\sigma_t^2(f) &= \hat{\sigma}_t(f[e_0^3, e_0^3, e_1^3])\\ &= \sigma_t(f[e_0^3, e_0^3, e_1^3])\end{aligned}$$

$$= (f[e_0^3, e_0^3, e_1^3])[e_0^3, e_0^3, e_1^3]$$

$$= f[e_0^3, e_0^3, e_0^3].$$

Therefore, $\sigma_t^3(f) = \hat{\sigma}_t(f[e_0^3, e_0^3, e_0^3]) = (f[e_0^3, e_0^3, e_0^3])[e_0^3, e_0^3, e_0^3] = f[e_0^3, e_0^3, e_0^3]$. Since $f[e_0^3, e_0^3, e_0^3]$ is an idempotent, $\langle \sigma_t \rangle = \{\sigma_t, \sigma_t^2\}$. This means that $|\sigma_t|$ is two.

For the proofs of (ii), (iii) and (iv), the procedures are similar to the proof of (i). □

Similarly, we obtain the following results:

Proposition 2.3. *Let* $\sigma_t \in Cohyp(3)$ *and* $t = f[t_1, t_2, t_3]$. *Then*

(i) *If* $t = f[e_0^3, e_2^3, e_0^3]$, *then* $|\sigma_t|$ *is two*.

(ii) *If* $t = f[e_2^3, e_0^3, e_0^3]$, *then* $|\sigma_t|$ *is two*.

(iii) *If* $t = f[e_2^3, e_2^3, e_0^3]$, *then* $|\sigma_t|$ *is two*.

(iv) *If* $t = f[e_2^3, e_0^3, e_0^3]$ *then* $|\sigma_t|$ *is two*.

Proposition 2.4. *Let* $\sigma_t \in Cohyp(3)$ *and* $t = f[t_1, t_2, t_3]$. *Then*

(i) *If* $t = f[e_2^3, e_1^3, e_1^3]$, *then* $|\sigma_t|$ *is two*.

(ii) *If* $t = f[e_1^3, e_2^3, e_2^3]$, *then* $|\sigma_t|$ *is two*.

(iii) *If* $t = f[e_2^3, e_2^3, e_1^3]$, *then* $|\sigma_t|$ *is two*.

(iv) *If* $t = f[e_1^3, e_2^3, e_1^3]$, *then* $|\sigma_t|$ *is two*.

And also for the case that $\sigma_t \in Cohyp(3)$ where $t = f[t_1, t_2, t_3]$ and $t_1, t_2, t_3 \in \{e_0^3, e_1^3, e_2^3\}$ being all different, we obtain the following results.

Proposition 2.5. *Let* $\sigma_t \in Cohyp(3)$ *and* $t = f[t_1, t_2, t_3]$. *Then*

(i) *If* $t = f[e_0^3, e_2^3, e_1^3]$, *then* $|\sigma_t|$ *is two*.

(ii) *If* $t = f[e_1^3, e_0^3, e_2^3]$, *then* $|\sigma_t|$ *is two*.

(iii) *If* $t = f[e_1^3, e_2^3, e_0^3]$, *then* $|\sigma_t|$ *is three*.

(iv) *If* $t = f[e_2^3, e_0^3, e_1^3]$, *then* $|\sigma_t|$ *is three*.

(v) *If* $t = f[e_2^3, e_1^3, e_0^3]$, *then* $|\sigma_t|$ *is two*.

Now, we have to consider $\sigma_t \in Cohyp(3)$ such that $t = f[t_1, t_2, t_3]$ and $t_j \notin E(t)$ for some $j \in \{1, 2, 3\}$.

Proposition 2.6. *Let* $t \in CT_{(3)}$ *such that* $E(t) = \{e_0^3, e_1^3\}$. *Then the following hold*:

(i) *If* $t = f[e_1^3, e_0^3, t_3]$ *such that* $t_3 \notin \{e_0^3, e_1^3\}$, *then* $|\sigma_t|$ *is two*.

