Far East Journal of Mathematical Sciences (FJMS)

Published Online: September 2015
http://dx.doi.org/10.17654/FJMSOct2015_397_425
Volume 98, Number 4, 2015, Pages 397-425 **ISSN: 0972-0871**

ON HAMILTONIAN COLORINGS OF FUZZY GRAPHS

A. Edward Samuel and C. Kayalvizhi

Ramanujan Research Centre
P.G. and Research Department of Mathematics
Government Arts College (Autonomous)
Kumbakonam, Tamil Nadu, India
e-mail: aedward74_thrc@yahoo.co.in

Abstract

In this paper, we define hamiltonian chromatic number of fuzzy graph, hamiltonian connected and semi-hamiltonian-connected fuzzy graph. Further we introduce a hamiltonian fuzzy coloring of a connected fuzzy graph $\tilde{G}$ of order n is an assignment c of color (positive integer) to the vertices of $\tilde{G}$ such that $D(u, v) + |c(u) - c(v)| \geq n - 1$ for every two distinct vertices u and v of $\tilde{G}$, where $D(u, v)$ is the length of the longest $u - v$ path in $\tilde{G}$. The circumference $cir(\tilde{G})$ of a fuzzy graph $\tilde{G}$ is the length of a longest cycle in $\tilde{G}$. Color sequences of fuzzy graphs provide some interesting theorems, corollaries, propositions as discussed here.

Received: April 8, 2015; Accepted: June 24, 2015

2010 Mathematics Subject Classification: 05C72.

Keywords and phrases: fuzzy graph, fuzzy bipartite and complete bipartite fuzzy graph, hamiltonian fuzzy graph, hamiltonian fuzzy path and cycle, hamiltonian connected and semi-hamiltonian-connected fuzzy graph, hamiltonian fuzzy coloring, hamiltonian chromatic number of a fuzzy graph.

Communicated by John N. Mordeson; Editor: Advances in Fuzzy Sets and Systems: Published by Pushpa Publishing House.

1. Introduction

Fuzzy graph theory is now finding numerous applications modern science and technology especially in the fields of information theory, neural network, expert systems and cluster analysis, medical diagnosis, etc. Bhattacharya [1] has established some connectivity concepts regarding fuzzy cut nodes and fuzzy bridges. Rosenfeld [21] has obtained the fuzzy analogues of several basic graph theoretic concepts like bridges, paths, cycles, trees, and connectedness and established some of the properties. The concepts of decomposition of graphs into hamiltonian cycles, hamiltonian paths decomposition of regular graphs was introduced by Markstrom [13]. Hamiltonian fuzzy path and hamiltonian fuzzy cycle of graphs was introduced by Nirmala and Vijaya [20].

For a connected graph *G* of order *n* and diameter *d* and an integer *k* with $1 \leq k \leq d$, a radio *k-coloring* of is defined in [3] as an assignment *c* of colors (positive integers) to the vertices of *G* such that $d(u, v) + |c(u) - c(v)| \geq 1 + k$ for every two distinct vertices *u* and *v* of *G*. The *value* $rc_k(c)$ of a radio *k*-coloring *c* of *G* is the maximum color assigned to a vertex of *G*; while the *radio k-chromatic number* $rc_k(G)$ of *G* is $\min\{rc_k(c)\}$ over all radio *k*-colorings *c* of *G* is a *minimum radio k-coloring* if $rc_k(c) = rc_k(G)$. These concepts were inspired by the so-called channel assignment problem, where channels are assigned to FM radio stations according to the distances between the stations (and some other factors as well).

Since $rc_1(G)$ is the chromatic number $\chi(G)$, radio *k*-colorings provide a generalization of ordinary colorings of graphs. The *radio d-chromatic number* was studied in [3, 4] and also called the *radio number*. Radio *d*-colorings are also referred to as *radio labelings* since no two vertices can be covered the same in a radio *d*-coloring. Thus, in a radio labeling of a connected graph of diameter *d*, the labels (colors) assigned to adjacent vertices must differ by at least *d*, the labels assigned to two vertices whose distance is 2 must differ by at least $d - 1$, and so on, up to the vertices whose distance is *d*, that is, *antipodal vertices*, whose labels are only required to be different. A radio $(d-1)$-coloring is less restrictive in that colors assigned to two vertices whose distance is *i*, where $1 \leq i \leq d$, are only required to differ by at least $d - i$. In particular, antipodal vertices can be colored in same. For this reason, radio $(d-1)$-colorings are also called *radio antipodal colorings* or, more simply, *antipodal colorings*. Antipodal colorings of graphs were studied in [5, 6], where $rc_{d-1}(G)$ was written as $ac(G)$.

Radio *k*-coloring of paths were studied in [7] for all possible values of *k*. In the case of an antipodal coloring of the path P_n of order $n \geq 3$ (and diameter $n-1$), only the end-vertices of P_n are permitted to be colored the same since the only pair of antipodal vertices in P_n are its two end-vertices. Of course, the two end-vertices of P_n are connected by a hamiltonian path. As mentioned earlier, if *u* and *v* are any two distinct vertices of P_n and $d(u, v) = i$, then $|c(u) - c(v)| \geq n - 1 - i$. Since P_n is a tree, not only is *i* the length of a shortest $u - v$ path in P_n, it is, in fact, the length of any $u - v$ path in P_n since every two vertices are connected by a unique path. In particular, the length of a longest $u - v$ path in P_n is *i* as well.

Hamiltonian colorings were studied in [8-10] for an arbitrary connected graph *G*. While radio *k*-colorings of graphs *G* of order *n* concern the distances $d(u, v)$ between pairs *u*, *v* of distinct vertices of *G* and therefore paths of smallest length, much of the work concerning paths and cycles deals with those of greatest length. For distinct vertices *u* and *v*, let $D(u, v)$ denote the length of a longest $u - v$ path. Of course, if *G* is a tree, then $D(u, v) = d(u, v)$ for every pair *u*, *v* of distinct vertices of *G*.

In this paper, we focus on the hamiltonian fuzzy coloring of graphs by taking fuzzy graph. In Section 2, we review the basic definition of fuzzy graphs. In Section 3, we introduce the hamiltonian fuzzy coloring, hamiltonian connected and semi-hamiltonian-connected fuzzy graph, hamiltonian chromatic number of a fuzzy graph and using some related theorems, corollaries, propositions, in Section 4, on the circumference of fuzzy graphs having many vertices with prescribed colors, in Section 5, circumference of the definition and related to lemma, some theorems, corollaries and finally in Section 6, on the circumference and color sequences of fuzzy graph using the theorems, definition of closure, and corollaries.

