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CLASSIFICATION SCHEMES FOR POSITIVE SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS ON A MEASURE CHAIN

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Abstract

Classification schemes for nonoscillatory solutions of neutral differential equations are given, and necessary as well as sufficient or sufficient conditions are provided.

1. Introduction

In this paper, we are concerned with the nonlinear differential equations on a measure chain of the form

$$(x(t) - x(t - \tau))^{\Delta} + F(t, x(t - \delta)) = 0, \quad t \geq t_0, \tag{1}$$

where $\tau > 0$, $\sigma \geq 0$, $F(t, x)$ is a real-valued function defined on $\{t : t \geq t_0\}$ $\times R$ which is nondecreasing and continuous in the second variable x and satisfies $F(t, x) > 0$ for $x > 0$.

In this paper, we consider classification schemes for all eventually positive solutions of (1) and our classification schemes are more full and clear. We need some preliminary definitions.

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Definition 1. Let T be a closed subset of the real number R with the property that

$$\sigma(t) := \inf\{\tau > t : \tau \in T\} \in T$$

and

$$\rho(t) := \sup\{\tau < t : t \in T\} \in T,$$

for all $t \in T$ with $t < \sup T$ and $t > \inf T$, respectively. We assume throughout that T has the topology that it inherits from the standard topology on the real number R. If $\sigma(t) > t$, then we say t is *right scattered*, while if $\rho(t) < t$, then we say t is *left scattered*. If $\sigma(t) = t$, then we say t is *right dense*, while if $\rho(t) = t$, then we say t is *left dense*.

Throughout this paper, we always assume that a is point in T and $\sup T = \infty$. Define the interval in T,

$$[a, \infty) := \{t \in T \text{ such that } a \leq t\}.$$

Other types of intervals are defined similarly.

Definition 2. Assume $x : T \to R$ and fix $t \in T$. Then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the *delta derivative* of $x(t)$.

It can be shown that if $x : T \to R$ is continuous at $t \in T$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $T = Z$, where Z is the set of integers, then

$$x^{\Delta}(t) = x(t+1) - x(t).$$

Of course, if $T = R$, then $x^{\Delta}(t) = x'(t)$.

Definition 3. If $F^{\Delta}(t) = f(t)$, then we define an integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

and we stipulate that $\int_a^{\infty} \Delta t = \infty$.

2. Classification of Positive Solutions

In this section, we will show that all positive solutions of (1) can be divided into four types. First of all, we give their definitions.

Definition 1. A positive solution $x(t)$ of equation (1) is called to belong to *A-type*, if it can be expressed in the form

$$x(t) = \alpha t + \beta(t), \tag{2}$$

where $\alpha > 0$ is a constant, $\beta(t)$ is bounded.

Definition 2. A positive solution $x(t)$ of equation (1) is called to belong to *B-type*, if it can be expressed in the form

$$x(t) = \alpha t + \theta(t), \tag{3}$$

where $\alpha > 0$ is a constant, $\theta(t)$ is unbounded and $\lim_{t\to\infty} \theta(t)/t = 0$.

Definition 3. A positive solution $x(t)$ of equation (1) is called to belong to *C-type*, if it can be expressed in the form $x(t) = \beta(t)$, and *D-type* if $x(t) = \theta(t)$.

Theorem 1. *If $x(t)$ is an eventually positive solution of* (1), *then it belongs to one of A-type, B-type, C-type or D-type.*

Proof. Define

$$z(t) = x(t) - x(t - \tau). \tag{4}$$

In view of (1), we have $z^{\Delta}(t) < 0$. Thus, $z(t) > 0$ or $z(t) < 0$. The later case is impossible. Otherwise, there is a $t_1 \geq t_0$ such that $z^{\Delta}(t) < 0$, $z(t) < 0$

and $x(t-\tau) > 0$ for $t \geq t_1$, then

$$x(t_1 + j\tau) = \sum_{i=1}^{j} z(t_1 + i\tau) + x(t_1)$$

$$\leq jz(t_1) + x(t_1) \to -\infty \text{ as } j \to \infty,$$

which contradicts the assumed positivity of $x(t)$. It follows that $z(t) > 0$ and let $\lim_{t\to\infty} z(t) = l$. Then either $l > 0$ or $l = 0$. It is obvious that $z(t) \geq l$. Let

$$M = \max\{x(t) : t_1 - \tau \leq t \leq t_1\} \text{ and } m = \min\{x(t) : t_1 - \tau \leq t \leq t_1\}.$$