(ii) *If* $t = f[e_1^3, e_1^3, t_3]$ *such that* $t_3 \notin \{e_0^3, e_1^3\}$, *then* $|\sigma_t|$ *is two*.

(iii) *If* $t = f[e_0^3, e_0^3, t_3]$ *such that* $t_3 \notin \{e_0^3, e_1^3\}$, *then* $|\sigma_t|$ *is two*.

Proof. (i) Since $\sigma_t(f) = t$ and $t = f[e_1^3, e_0^3, t_3]$,

$$\begin{aligned}
\sigma_t^2(f) &= \hat{\sigma}_t(f[e_1^3, e_0^3, t_3]) \\
&= \sigma_t(f)[\hat{\sigma}_t(e_1^3), \hat{\sigma}_t(e_0^3), \hat{\sigma}_t(t_3)] \\
&= (f[e_1^3, e_0^3, t_3])[e_1^3, e_0^3, \hat{\sigma}_t(t_3)] \\
&= f[e_1^3[e_1^3, e_0^3, \hat{\sigma}_t(t_3)], e_0^3[e_1^3, e_0^3, \hat{\sigma}_t(t_3)], t_3[e_1^3, e_0^3, \hat{\sigma}_t(t_3)]] \\
&= f[e_0^3, e_1^3, t_3[e_1^3, e_0^3, \hat{\sigma}_t(t_3)]].
\end{aligned}$$

Since $E(t) = \{e_0^3, e_1^3\}$, by Proposition 2.1, we get that σ_t^2 is an idempotent, so the subsemigroup $\langle \sigma_t \rangle = \{\sigma_t, \sigma_t^2\}$, i.e., $|\sigma_t|$ is two.

(ii) Since $\sigma_t(f) = t$ and $t = f[e_1^3, e_1^3, t_3]$,

$$\begin{aligned}\sigma_t^2(f) &= \hat{\sigma}_t(f[e_1^3, e_1^3, t_3]) \\ &= \sigma_t(f)[\hat{\sigma}_t(e_1^3), \hat{\sigma}_t(e_1^3), \hat{\sigma}_t(t_3)] \\ &= (f[e_1^3, e_1^3, t_3])[e_1^3, e_1^3, \hat{\sigma}_t(t_3)] \\ &= f[e_1^3[e_1^3, e_1^3, \hat{\sigma}_t(t_3)], e_1^3[e_1^3, e_1^3, \hat{\sigma}_t(t_3)], t_3[e_1^3, e_1^3, \hat{\sigma}_t(t_3)]] \\ &= f[e_1^3, e_1^3, t_3[e_1^3, e_1^3, \hat{\sigma}_t(t_3)]],\end{aligned}$$

so

$$\begin{aligned}\sigma_t^3(f) &= \hat{\sigma}_t^2(f[e_1^3, e_1^3, t_3]) \\ &= \sigma_t^2(f)[\hat{\sigma}_t(e_1^3), \hat{\sigma}_t(e_1^3), \hat{\sigma}_t(t_3)] \\ &= \sigma_t^2(f)[e_1^3, e_1^3, \hat{\sigma}_t(t_3)].\end{aligned}$$

Since $E(t) = \{e_0^3, e_1^3\}$, also $E(t_3) \subseteq \{e_0^3, e_1^3\}$, so $E(t_3[e_1^3, e_1^3, \hat{\sigma}_t(t_3)])$ $= \{e_1^3\}$. Therefore, $E(\sigma_t^2(f)) = \{e_1^3\}$. This implies that $\sigma_t^2(f)[e_1^3, e_1^3, \hat{\sigma}_t(t_3)]$ $= \sigma_t^2(f)$. Then $\sigma_t^3 = \sigma_t^2$. Therefore, $\langle \sigma_t \rangle = \{\sigma_t, \sigma_t^2\}$, i.e., $|\sigma_t|$ is two.