2. Preliminary Definitions

The following basic definitions are taken from [11-20]. A *fuzzy graph* $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where for all $u, v \in V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. The fuzzy graph $H = (\tau, \rho)$ is called a *fuzzy subgraph* of $G = (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all $u \in V$ and $\rho(uv) \leq \mu(uv)$ for all $uv \in V$. A fuzzy graph $H = (\tau, \rho)$ is said to be a *spanning fuzzy subgraph* of $G = (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all u. In this case the two graphs have same fuzzy node set; they differ only is the arc weights. A fuzzy graph $G = (\sigma, \mu)$ is a *complete fuzzy graph* if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. The *complement of a fuzzy graph* $G = (\sigma, \mu)$ is a fuzzy graph $G^C = (\sigma^C, \mu^C)$, where $\sigma^C = \sigma$ and $\mu^C(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all u, v in V. Two vertices u and v in $\hat{G}$ is called *adjacent* if $\left(\frac{1}{2}\right)\min\{\sigma(u), \sigma(v)\} \leq \mu(u, v)$. The edge uv of $\hat{G}$ is called *strong* if u and v are adjacent, otherwise it is called *walk*. Two nodes u and v are said to be *neighbors* if $\mu(u, v) > 0$. Two edges $v_i v_j$ and $v_j v_k$ are said to be *incident* if $2\min\{\mu(v_i v_j), \mu(v_j v_k)\} \leq \sigma(v_j)$ for $j = 1, 2, ..., |v|$, $1 \leq i$, $k \leq |v|$. Let $G = (\sigma, \mu)$ be a fuzzy graph and τ be any fuzzy subset of σ, i.e., $\tau(u) \leq \sigma(u)$ for all u. Then the fuzzy subgraph of $G = (\sigma, \mu)$ *induced* by τ is the maximal fuzzy subgraph of $G = (\sigma, \mu)$ that has fuzzy node set τ. Evidently, this is just the fuzzy graph (τ, ρ), where $\rho(u, v) = \tau(u) \wedge \tau(v) \wedge \mu(u, v)$ for all $u, v \in V$. Let $G = (\sigma, \mu)$ be a fuzzy graph. The *degree of vertex* u is $d_G(u) = \sum_{u \neq v} \mu(uv)$. Since $\mu(uv) > 0$ for $uv \in E$ and $\mu(uv) = 0$ for $uv \notin E$, i.e., equivalent to $d_G(u) = \sum_{uv \in E} \mu(uv)$. Minimum degree of $G = \delta(G) = \wedge \{d(v) | v \in V\}$. Maximum degree of $G = \Delta(G) = \vee\{d(v) | v \in V\}$. The *order* and *size* of a fuzzy graph, $G = (V, \sigma, \mu)$ are defined $O(G) = \sum_{v \in V} \sigma(v)$ and $S(G) = \sum_{uv \neq E} \mu(u, v)$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Vertex coloring of G is a mapping $C : V(G) \to \mathbb{N}$ with $\mathbb{N}$ is a set of natural numbers such that $C(x) \neq C(y)$ if $(x, y) \in E(G)$. Given an integer k, a coloring of G is a mapping $C : V(G) \to \{1, 2, ..., k\}$ such that $C(x) \neq C(y)$ if $(x, y) \in E(G)$. A family $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ of fuzzy sets on V is called a *k-fuzzy coloring* of $\hat{G} = (V, \sigma, \mu)$ if

(a) $V\Gamma = \sigma$,

(b) $\gamma_i \wedge \gamma_j = 0$,

(c) For every strong edge *xy* of G, $\min\{\gamma_i(x), \gamma_i(y)\} = 0$ $(1 \leq i \leq k)$.

The minimum number k for which there exists a k-fuzzy coloring is called the *fuzzy chromatic number* of G, denoted as $\chi^f(G)$. A fuzzy graph is said to be *connected fuzzy graph* if there is at least one path between every pair of vertices in fuzzy graph. A path P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, ..., n$ is called a *fuzzy path* and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a *cycle* and cycle P is called a *fuzzy cycle* (f-cycles) if it contains more than one weakest arc. A fuzzy cycle of length n is denoted by C_n. Two nodes of a fuzzy graph are said to be *fuzzy independent* if there is no strong arc between them. A subset S of V is said to be a *fuzzy independent set* of G if any two nodes of S are fuzzy independent. A fuzzy graph $G : (\sigma, \mu)$ is *fuzzy bipartite* if it has a spanning fuzzy subgraph $H : (\tau, \pi)$ which is bipartite where for all edges (u, v) not in $H : (\tau, \pi)$, weight of (u, v) in G is strictly less than the strength of pair

(u, v) in H, i.e., $\mu(u, v) < \pi^{\infty}(u, v)$. A fuzzy graph $G : (\sigma, \mu)$ is fuzzy bipartite then the node set V can be partitioned into nonempty sets V_1 and V_2 such that V_1 and V_2 are fuzzy independent sets. These V_1 and V_2 are called *fuzzy bipartition* of V. Thus every strong arc of $G : (\sigma, \mu)$ has one end in V_1 and the other end in V_2. The size of a fuzzy bipartite graph is defined to be the sum of the weights of all strong arcs of it. A fuzzy bipartite graph $G : (\sigma, \mu)$ with fuzzy bipartition (V_1, V_2) is said to be *complete fuzzy bipartite graph* if for each node of V_1, every node of V_2 is a strong neighbor. A connected fuzzy graph $G : (\sigma, \mu)$ is called a *fuzzy tree* if it has a spanning fuzzy subgraph $F : (\sigma, \pi)$ which is a tree, where for all arcs (u, v) not in F, i.e., $\mu(u, v) < \mu^{\infty}(u, v)$. Equivalently, there is a path in F between u and v whose strength exceeds $\mu(u, v)$. Fuzzy spanning tree is a fuzzy tree which covers all the vertices of a fuzzy graph, note that fuzzy trees has no circuits and it is fine to have vertices with degree higher than two. A fuzzy graph $G : (\sigma, \mu)$ be a fuzzy path P covers all the vertices of G exactly once then the path is called *hamiltonian fuzzy path*. A fuzzy graph $G : (\sigma, \mu)$ be a fuzzy cycle C covers all the vertices of G exactly once expect the end vertices then the cycles is called *hamiltonian fuzzy cycle*. Fuzzy hamiltonian circuit in a connected fuzzy graph is defined as a closed walk that traverses every vertex of G exactly once, except the starting vertex at which that walk also terminates.

3. Hamiltonian Fuzzy Coloring

The *distance* $d(u, v)$ from a vertex u to a vertex v in a connected fuzzy graph $\tilde{G}$ is minimum of the lengths of the $u - v$ paths in $\tilde{G}$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. The distance $d(u, v)$ by following four properties in a connected fuzzy graph $\tilde{G}$:

(1) $d(u, v) \geq 0$ for every two vertices u and v of $\tilde{G}$;

(2) $d(u, v) = 0$ if and only if $u = v$;

(3) $d(u, v) = d(v, u)$ for all $u, v \in V(\tilde{G})$ (the *symmetric* property);

(4) $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V(\tilde{G})$ (the *triangle inequality*).

Since *d* satisfies the four properties, *d* is a *metric* on $V(\tilde{G})$ and $(V(\tilde{G}), d)$ is a *metric space*. Since *d* is symmetric, we can speak of the distance between two vertices *u* and *v* rather than the distance from *u* to *v*. The *eccentricity* of a vertex *v* in a connected fuzzy graph $\tilde{G}$ is the distance between vertex *v* is $ecc(v) = \max\{d(v, w) : w \in V\}$. The *radius* of $\tilde{G}$ is $rad(\tilde{G}) = \min\{ecc(v) : v \in V\}$ and the *diameter* $\tilde{G}$ is $diam(\tilde{G}) = \max\{ecc(v) : v \in V\}$.

A vertex *v* with $ecc(v) = rad(\tilde{G})$ is called a *central vertex* of $\tilde{G}$. A vertex *v* with $ecc(v) = diam(\tilde{G})$ is called a *peripheral vertex* of $\tilde{G}$. Two vertices *u* to *v* of $\tilde{G}$ with $d(u, v) = diam(\tilde{G})$ are *antipodal vertices* of $\tilde{G}$. Necessarily, if *u* and *v* are antipodal vertices in $\tilde{G}$, then each of *u* and *v* is a peripheral vertex. The *girth* of a fuzzy graph $\tilde{G}$ with cycle is the length of a smallest cycle in a connected fuzzy graph $\tilde{G}$. The *detour distance* $D(u, v)$ from a vertex *u* to *v* in $\tilde{G}$ is the length of a longest $u - v$ path in $\tilde{G}$. Thus $D(u, u) = 0$ and if $u \neq v$, then $1 \leq D(u, v) \leq n - 1$. A $u - v$ path of the length $D(u, v)$ is called a $u - v$ *detour*. If $D(u, v) = n - 1$, then $\tilde{G}$ contains a spanning $u - v$ path. The fuzzy graph $K_{1, t}$ is called *star*. The fuzzy graph $K_{s, t}$ has order $s + t$ and size *st*. A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a *double star*. Thus a double star is a tree of diameter 3.

If fuzzy graph $\tilde{G}$ has a spanning cycle *z*, then $\tilde{G}$ is called *hamiltonian fuzzy graph*. A fuzzy graph $\tilde{G}$ is a hamiltonian connected, if for every pair *u*, *v* of vertices of $\tilde{G}$, there is a hamiltonian $u - v$ path in $\tilde{G}$. Necessarily, every hamiltonian connected fuzzy graph of order 3 or more is hamiltonian but the converse is not true. The cubic fuzzy graph $\tilde{G}_1 = C_3 \times k_2$ is *hamiltonian connected fuzzy graph*, while the cubic fuzzy graph $\tilde{G}_2 = C_4 \times k_2 = \tilde{Q}$ is not hamiltonian connected fuzzy graph. The graph $\tilde{G}_2$ contains no hamiltonian $u - v$ path. A connected fuzzy graph $\tilde{G}$ of order $n \geq 3$ is called *semi-hamiltonian connected fuzzy graph* if

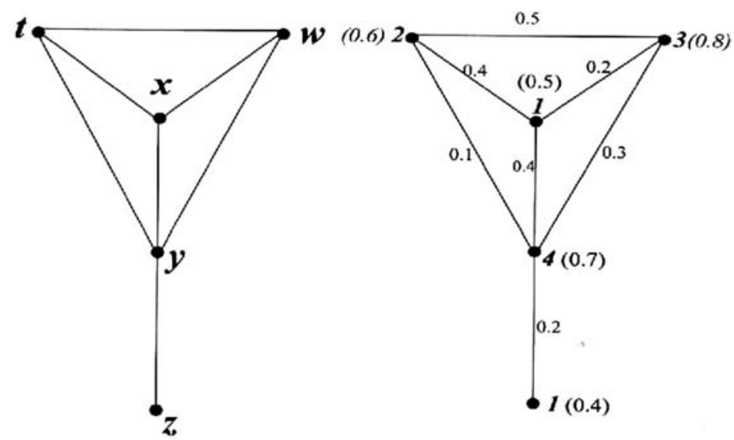
$$D(u, v) = \begin{cases} n - 2 & \text{if } uv \in E(\tilde{G}), \\ n - 1 & \text{if } uv \notin E(\tilde{G}). \end{cases}$$

Moreover, semi-hamiltonian-connected fuzzy graphs, a vertex coloring c is a hamiltonian fuzzy coloring if and only if c is a proper coloring. If fuzzy graph $\tilde{G}$ has a spanning cycle z, then $\tilde{G}$ is called *hamiltonian fuzzy graph*. A *hamiltonian fuzzy coloring* of a connected fuzzy graph $\tilde{G}$ of order n is a vertex coloring c such that $D(u, v) + |c(u) - c(v)| \geq n - 1$ for every two distinct vertices u and v of $\tilde{G}$. The *hamiltonian chromatic number of a fuzzy graph* $\tilde{G}$ is denoted by $hc(\tilde{G})$. The smallest value among all hamiltonian fuzzy colorings of $\tilde{G}$.

A hamiltonian fuzzy coloring c of a connected fuzzy graph $\tilde{G}$ of order n is a function $c : V(\tilde{G}) \to N$ for which

$$|c(u) - c(v)| + D(u, v) \geq n - 1. \tag{1}$$

For every pair u, v of distinct vertices of $\tilde{G}$. If c is hamiltonian fuzzy coloring of a connected fuzzy graph $\tilde{G}$ and u and v are two distinct vertices of $\tilde{G}$ with $c(u) = c(v)$, then $\tilde{G}$ contains a hamiltonian $u - v$ path. For a hamiltonian fuzzy coloring c, $hc(c)$ denotes the largest color assigned to any vertex of $\tilde{G}$; while the hamiltonian chromatic number $hc(\tilde{G})$ is the minimum value of $hc(c)$ over all hamiltonian fuzzy colorings c of $\tilde{G}$. Hence $hc(\tilde{G}) = 1$ if and only if $\tilde{G}$ is hamiltonian-connected. Thus the hamiltonian chromatic number of a connected fuzzy graph $\widetilde{G}$ can be thought of as a measure of how close $\widetilde{G}$ is to be hamiltonian-connected, namely, the closer $hc(\widetilde{G})$ is to 1, the closer $\widetilde{G}$ is to be hamiltonian-connected. If $hc(c) = hc(\widetilde{G})$, then c is a minimum hamiltonian fuzzy coloring.



(a) (b)

Figure 3.1

Figure 3.1(a) shows a fuzzy graph $\widetilde{H}$ of order 5. A vertex coloring c of $\widetilde{H}$ is shown in Figure 3.1(b). Since $D(u, v) + |c(u) - c(v)| \geq 4$ for every two distinct vertices u and v of $\widetilde{H}$, it follows that c is a hamiltonian fuzzy coloring and so $hc(\widetilde{H}) = 4$. Hence $hc(\widetilde{H}) \leq 4$. Because no two of the vertices t, w, x and y are connected by a hamiltonian fuzzy path, these vertices must be assigned distinct colors and so $hc(\widetilde{H}) \geq 4$. Thus $hc(\widetilde{H}) = 4$.

Theorem 3.1. *For every integer* $n \geq 3$, $hc(K_1, n-1) = (n-2)^2 + 1$.

Proof. Since $hc(K_{1,2}) = 2$, we may assume that $n \geq 4$.

Let $\widetilde{G} = K_{1,n-1}$, where $V(\widetilde{G}) = \{v_1, v_2, ..., v_n\}$ and v_n is the central vertex. Define the coloring c of $\widetilde{G}$ by $c(v_n) = 1$ and

$$c(v_i) = (n-1) + (i-1)(n-3) \quad \text{for } 1 \leq i \leq n-1.$$

Then c is a hamiltonian fuzzy coloring of $\tilde{G}$ and

$$hc(\tilde{G}) \leq hc(c) = c(v_{n-1}) = (n-1) + (n-2)(n-3) = (n-2)^2 + 1.$$

It remains to show that $hc(\tilde{G}) \geq (n-2)^2 + 1.$

Let c be a hamiltonian fuzzy coloring of $\tilde{G}$ such that $hc(c) = hc(\tilde{G}).$ Because $\tilde{G}$ contains no hamiltonian fuzzy path, c assigns distinct colors to the vertices of $\tilde{G}.$ We may assume that

$$c(v_1) < (v_2) < \cdots < c(v_{n-1}).$$

We now consider three cases, depending on the assigned to the central vertex v_n.

Case 1. $c(v_n) = 1.$ Since

$$D(v_1, v_n) = 1 \text{ and } D(v_i, v_{i+1}) = 2 \quad \text{for } 1 \leq i \leq n-2.$$

It follows that

$$c(v_{n-1}) \geq 1 + (n-2) + (n-2)(n-3) = (n-2)^2 + 1$$

and so $hc(\tilde{G}) = hc(c) = c(v_{n-1}) \geq (n-2)^2 + 1.$

Case 2. $hc(v_n) = hc(c).$ Thus, in this case,

$$1 = c(v_1) < (v_2) < \cdots < c(v_{n-1}) < c(v_n).$$

Hence

$$c(v_n) \geq 1 + (n-2)(n-3) + (n-2) = (n-2)^2 + 1$$

and so

$$hc(\tilde{G}) = hc(c) = c(v_n) \geq (n-2)^2 + 1.$$

Case 3. $c(v_j) < c(v_n) < c(v_{j+1})$ for some integer j with $i \leq j \leq n-2.$ Thus $c(v_1) = 1$ and $c(v_{n-1}) = hc(c).$ In this case,

$$c(v_j) \geq 1 + (j-1)(n-3),$$

$$c(v_n) \geq c(v_j) + (n-2),$$

$$c(v_{j+1}) \geq c(v_n) + (n-2), \text{ and}$$

$$c(v_{n-1}) \geq c(v_{j+1}) + [(n-1) - (j+1)](n-3).$$

Therefore,

$$\begin{aligned} c(v_{n-1}) &\geq 1 + (j-1)(n-3) + 2(n-2) + (n-j-2)(n-3) \\ &= (2n-3) + (n-3)^2 \\ &= (n-2)^2 + 2 \\ &> (n-2)^2 + 1 \end{aligned}$$

and so $hc(\tilde{G}) = hc(c) = c(v_{n-1}) > (n-2)^2 + 1.$

Hence in any case, $hc(\tilde{G}) \geq (n-2)^2 + 1$ and so $hc(\tilde{G}) = (n-2)^2 + 1.$

□

Theorem 3.2. *For every integer* $n \geq 3$, $hc(\tilde{C}_n) = n - 2.$

Proof. Since we noted that $hc(\tilde{C}_n) = n-2$ for $n = 3, 4, 5,$ we may assume that $n \geq 6.$

Let $\tilde{C}_n = (v_1, v_2, \ldots, v_n, v_1)$. Because the vertex coloring c of $\tilde{C}_n$ defined by $c(v_1) = c(v_2) = 1,$ $c(v_{n-1}) = c(v_n) = n-2,$ and $c(v_i) = i-1$ for $3 \leq i \leq n-2$ is a hamiltonian fuzzy coloring, it follows that $hc(\tilde{C}_n) \leq n-2.$

Assume to the contrary that $hc(\tilde{C}_n) < n-2$ for some integer $n \geq 6.$ Then there exists a hamiltonian $(n-3)$-coloring c of $\tilde{C}_n$. We consider two cases, according to whether n is odd or n is even.