(iii) Since $\sigma_t(f) = t$ and $t = f[e_0^3, e_0^3, t_3]$,

$$\begin{aligned}\sigma_t^2(f) &= \hat{\sigma}_t(f[e_0^3, e_0^3, t_3]) \\ &= \sigma_t(f)[\hat{\sigma}_t(e_0^3), \hat{\sigma}_t(e_0^3), \hat{\sigma}_t(t_3)] \\ &= (f[e_0^3, e_0^3, t_3])[e_0^3, e_0^3, \hat{\sigma}_t(t_3)] \\ &= f[e_0^3[e_0^3, e_0^3, \hat{\sigma}_t(t_3)], e_0^3[e_0^3, e_0^3, \hat{\sigma}_t(t_3)], t_3[e_0^3, e_0^3, \hat{\sigma}_t(t_3)]] \\ &= f[e_0^3, e_0^3, t_3[e_0^3, e_0^3, \hat{\sigma}_t(t_3)]],\end{aligned}$$

so

$$\sigma_t^3(f) = \hat{\sigma}_t^2(f[e_0^3, e_0^3, t_3])$$

$$= \sigma_t^2(f)[\hat{\sigma}_t(e_0^3), \hat{\sigma}_t(e_0^3), \hat{\sigma}_t(t_3)]$$

$$= \sigma_t^2(f)[e_0^3, e_0^3, \hat{\sigma}_t(t_3)].$$

Since $E(t) = \{e_0^3, e_1^3\}$, also $E(t_3) \subseteq \{e_0^3, e_1^3\}$, so $E(t_3[e_0^3, e_0^3, \hat{\sigma}_t(t_3)]) = \{e_0^3\}$. Therefore, $E(\sigma_t^2(f)) = \{e_0^3\}$. This gives us that $\sigma_t^2(f)[e_0^3, e_0^3, \hat{\sigma}_t(t_3)] = \sigma_t^2(f)$. Then $\sigma_t^3 = \sigma_t^2$. Therefore, $\langle \sigma_t \rangle = \{\sigma_t, \sigma_t^2\}$, i.e., $|\sigma_t|$ is two. □

Similarly, we obtain the following results:

Proposition 2.7. *Let $t \in CT_{(3)}$ such that $E(t) = \{e_0^3, e_2^3\}$. Then the following hold*:

(i) *If $t = f[e_2^3, t_2, e_0^3]$ such that $t_2 \notin \{e_0^3, e_2^3\}$, then $|\sigma_t|$ is two.*

(ii) *If $t = f[e_2^3, t_2, e_2^3]$ such that $t_2 \notin \{e_0^3, e_2^3\}$, then $|\sigma_t|$ is two.*

(iii) *If $t = f[e_0^3, t_2, e_0^3]$ such that $t_2 \notin \{e_0^3, e_2^3\}$, then $|\sigma_t|$ is two.*

Proposition 2.8. *Let $t \in CT_{(3)}$ such that $E(t) = \{e_1^3, e_2^3\}$. Then the following hold*:

(i) *If $t = f[t_1, e_2^3, e_1^3]$ such that $t_1 \notin \{e_1^3, e_2^3\}$, then $|\sigma_t|$ is two.*

(ii) *If $t = f[t_1, e_2^3, e_2^3]$ such that $t_1 \notin \{e_1^3, e_2^3\}$, then $|\sigma_t|$ is two.*

(iii) *If $t = f[t_1, e_1^3, e_1^3]$ such that $t_1 \notin \{e_1^3, e_2^3\}$, then $|\sigma_t|$ is two.*

Lemma 2.3. *Let $t \in CT_{(3)}$ and $E(t) = \{e_0^3, e_1^3\}$. If $s_1, s_2, s_3 \in CT_{(3)}$ and $s_1 \notin E(t)$ or $s_2 \notin E(t)$, then $t[s_1, s_2, s_3] \neq t$.*

Proof. Let $s_1 \notin E(t)$. We give a proof by induction on the complexity of the coterm t.