Case 1. n is odd.

Then $n = 2k + 1$ for some integer $k \geq 3$. Hence there exists a hamiltonian $(2k - 2)$-coloring c of $\tilde{C}_n$.

Let $A = \{1, 2, ..., k - 1\}$ and $B = \{k, k + 1, ..., 2k - 2\}$.

For every vertex u of $\tilde{C}_n$, there are two vertices v of $\tilde{C}_n$ such that $D(u, v)$ is minimum (and $d(u, v)$ is maximum), namely $D(u, v) = d(u, v) + 1 = k + 1$.

For $u = v_i$, these two vertices v are v_{i+k} and v_{i+k+1} (where the addition in $i + k$ and $i + k + 1$ is performed modulo n).

Since c is a hamiltonian fuzzy coloring, $D(u, v) + |c(u) - c(v)| \geq n - 1 = 2k$. Because $D(u, v) = k + 1$, it follows that $|c(u) - c(v)| \geq k - 1$. Therefore, if $c(u) \in A$, then the colors of these two vertices v with this property must belong to B.

In particular, if $c(v_i) \in A$, then $c(v_{i+k}) \in B$. Suppose that there are a vertices of $\tilde{C}_n$ whose colors belong to A and b vertices of $\tilde{C}_n$ whose colors belong to B. Then $b \geq a$. However, if $c(v_i) \in B$, then $c(v_{i+k}) \in A$, implying that $a \geq b$ and so $a = b$. Since $a + b = n$ and n is odd, this is impossible.

Case 2. n is even.

Then $n = 2k$ for some integer $k \geq 3$. Hence there exists a hamiltonian $(2k - 3)$-coloring c of $\tilde{C}_n$.

For every vertex u of $\tilde{C}_n$, there is a unique vertex v of $\tilde{C}_n$ for which $D(u, v)$ is minimum (and $d(u, v)$ is maximum), namely $D(u, v) = d(u, v) = k$. For $u = v_i$, this vertex v is v_{i+k} (where the addition in $i + k$ is performed modulo n).

Since c is a hamiltonian fuzzy coloring, $D(u, v) + |c(u) - c(v)| \geq n - 1 = 2k - 1$. Because $D(u, v) = k$, it follows that $|c(u) - c(v)| \geq k - 1$. This implies, however, that if $c(u) = k - 1$, then there is no color that can be assigned to v to satisfy this requirement. Hence no vertex of $\tilde{C}_n$ can be assigned the color $k-1$ by c.

Let $A = \{1, 2, ..., k-2\}$ and $B = \{k, k+1, ..., 2k-3\}$.

Thus $|A| = |B| = k-2$. If $c(v_i) \in A$, then $c(v_{i+k}) \in B$. Also, if $c(v_i) \in B$, then $c(v_{i+k}) \in A$. Hence there are k vertices of $\tilde{C}_n$ assigned color from A and k vertices of $\tilde{C}_n$ assigned color from B.

Consider two adjacent vertices of $\tilde{C}_n$, one of which is assigned a color from A and other is assigned a color from B. We may assume that $c(v_1) \in A$ and $c(v_2) \in B$. Then $c(v_{k+1}) \in B$. Since $D(v_2, v_{k+1}) = k+1$, it follows that $|c(v_2) - c(v_{k+1})| \geq k-2$. Because $c(v_2), c(v_{k+1}) \in B$, this implies that one of $c(v_2)$ and $c(v_{k+1})$ is at least $2k-2$. This is a contradiction. □

Proposition 3.3. *If $\tilde{H}$ is a spanning connected subgraph of a fuzzy graph $\tilde{G}$, then $hc(\tilde{G}) \leq hc(\tilde{H})$.*

Proof. Suppose that the order of $\tilde{H}$ is n. Let c be a hamiltonian fuzzy coloring of $\tilde{H}$ such that $hc(c) \leq hc(\tilde{H})$. Then $D_{\tilde{H}}(u, v) + |c(u) - c(v)| \geq n-1$ for every two distinct vertices u and v of $\tilde{H}$.

Since $D_{\tilde{G}}(u, v) \geq D_{\tilde{H}}(u, v)$ for every two distinct vertices u and v of $\tilde{H}$ (and of $\tilde{G}$), it follows that $D_{\tilde{G}}(u, v) + |c(u) - c(v)| \geq n-1$ and so c is a hamiltonian fuzzy coloring of $\tilde{G}$ as well.

Hence $hc(\tilde{G}) \leq hc(c) = hc(\tilde{H})$. □

Combining Theorem 3.2 and Proposition 3.3, we have the following corollary.

Corollary 3.4. *If $\tilde{G}$ is a hamiltonian fuzzy graph of order $n \geq 3$, then $hc(\tilde{G}) \leq n-2$.*

The following result gives the hamiltonian chromatic number of a related class of fuzzy graphs.

Proposition 3.5. *Let* $\tilde{H}$ *be a hamiltonian fuzzy graph of order* $n - 1 \geq 3$. *If* $\tilde{G}$ *is a fuzzy graph obtained by adding a pendant edge to* $\tilde{H}$, *then* $hc(\tilde{G}) = n - 1$.

Proof. Suppose that $\tilde{C}_n = (v_1, v_2, ..., v_{n-1}, v_1)$ is a hamiltonian fuzzy cycle of $\tilde{H}$ and $v_1 v_n$ is the pendent edge of $\tilde{G}$. Let c be a hamiltonian fuzzy coloring of $\tilde{G}$. Since $D_{\tilde{G}}(u, v) \leq n - 2$ for every two distinct vertices u and v of $\tilde{C}$, no two vertices of $\tilde{C}$ can be assigned the same color by c. Consequently, $hc(c) \geq n - 1$ and so $hc(\tilde{G}) \geq n - 1$.

Now define a coloring c' of $\tilde{G}$ by

$$c'(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq n-1, \\ n-1 & \text{if } i = n. \end{cases}$$

We claim that c' is a hamiltonian fuzzy coloring of $\tilde{G}$. First let v_j and v_k be two vertices of $\tilde{C}$, where $1 \leq j < k \leq n-1$. Then $| c'(v_j) - c'(v_k) | = k - j$ and

$$D(v_j, v_k) = \max\{k - j, (n-1) - (k-j)\}.$$

In either case, $D(v_j, v_k) \geq n - 1 + j - k$ and so

$$D(v_j, v_k) + | c'(v_j) - c'(v_k) | \geq n - 1.$$

For $1 \leq j \leq n-1$, $| c'(v_j) - c'(v_n) | = n - 1 - j$, while

$$D(v_j, v_n) \geq \max\{j, n - j + 1\}$$

and so $D(v_j, v_n) \geq j$. Therefore,

$$D(v_j, v_n) + | c'(v_j) - c'(v_n) | \geq n - 1.$$

Hence, as claimed, c' is a hamiltonian fuzzy coloring of $\tilde{G}$ and so $hc(\tilde{G}) \leq h(c') = c'(v_n) = n - 1.$ □

Theorem 3.6. *For every connected fuzzy graph* $\tilde{G}$ *of order* $n \geq 2$, $hc(\tilde{G}) \leq (n-2)^2 + 1.$

Proof. First, if $\tilde{G}$ contains a vertex of degree $n - 1$, then $\tilde{G}$ contains the star $K_{1, n-1}$ as a spanning fuzzy subgraph. Since $hc(K_{1, n-1}) = (n-2)^2 + 1$, it follows by Proposition 3.3 that $hc(\tilde{G}) \leq (n-2)^2 + 1.$

Hence we may assume that $\tilde{G}$ contains a spanning fuzzy tree $\tilde{T}$ that is not a star and so its complement $\tilde{T}^C$ contains a hamiltonian fuzzy path $\tilde{P} = \{v_1, v_2, ..., v_n\}$. Thus $v_i v_{i+1} \notin E(\tilde{T})$ for $1 \leq i \leq n-1$ and so $D_{\tilde{T}}(v_i v_{i+1}) \geq 2.$ Define a vertex coloring c of $\tilde{T}$ by

$$c(v_i) = (n-2) + (i-2)(n-3) \quad \text{for } 1 \leq i \leq n.$$

Hence

$$hc(c) = c(v_n) = (n-2) + (n-2)(n-3) = (n-2)^2.$$

Therefore, for integers i and j with $1 \leq i < j \leq n$,

$$| c(v_i) - c(v_j) | = (j - i)(n-3).$$

If $j = i + 1$, then

$$D(v_i, v_j) + | c(v_i) - c(v_j) | \geq 2 + (n-3) = n - 1.$$

While if $j \geq i + 2$, then

$$D(v_i, v_j) + | c(v_i) - c(v_j) | \geq 1 + 2(n-3) = 2n - 5 \geq n - 1.$$

Thus c is a hamiltonian fuzzy coloring of $\tilde{T}$. Therefore,

$$hc(\tilde{G}) \leq hc(\tilde{T}) \leq hc(c) = c(v_n) = (n-2)^2 < (n-2)^2 + 1,$$

which completes the proof. □

Theorem 3.6 shows how large the hamiltonian chromatic number of a fuzzy graph $\tilde{G}$ of order n can be. If $\tilde{G}$ is hamiltonian however, then by Corollary 3.4 its hamiltonian chromatic number cannot exceed $n - 2$. Moreover, if the hamiltonian chromatic number is small relative to n, then $\tilde{G}$ must contain cycles of relatively large length.

The concept of hamiltonian colorings of graph was introduced in [9], hamiltonian chromatic numbers of several well known graphs were established, including complete bipartite graphs, cycles, and Petersen graph.

To be sure, if $\tilde{G}$ is a non-hamiltonian fuzzy graph of order $n \geq 3$, then $\tilde{G}$ is not hamiltonian-connected. Since for every pair u, v of adjacent vertices, $\tilde{G}$ does not contain a hamiltonian $u - v$ path. On the other hand, if u and v are nonadjacent vertices of $\tilde{G}$, then $\tilde{G}$ may contain a hamiltonian $u - v$ path. For such a fuzzy graph then, $D(u, v) \leq n - 2$ if u and v are adjacent and $D(u, v) \leq n - 1$ if u and v are not adjacent. We define a connected fuzzy graph $\tilde{G}$ of order $n \geq 3$ to be semi-hamiltonian-connected if

$$D(u, v) = \begin{cases} n - 2 & \text{if } uv \in E(\tilde{G}), \\ n - 1 & \text{if } uv \notin E(\tilde{G}). \end{cases}$$

Now, let c be a hamiltonian fuzzy coloring of a semi-hamiltonian-connected fuzzy graph $\tilde{G}$ order $n \geq 3$. Then $|c(u) - c(v)| + D(u, v) \geq n - 1$ for every pair u, v of distinct vertices of $\tilde{G}$. Hence if u and v are adjacent, then $|c(u) - c(v)| \geq 1$; while if u and v are not adjacent vertices, then $|c(u) - c(v)| \geq 0$. That is, two vertices must be assigned distinct colors if the vertices are adjacent and may be assigned the same color if they are not adjacent. In other words, every hamiltonian fuzzy coloring of a semi-hamiltonian-connected fuzzy graph $\tilde{G}$ of order $n \geq 3$ is an ordinary coloring of $\tilde{G}$ and so $hc(\tilde{G}) = \chi^f(\tilde{G})$. Thus we have the following.

Proposition 3.7. *If $\tilde{G}$ is a semi-hamiltonian-connected fuzzy graph of order $n \geq 3$, then $hc(\tilde{G}) = \chi^f(\tilde{G})$.*

The fuzzy graph P_3 and the Petersen graph are semi-hamiltonian-connected and so their hamiltonian chromatic number equals their chromatic number, which is 2 and 3, respectively. Whether there are other semi-hamiltonian-connected fuzzy graphs is not known. If $\tilde{G}$ is a connected non-hamiltonian fuzzy graph of order $n \geq 3$ such that $\tilde{G}$ has a hamiltonian $u - v$ path for every pair u, v of nonadjacent vertices, then $\tilde{G}$ need not be semi-hamiltonian-connected.

For example, for $1 \leq m \leq n - m - 1$, the fuzzy graph $\tilde{G} = K_1 + (K_m \cup K_{n-m-1})$ has this property but is not semi-hamiltonian-connected. On the other hand, the fuzzy graph $\tilde{G} = K_{r,r}$, $r \geq 2$ with $n = 2r$, has the property that if

$$D(u, v) = \begin{cases} n - 1 & \text{if } uv \in E(\tilde{G}), \\ n - 2 & \text{if } uv \notin E(\tilde{G}). \end{cases}$$

Thus, two vertices of $\tilde{G} = K_{r,r}$ must be assigned distinct colors in any hamiltonian fuzzy coloring if they are not adjacent and may be assigned the same color if they are adjacent, that is, $hc(\tilde{G}) = \chi^f(\tilde{G}) = r$. The fuzzy graph, and $K_{r,r}$, $r \geq 2$, have the property that the number, $D(u, v)$ have two distinct values, one if u and v are adjacent and another if u and v are not adjacent. For each of these fuzzy graphs $\tilde{G}$ of order n, one of the values of $D(u, v)$ is $n - 1$ and the other is $n - 2$.

4. On the Circumference of Fuzzy Graphs Having Many Vertices with Prescribed Colors

Let c be a hamiltonian fuzzy coloring of a connected fuzzy graph $\tilde{G}$. For integers i and j with $1 \leq i \leq j \leq hc(c)$, we define $V(c; i, j) = \{u \in V(\tilde{G}) : i \leq c(u) \leq j\}$.

Let U be a set of vertices of $\tilde{G}$. If $|U| \geq 2$, then we define $dis(c; U)$

$= \min\{|c(u) - c(v)|\},$ where the minimum is taken over all distinct pairs u, v of vertices in U. If $|U| \leq 1,$ we define $dis(c; U) = hc(c)$. If $U = V(c; i, j),$ then we write $dis(c; U) = dis(c; i, j)$.

More simply, we write

$$V(i, j) = V(c; i, j)$$

$$dis(U) = dis(c; U) \text{ and}$$

$$dis(i, j) = dis(c; i, j)$$

if the hamiltonian fuzzy coloring c under discussion is clear.

5. Circumference

The length of a longest cycle in a connected fuzzy graph is called the *circumference* of $\tilde{G}$ and is denoted by $cir(\tilde{G})$.

If $\tilde{G}$ is a fuzzy tree, then we write $cir(\tilde{G}) = 0.$ For a hamiltonian fuzzy coloring of a connected fuzzy graph $\tilde{G}$ of order n, we now show that if the sets $V(i, j)$ are sufficiently large (as a function of n and $dis(i, j)),$ then $cir(\tilde{G})$ is large as well. First, we present a lemma.

Lemma 5.1. *Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 3,$ let c be a hamiltonian fuzzy coloring of $\tilde{G},$ and let k be an integer with $0 \leq k \leq n - 3.$ Assume that $cir(\tilde{G}) < n - k.$ Then $V(i, i + k)$ is an independent set in $\tilde{G}$ for every i with $1 \leq i \leq hc(c) - k.$*

Proof. Let i be an integer with $1 \leq i \leq hc(c) - k.$

Since $cir(\tilde{G}) < n - k,$ it follows that $D(u, v) \leq n - k - 2$ for every pair u, v of adjacent vertices of $\tilde{G}.$ Since c is a hamiltonian fuzzy coloring of $\tilde{G}.$

It follows that $|c(u) - c(v)| \geq k + 1$ for every pair u, v of adjacent vertices of $\tilde{G}.$

Moreover, $|c(u') - c(v')| \leq k$ for each pair u', v' of vertices in $V(i, i+k)$.