If $t = f[t_1, t_2, t_3]$ such that $t_1, t_2, t_3 \in \{e_0^3, e_1^3\}$ and $e_1^3, e_1^3 \in E(t_1) \cup E(t_2) \cup E(t_3)$, then

$$t[s_1, s_2, s_3] = (f[t_1, t_2, t_3])[s_1, s_2, s_3]$$

$$= f[t_1[s_1, s_2, s_3], t_2[s_1, s_2, s_3], t_3[s_1, s_2, s_3]].$$

Since $t_1, t_2, t_3 \in \{e_0^3, e_1^3\}$, $t_j[s_1, s_2, s_3] = s_1$ for some $j \in \{1, 2, 3\}$, so $t = f[t_1, t_2, t_3] \neq t[s_1, s_2, s_3]$. If $t = f[t_1, t_2, t_3]$ and assume that $t_i[s_1, s_2, s_3] \neq t_i$ for some $i \in \{1, 2, 3\}$, then

$$t[s_1, s_2, s_3] = (f[t_1, t_2, t_3])[s_1, s_2, s_3]$$

$$= f[t_1[s_1, s_2, s_3], t_2[s_1, s_2, s_3], t_3[s_1, s_2, s_3]]$$

$$\neq f[t_1, t_2, t_3] = t.$$

□

Then we obtain that

Theorem 2.4. *Let $t \in CT_{(3)}$ and $E(t) = \{e_0^3, e_1^3\}$. If $t = f[t_1, t_2, t_3]$ and $t_1 \notin E(t)$ or $t_2 \notin E(t)$, then $\sigma_t^m \neq \sigma_t^n$ for all $m, n \in \mathbb{N}$ and $m \neq n$.*

Proof. Let $t_1 \notin E(t)$ and $m, n \in \mathbb{N}$ such that $m > n$. Then there is $k \in \mathbb{N}$ such that $m = n + k$.

If $k = 1$, then

$$\sigma_t^m(f) = \sigma_t^{n+1}(f)$$

$$= \hat{\sigma}_t^n(f[t_1, t_2, t_3])$$

$$= \sigma_t^n(f)[\hat{\sigma}_t^n(t_1), \hat{\sigma}_t^n(t_2), \hat{\sigma}_t^n(t_3)].$$

Since $t_1 \notin E(t)$, $\hat{\sigma}_t^n(t_1) \neq e_0^3$ and $\hat{\sigma}_t^n(t_1) \neq e_1^3$. By Lemma 2.3, we get that $\sigma_t^n(f)[\hat{\sigma}_t^n(t_1), \hat{\sigma}_t^n(t_2), \hat{\sigma}_t^n(t_3)] \neq \sigma_t^n(f)$. This means that $\sigma_t^{n+1}(f) \neq \sigma_t^n(f)$. Assuming that $k > 1$, let $\hat{\sigma}_t^{k-1}(f) = f[w_1, w_2, w_3]$ for some $w_1, w_2, w_3 \in CT_{(3)}$. Since $E(t) = \{e_0^3, e_1^3\}$, $\hat{\sigma}_t(e_0^3) = e_0^3$ and $\hat{\sigma}_t(e_1^3) = e_1^3$, $E(\hat{\sigma}_t^{k-1}(f)) = \{e_0^3, e_1^3\}$. This implies that $E(w_1) \subseteq \{e_0^3, e_1^3\}$, $E(w_2) \subseteq \{e_0^3, e_1^3\}$ and $E(w_3) \subseteq \{e_0^3, e_1^3\}$. Since $t_1 \notin E(t)$, $\sigma_t^{k-1}(t_1) \neq e_0^3$ and $\sigma_t^{k-1}(t_1) \neq e_1^3$, so $w_1 \notin E(t)$ and $w_1[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)] \neq e_0^3$ and $w_1[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)] \neq e_1^3$. By Lemma 2.3, we get that