Therefore, $V(i, i+k)$ is an independent set in $\tilde{G}$. □

Theorem 5.2. *Let* $\tilde{G}$ *be a connected fuzzy graph of order* $n \geq 3$, *let c be a hamiltonian fuzzy coloring of* $\tilde{G}$, *and let i, j be a pair of integers with* $1 \leq i \leq j \leq hc(c)$ *and* $j - i \leq n - 3$.

If $|V(i, j)| \geq \dfrac{n + dis(i, j) + 2}{2}$, *then* $cir(\tilde{G}) \geq n - (j - i)$.

Proof. Assume that $cir(\tilde{G}) \leq n - (j - i) - 1$.

If $|V(i, j)| \leq 1$, then $|V(i, j)| < \dfrac{(n + dis(i, j) + 2)}{2}$, a contradiction.

Hence we may assume that $|V(i, j)| \geq 2$.

Since $cir(\tilde{G}) < n - (j - i)$, by Lemma 5.1, the set $V(i, j)$ is an independent set in $\tilde{G}$.

Now, let $W = V(\tilde{G}) - V(i, j)$.

Since $|V(i, j)| \geq 2$, there exist distinct vertices *x* and *y* in $V(i, j)$ with $|c(x) - c(y)| = dis(i, j)$ and so $D(x, y) \geq n - 1 - dis(i, j)$.

Hence there exists an $x - y$ path $\tilde{P}$ containing at least $n - dis(i, j)$ vertices of $\tilde{G}$.

On the other hand, since $V(i, j)$ is independent in $\tilde{G}$, the vertices *x* and *y* are in $V(i, j)$, and $\tilde{P}$ contains at most $|W|$ vertices that are not in $V(i, j)$, it follows that $\tilde{P}$ contains at most $|W| + 1$ vertices of $V(i, j)$.

Consequently, $\tilde{P}$ contains at most $2|W| + 1$ vertices. Thus $n - dis(i, j) \leq 2|W| + 1$, which implies that

$$|V(i,\ j)| < \frac{(n + dis(i,\ j) + 2)}{2},\ \text{a contradiction.} \qquad \square$$

Corollary 5.3. *Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 3$. If there exists a hamiltonian fuzzy coloring of $\tilde{G}$ such that at least $(n+2)/2$ vertices of $\tilde{G}$ are colored the same, then $\tilde{G}$ is hamiltonian.*

Proof. Let c be a hamiltonian fuzzy coloring of $\tilde{G}$ such that at least $(n+2)/2$ vertices of $\tilde{G}$ are colored the same, say i. Then

$$|V(i,\ i)| \geq \frac{n+2}{2} = \frac{n + dis(i,\ i) + 2}{2}.$$

It then follows from Theorem 5.2 that $cir(\tilde{G}) \geq n$ and so $\tilde{G}$ is hamiltonian.

$\square$

To see that Corollary 5.3 cannot be improved, consider the fuzzy graph $\tilde{G} = k_{r,\, r+1}$, where $r \geq 2$, with partite sets V_1 and V_2 such that $|V_1| = r$ and $|V_2| = r+1$. Then $\tilde{G}$ has order $n = 2r+1$ and

$$D(u,\ v) = \begin{cases} 2r - 2 & \text{if } u,\ v \in V_1, \\ 2r - 1 & \text{if } uv \in E(\tilde{G}), \\ 2r & \text{if } u,\ v \in V_2. \end{cases}$$

Observe that a coloring c is a hamiltonian fuzzy coloring of $\tilde{G}$ if and only if

$$|c(u) - c(v)| \geq \begin{cases} 2 & \text{if } u,\ v \in V_1, \\ 1 & \text{if } uv \in E(\tilde{G}), \\ 0 & \text{if } u,\ v \in V_2. \end{cases}$$

Let $V_1 = \{v_1,\ v_2,\ \ldots,\ v_r\}$. Define a hamiltonian fuzzy coloring c of $\tilde{G}$ by $c(u) = 1$ for all $u \in V_2$ and $c(v_i) = 2i$ for $1 \leq i \leq r$.

The exactly $r+1 = (n+1)/2$ vertices of $\tilde{G}$ are colored the same, but $\tilde{G}$ is not hamiltonian.

Corollary 5.4. *Let* $\tilde{G}$ *be a connected fuzzy graph of order* $n \geq 4$. *If there exist a hamiltonian fuzzy coloring c of* $\tilde{G}$ *and an integer i with* $1 \leq i \leq hc(c)$ *such that at least* $(n+2)/2$ *vertices of* $\tilde{G}$ *are colored i or* $i+1$, *then* $cir(\tilde{G}) \geq n-1$.

Proof. If there exist a hamiltonian fuzzy coloring c of $\tilde{G}$ and an integer i with $1 \leq i \leq hc(c)$ such that at least $(n+2)/2$ vertices of $\tilde{G}$ are colored i or $i+1$, then $|V(i, i+1)| \geq \max\{3, (n+2)/2\}$ and therefore, $dis(i, i+1) = 0$. It then follows from Theorem 5.2 that $cir(\tilde{G}) \geq n-1$. □

We now present another lower bound for the circumference of a connected fuzzy graph.

Theorem 5.5. *Let* $\tilde{G}$ *be a connected fuzzy graph of order* $n \geq 3$ *and let k be an integer such that* $0 \leq k \leq n-3$. *If there exists a hamiltonian fuzzy coloring c of* $\tilde{G}$ *such that*

(a) *The set* $V(1, k+1)$ *and* $V(hc(c)-k, hc(c))$ *from a partition of* $V(\tilde{G})$, *and*

(b) *There exists* $U \in \{V(1, k+1), V(hc(c)-k, hc(c))\}$ *such that* $|U| \geq 2$ *and*

$$|U| \leq \frac{n - dis(U)}{2}, \tag{2}$$

then $cir(\tilde{G}) \geq n-k$.

Proof. Let $W = V(\tilde{G}) - U$. We wish to prove that $cir(\tilde{G}) \geq n-k$.

Assume to the contrary that $cir(\tilde{G}) < n-k$. By Lemma 5.1, the sets U and W are independent in $\tilde{G}$. Since U and W are disjoint and $V(\tilde{G}) = V \cup W$, it follows that $\tilde{G}$ is a bipartite fuzzy graph with partite sets U and W.

Since $|U| \geq 2$, there exist two distinct vertices $u, v \in U$ such that $|c(u) - c(v)| = dis(U)$ and so $D(u, v) \geq n - 1 - dis(U)$.

Thus $\tilde{G}$ contains a $u-v$ path $\tilde{P}$ of length at least $n-1-dis(U)$ and so at most $dis(U)$ vertices of $\tilde{G}$ do not belong to $\tilde{P}$.

Since $\tilde{G}$ is bipartite with partite sets U and W and $u, v \in U$, there exists an integer j with $2 \leq j \leq |U|$ such that $\tilde{P}$ contains exactly j vertices of U and exactly $j-1$ vertices of W. Thus $2|U|-1 \geq 2j-1 \geq n-dis(U)$.

This means that $|U| \geq (n+1-dis(U))/2$, which contradicts (2).

Therefore, $cir(\tilde{G}) \geq n-k$. □

6. On the Circumference and Color Sequences of Fuzzy Graph

For a hamiltonian fuzzy coloring c of a connected fuzzy graph $\tilde{G}$, let $\mathscr{C}$ be the set of all colors assigned to the vertices of $\tilde{G}$, that is $\mathscr{C} = \{c(v) : v \in (\tilde{G})\}$. If $\mathscr{C} = \{c_1, c_2, ..., c_p\}$, where $c_1 < c_2 < \cdots < c_p = hc(c)$, then $Seq(c) = (c_1, c_2, ..., c_p)$ is called the *color sequence* of c. Similarly, as in [8], a set $S = \{u, v\}$ of two distinct vertices of $\tilde{G}$ is called *c-pair* if $c(u) = c(v)$. We define $c(S) = c(u) = c(v)$. A set $S = \{u, v\}$ of two distinct vertices of $\tilde{G}$ is called *c-semipair* if $|c(u) - c(v)| \leq 1$. For integers a and b with $a \leq b$, the integer interval $[a \cdots b]$ is defined as $\{x \in Z : a \leq x \leq b\}$.