$$\begin{aligned}
\sigma_t^m(f) &= \sigma_t^{n+k}(f) \\
&= \hat{\sigma}_t^n(\sigma_t^k(f)) \\
&= \hat{\sigma}_t^n(\hat{\sigma}_t^{k-1}(f[t_1, t_2, t_3])) \\
&= \hat{\sigma}_t^n(\sigma_t^{k-1}(f)[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]) \\
&= \hat{\sigma}_t^n((f[w_1, w_2, w_3])[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]) \\
&= \sigma_t^n(f)[\hat{\sigma}_t^n(w_1[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]), \\
&\quad \hat{\sigma}_t^n(w_2[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)]), \\
&\quad \hat{\sigma}_t^n(w_3[\hat{\sigma}_t^{k-1}(t_1), \hat{\sigma}_t^{k-1}(t_2), \hat{\sigma}_t^{k-1}(t_3)])] \\
&\neq \sigma_t^n(f).
\end{aligned}$$

Therefore, $\sigma_t^m \neq \sigma_t^n$ for all $m, n \in \mathbb{N}$. □

These give us that

Corollary 2.1. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_0^3, e_1^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_1 \notin E(t)$ *or* $t_2 \notin E(t)$, *then* σ_t *has infinite order.*

For any $t \in CT_{(3)}$ such that $t = f[t_1, t_2, t_3]$, if we use the same procedure as Lemma 2.3 and Theorem 2.4 for the following cases:

(i) $E(t) = \{e_0^3, e_2^3\}$, $t_1 \notin E(t)$ or $t_3 \notin E(t)$,

(ii) $E(t) = \{e_1^3, e_2^3\}$, $t_2 \notin E(t)$ or $t_3 \notin E(t)$ and

(iii) $E(t) = \{e_0^3, e_1^3, e_2^3\}$, $t_i \notin E(t)$ for some $i \in \{1, 2, 3\}$,

then we obtain the following results:

Corollary 2.2. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_0^3, e_2^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_1 \notin E(t)$ *or* $t_3 \notin E(t)$, *then* σ_t *has infinite order.*

Corollary 2.3. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_1^3, e_2^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_2 \notin E(t)$ *or* $t_3 \notin E(t)$, *then* σ_t *has infinite order.*

Corollary 2.4. *Let* $t \in CT_{(3)}$ *and* $E(t) = \{e_1^3, e_2^3, e_2^3\}$. *If* $t = f[t_1, t_2, t_3]$ *and* $t_i \notin E(t)$ *for some* $i \in \{1, 2, 3\}$, *then* σ_t *has infinite order.*

We summarize all results of the order of $Cohyp(3)$ as follow:

Theorem 2.5. *The order of* $\sigma_t \in Cohyp(3)$, *where* $t \in CT_{(3)}$ *is* 1, 2, 3 *or infinite.*

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References

[1] K. Boonchari and K. Saengsura, Regular and idempotent of cohypersubstitutions of type $\tau = (3)$, Asian-European J. Math. 6(2) (2013), 10 pp.

[2] B. Csákány, Completeness in coalgebras, Acta Sci. Math. 48 (1985), 75-84.

[3] K. Denecke and K. Saengsura, Menger algebras and clones of cooperations, Algebra Colloq. 15(2) (2008), 223-234.

[4] K. Denecke and K. Saengsura, Cohyperidentities and *M*-solid classes of coalgebras, Discrete Math. 309(4) (2009), 772-783.

[5] K. Drbohlav, On quasicovarieties, Acta Fac. Rerum Natur. Univ. Comenian. Math. Mimoriadne Čislo, 1971, pp. 17-20.

[6] J. M. Howie, Fundamentals of Semigroup Theory, Oxford Science Publications, Clarendon Press, Oxford, 1995.

[7] K. Khamthong, S. Jermjitpornchai and K. Saengsura, The order of cohypersubstitutions of type $\tau = (2)$, Far East J. Math. Sci. (FJMS) 96(7) (2015), 879-886.

[8] M. Krapeedaeng and K. Saengsura, Idempotent and regular elements of cohypersubstitutions of type $\tau = (2)$, Far East J. Math. Sci. (FJMS) 75(2) (2013), 345-357.