Theorem 6.1. *For a connected fuzzy graph* $\tilde{G}$ *of order* $n \geq 4$ *and an integer* k *with* $0 \leq k \leq n-3$, *let* c *be a hamiltonian fuzzy coloring of* $\tilde{G}$ *with* $Seq(c) = (c_1, c_2, ..., c_p)$, *where* $p \geq 2$, *such that*

$$\mathscr{C} \subseteq [c_1 \cdots c_1 + k] \cup [c_p - k \cdots c_p]. \tag{3}$$

If at least one of the tree conditions

(a) $k = 0$;

(b) $c_p - k \leq c_1 + k$ *and* $\mathscr{C} \cap [c_p - k \cdots c_1 + k]$ *is nonempty;*

(c) *there exist* c-*semipairs* S *and* S', *at least one of which is a* c-*pair, such that the colors of the vertices of* S *are at most* $c_1 + k$ *and the colors of the vertices of* S' *are at least* $c_p - k$, *is satisfied, then* $cir(\tilde{G}) \geq n - k$.

Proof. We may assume, without loss of generality, that $c_1 = 1$. Since $p \geq 2$, it follows that $c_p > 1$. Define $V_1 = V(1, k+1)$, $V_2 = V(c_p - k, c_p)$, $W_1 = V(1, 1)$, $W_p = (c_p, c_p)$.

Thus W_1 and W_2 are nonempty, as are V_1 and V_2. Moreover, if (a) holds, then $V_1 = W_1$ and $V_2 = W_2$. More generally, $W_i \subseteq V_i$ for $i = 1, 2$ and $V_1 \cup V_2 = V(\tilde{G})$ by (3).

We wish to prove that $cir(\tilde{G}) \geq n - k$.

Assume to the contrary that $cir(\tilde{G}) < n - k$. By Lemma 5.1, V_1 and V_2 are independent sets in $\tilde{G}$. Since $V_1 \cup V_2 = V(\tilde{G})$, it follows that $V_1 \cap V_2$ is a set of isolated vertices of $\tilde{G}$. However, since $\tilde{G}$ is a nontrivial connected fuzzy graph $\tilde{G}$ has no isolated vertices and so $V_1 \cap V_2 = \varnothing$.

Thus condition (b) does not hold, implying that at least one of conditions (a) and (c) holds. Since $V(\tilde{G})$ is partitioned into the independent sets V_1 and V_2, it follows that $\tilde{G}$ is a bipartite fuzzy graph with partite sets V_1 and V_2.

Because $n \geq 4$, it follows that if (a) holds, then $|W_1| \geq 2$ or $|W_2| \geq 2$ and so either V_1 or V_2 contains a c-pair.

On the other hand, if (c) holds, then V_1 or V_2 contains a c-pair.

In either case, at least once of V_1 and V_2 contains a c-pair. Let $\{i, j\} = \{1, 2\}$ such that V_j contains a c-pair, say $\{x, y\}$.

Since c is a hamiltonian fuzzy coloring, $D(x, y) = n - 1$ and so there exists a hamiltonian $x - y$ path in $\tilde{G}$. Since $x, y \in V_j$ and $\tilde{G}$ is a bipartite fuzzy graph with partite sets V_i and V_j, we have $|V_j| = |V_i| + 1$, which implies that $D(x', y') \leq n - 3$ for every pair x', y' of distinct vertices in V_i. Thus V_i contains no c-pair. Since $n \geq 4$, it follows that $|V_i| \geq 2$.

Therefore, $V_i \neq W_i$ and (a) does not hold. Hence (c) holds.

Consequently, V_i contains a c-semipair, say $\{x^*, y^*\}$. Since $|c(x^*) - c(y^*)| \leq 1$, we have $D(x^*, y^*) \geq n - 2$, which is a contradiction.

□

Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 4$, let k be an integer with $0 \leq k \leq n - 3$, and let c be a hamiltonian fuzzy coloring of $\tilde{G}$. Now, suppose that the sets $V_1 = V(1, k+1)$ and $V_2 = V(hc(c) - k, hc(c))$ from a partition of $V(\tilde{G})$, where, say $|V_1| \geq |V_2|$. Thus $|V_1| \geq n/2$. Suppose that we wish to apply Theorem 6.1 to such a fuzzy graph $\tilde{G}$. If the set V_1 contains a c-pair, so that $dis(1, k+1) = 0$, and $|V_1| \geq (n+2)/2$, then $cir(\tilde{G}) \geq n - k$ by Theorem 5.2. If, on the other hand, V_1 does not contain a c-pair but contains a c-semipair, so that $k \geq 1$ and $dis(1, k+1) = 1$ and $|V_1| \geq (n+3)/2$, then $cir(\tilde{G}) \geq n - k$ by Theorem 5.2. Hence to apply Theorem 6.1 to a fuzzy graph $\tilde{G}$ satisfying the conditions described above, we need only deal with the situation where $n/2 \leq |V_1| \leq (n+2)/2$.

We have already noted that if some hamiltonian fuzzy coloring assigns the same color, namely 1, to every vertex in a connected fuzzy graph $\tilde{G}$ of order $n \geq 3$, then $\tilde{G}$ is a hamiltonian-connected. By Theorem 6.1(a), if there exists a hamiltonian fuzzy coloring that assigns one of two colors to every vertex of $\tilde{G}$, then $\tilde{G}$ is hamiltonian.

Corollary 6.2. *Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 4$. If there exists a hamiltonian fuzzy coloring c of $\tilde{G}$ such that $Seq(c) = 1$ or $Seq(c) = (1, r)$ for some $r \geq 2$, then $\tilde{G}$ is a hamiltonian.*

The following theorem and corollary are due to Bondy and Chvatal [2].

Closure 6.3. The fuzzy graph obtained from $\tilde{G}$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting fuzzy graph at each stage) until no such pair remains.

Theorem 6.4. *A fuzzy graph is hamiltonian if and only if its closure is hamiltonian.*

Corollary 6.5. *If the closure of a fuzzy graph* $\tilde{G}$ *of order at least* 3 *is complete, then* $\tilde{G}$ *is a hamiltonian.*

Thus Corollary 6.5 gives a sufficient condition for a fuzzy graph to be hamiltonian. Let $\tilde{G}_0$ be a hamiltonian-connected fuzzy graph of a even order $n = 2k \geq 6.$ Then $\tilde{G}_0$ contains a hamiltonian fuzzy cycle $u_1, v_1, u_2, v_2, \ldots,$ $u_k, v_k, u_1.$ We construct a new fuzzy graph $\tilde{G}$ from $\tilde{G}_0$ and k pairwise a vertex-disjoint complete fuzzy graphs of order $l \geq 3,$ which we denote by $F_1, F_2, \ldots, F_k$ by indentifying an edge of F_i with the edge $u_i v_i$ for each $i\ (1 \leq i \leq k).$ The fuzzy graph $\tilde{G}$ has order kl but it is not hamiltonian-connected, as there is no hamiltonian $u_i - v_i$ path for any $i\ (1 \leq i \leq k).$ On the other hand, there is a hamiltonian fuzzy coloring of $\tilde{G}$ with two colors, namely, assign $u_i\ (1 \leq i \leq k)$ the color $l - 1$ and assign all other vertices of $\tilde{G}$ the color 1. By the remark above, $\tilde{G}$ is a hamiltonian. We now consider the Bondy and Chvatal closure of this fuzzy graph $\tilde{G}.$ Let $x \in V(F_i) - \{u_i, v_i\}$ $(1 \leq i \leq k)$ and $y \notin V(F_i)$ be nonadjacent vertices in $\tilde{G}.$ Then $dig_{\tilde{G}} x = l - 1$ and $dig_{\tilde{G}} y \leq (2k - 1) + (l - 1) - 1.$ So

$$dig_{\tilde{G}} x + dig_{\tilde{G}} y \leq 2k + (2l - 4) = kl - (k - 2)(l - 2) < kl$$

which implies that no vertex in $V(F_i) - \{u_i, v_i\}$ can be adjacent to a vertex in $V(F_i)$ in the formation of the closure of $\tilde{G}.$ Thus the closure of $\tilde{G}$ is not complete and, even though Corollary 6.2 shows that $\tilde{G}$ is hamiltonian, Corollary 6.5 does not.

The closure of the complete bipartite fuzzy graph $K_{r,r}$, $r > 2$ is complete, however, therefore, by Corollary 6.5, $K_{r,r}$, is hamiltonian. On the other hand, there is no hamiltonian fuzzy coloring of $K_{r,r}$ that assigns one of two colors to each of its vertices. Hence $K_{r,r}$ cannot to show to be hamiltonian with the aid of Corollary 6.2. Therefore, Corollaries 6.2 and 6.5 are independent.

The next result gives a sufficient condition for a connected fuzzy graph of order $n \geq 5$ to contain a cycle of length $n-1$ or n.

Corollary 6.6. *Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 5$. If there exists a hamiltonian fuzzy coloring c of $\tilde{G}$ with $hc(c) \geq 4$ satisfying one of the following conditions*:

(1) $Seq(c) = (1, 2, hc(c)-1, hc(c))$;

(2) $Seq(c) = (1, hc(c)-1, hc(c))$ *and there exists a c-pair S with* $c(S) = 1$;

(3) $Seq(c) = (1, 2, hc(c))$ *and there exists a c-pair S with* $c(S) = hc(c)$;

then $cir(\tilde{G}) \geq n-1$.

The following result shows that, in general, the hamiltonian chromatic number of a fuzzy graph and the circumference cannot both be small.

Theorem 6.7. *If $\tilde{G}$ be a connected fuzzy graph of order $n \geq 4$ with* $2 \leq hc(\tilde{G}) \leq n-1$, *then* $cir(\tilde{G}) + hc(\tilde{G}) \geq n+2$.

Proof. Let c be a minimum hamiltonian coloring of $\tilde{G}$ with color set $\mathscr{C}$. Then 1 and $hc(c)$ belong to $\mathscr{C}$, and $hc(c) = hc(\tilde{G})$.

If $\mathscr{C} = \{1, hc(c)\}$, then $cir(\tilde{G}) = n$ by Corollary 6.2 and therefore, $cir(\tilde{G})$

$\geq n + 2 - hc(\tilde{G})$. So we may assume that $\mathscr{C} \neq \{1, hc(c)\}$. Then $2 \leq hc(c) - 1$ and the set $\mathscr{C} \cap [2 \cdots hc(c) - 1]$ is nonempty. It then follows from Theorem 6.1(b) that $cir(\tilde{G}) \geq n + 2 - hc(\tilde{G})$ again. □

On consequence of Theorem 6.7, we have the following:

Corollary 6.8. *Let $\tilde{G}$ be a connected fuzzy graph of order $n \geq 4$. If $hc(\tilde{G}) = 2$, then $\tilde{G}$ is hamiltonian. If $hc(\tilde{G}) = 3$, then $cir(\tilde{G}) \geq n - 1$.*

The inequality of Theorem 6.7 is also sharp for $hc(\tilde{G}) = 3$ since the Petersen fuzzy graph has hamiltonian chromatic number 3, order 10, and circumference 9, Figure 6.1 (every two nonadjacent vertices of $\tilde{P}$ are circumference while no two adjacent of $\tilde{P}$ connected by a path of length 9 but are connected by a path of length 8).

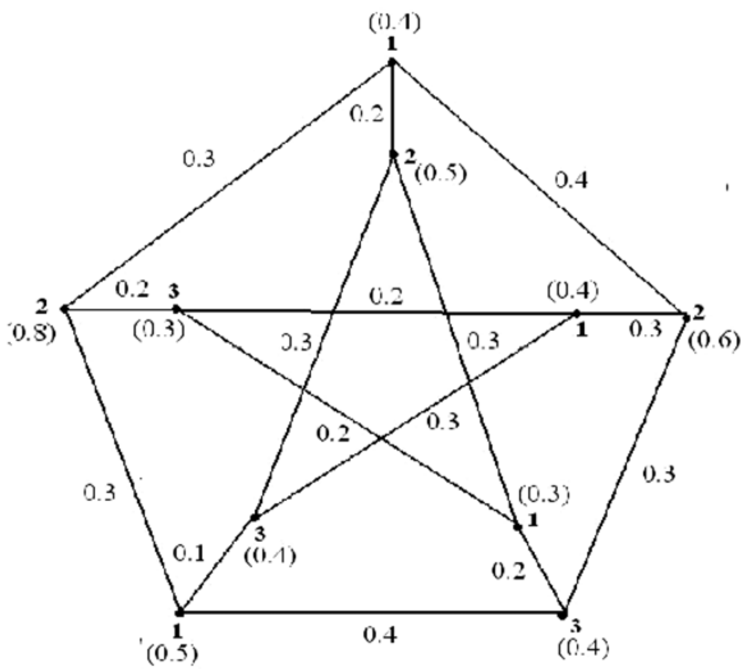


Figure 6.1

References

[1] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recog. Lett. 6 (1987), 297-302.

[2] J. A. Bondy and V. Chvatal, A method in graph theory, Discrete Math. 15 (1976), 111-136.

[3] G. Chartrand, D. Erwin and F. Harary, Radio labelings of graphs, Bull, Inst. Combin. Appl. 33 (2001), 77-85.

[4] G. Chartrand, D. Erwin and P. Zhang, A graph labeling problem suggested by FM channel restrictions, Bull, Inst. Combin. Appl. 43 (2005), 43-57.

[5] G. Chartrand, D. Erwin and P. Zhang, Radio antipodal colorings of cycles, Congr. Numer. 144 (2000), 129-141.

[6] G. Chartrand, D. Erwin and P. Zhang, Radio antipodal colorings of graphs, Math. Bohem. 127 (2007), 57-69.

[7] G. Chartrand, L. Nebesky and P. Zhang, Radio *k*-colorings of paths, Discuss. Math. Graph Theory 24 (2004), 5-21.

[8] G. Chartrand, L. Nebesky and P. Zhang, Hamiltonian colorings of graphs, Discrete Appl. Math. 146 (2005), 257-272.

[9] G. Chartrand, L. Nebesky and P. Zhang, On hamiltonian colorings of graphs, Discrete Math. 290 (2005), 133-143.

[10] G. Chartrand and P. Zhang, Chromatic Graph Theory, CRC Press, Taylor & Francis Group, 2009.

[11] C. Eslahchi and B. N. Onagh, Vertex-strength of fuzzy graphs, Inter. J. Math. Math. Sci. 2006 (2006), Article ID 43614, 9 pp.

[12] Isnaini Rosyida, S. Lavanya, Widodo, Ch. R. Indrati and K. A. Sugeng, An upper bound for fuzzy chromatic number of fuzzy graphs and their complement, AKCE Int. J. Graphs Comb. (submitted).

[13] Klas Markstrom, Even cycle decompositions of 4-regular graphs and line graphs, Discrete Math. 312(17) (2012), 2676-2681.

[14] S. Lavanya and R. Sattanathan, Fuzzy total coloring of fuzzy graphs, Inter. J. Inform. Tech. Knowledge Manage. 2(1) (2009), 37-39.

[15] S. Muñoz, M. T. Ortuño, J. Ramírez and J. Yáñez, Coloring fuzzy graphs, Omega 33(3) (2005), 211-221.

[16] A. Nagoor Gani, V. T. Chandrasekaran and R. Balasubramanian, Fuzzy bipartite graphs, J. Quantitative Methods 2(2) (2006), 54-60.

[17] A. Nagoor Gani and M. Basheer Ahamed, Order and size in fuzzy graph, Bull. Pure Appl. Sci. 22E(1) (2003), 145-148.

[18] G. Nirmala and K. Dhanabal, Special planar fuzzy graph configurations, Inter. J. Sci. Res. Publ. 12(7) (2012), 1-4.

[19] G. Nirmala and K. Uma, Fuzzy shortest route algorithm for telephone line connection, Inter. J. Sci. Res. Publ. 2(8) (2012), 1-5.

[20] G. Nirmala and M. Vijaya, Hamiltonian fuzzy cycles on K_{2n+1} fuzzy graph, Inter. J. Sci. Res. Publ. 2(11) (2012), 1-6.

[21] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and Their Application to Cognitive and Decision Processes, L. A. Zadeh, K. S. Fu and M. Shimura, eds., Academic Press, New York, 1975, pp. 77-95.