If $t_1 \leq t \leq t_1 + \tau$, then we have

$$x(t) = z(t) + x(t-\tau) \leq M + l + (z(t_1) - l)$$

and

$$x(t) \geq m + l + (z(t_1 + \tau) - l).$$

By induction, we have

$$x(t) \leq M + (k+1)l + \sum_{i=0}^{k} (z(t_1 + i\tau) - l) \tag{5}$$

and

$$x(t) \geq m + (k+1)l + \sum_{i=0}^{k} (z(t_1 + i\tau + \tau) - l) \tag{6}$$

for $t_1 + k\tau \leq t \leq t_1 + (k+1)\tau$ and $k = 1, 2, \dots$. Hence

$$\frac{x(t)}{t} \leq \frac{M}{t} + \frac{k+1}{t} l + \frac{\sum_{i=0}^{k} (z(t_1 + i\tau) - l)}{t_1 + k\tau}$$

and

$$\frac{x(t)}{t} \geq \frac{m}{t} + \frac{k+1}{t} l + \frac{\sum_{i=0}^{k} (z(t_1 + i\tau + \tau) - l)}{t_1 + k\tau + \tau}.$$

Note that

$$\lim_{t\to\infty}\frac{\sum_{i=t_1}^{t}(z(i)-l)}{t} = \lim_{t\to\infty}(z(t)-l) = 0.$$

Thus, we have

$$\lim_{t\to\infty}\frac{k+1}{t} = \frac{1}{\tau} \text{ and } \lim_{t\to\infty}\frac{x(t)}{t} = \frac{l}{\tau}. \tag{7}$$

If $l = 0$, then either $x \in C$ or $x \in D$. When $l > 0$, we have $x \in A$ or $x \in B$. The proof is complete.

3. Existence

First, we will give some lemmas which are important in proving our results on existence. In view of (7), we prove the following lemma easily.

Lemma 1. *If $x(t)$ is an eventually positive solution of* (1) *such that $\lim_{t\to\infty} x(t)/t = p$, then $\lim_{t\to\infty} z(t) = p\tau$.*

On the other hand, let

$$a(t) = \sum_{i=0}^{k}(z(t_1 + i\tau) - l) \text{ and } b(t) = \sum_{i=0}^{k}(z(t_1 + i\tau + \tau) - l).$$

Clearly, $a(t)$ and $b(t)$ are nondecreasing positive sequence. It is easy to see that $a(t)$ is bounded if and only if $b(t)$ is bounded. Thus, we have

$$\theta(t) = x(t) - \frac{l}{\tau}t \leq a(t) + \left[M + l\left(k + 1 - \frac{1}{\tau}t\right)\right]$$

and

$$\theta(t) \geq b(t) + \left[m + l\left(k + 1 - \frac{1}{\tau}t\right)\right].$$

If $\theta(t)$ is unbounded, then $a(t)$ is unbounded. This implies that $b(t)$ is unbounded.

Lemma 2. *If $x(t)$ is B-type solution of* (1), *then $\theta(t)$ is eventually positive.*

The following lemma can be seen in [1] and its proof will be omitted.

Lemma 3. *If* $f(t) \geq 0,\ t \in [t_1, s],\ s > 0,$ *then*

$$\sum_{i=0}^{\infty}\left(\int_{t_1+is}^{\infty} f(t)\Delta t\right)\tau < \infty \ \text{ if and only if } \int_{t_1}^{\infty} tf(t)\Delta t < \infty.$$

Lemma 4. *Suppose that*

$$\int_{t_0}^{\infty} F(t,\ \omega(t-\sigma))\Delta t < \infty \ \text{ for some } \omega > 0, \tag{8}$$

then for every positive number $\alpha < \omega,$ *the equation*

$$(x(t) - x(t-\tau) + F(t,\ \alpha(t-\sigma) + x(t-\delta))) = 0 \tag{9}$$

has an eventually positive solution $y(t)$ *such that* $\lim_{t\to\infty} y(t)/t = 0.$

Proof. Let $\alpha < \omega$ be a positive number. In view of (8), there exists a $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} F(t,\ \omega(t-\sigma))\Delta t < \frac{\tau}{2}(\omega - \alpha).$$

Set

$$H(t) = \begin{cases} \displaystyle\int_{t}^{\infty} F(t,\ \omega(t-\sigma))\Delta t, & t \geq t_1, \\[2ex] \dfrac{t - t_1 + \tau}{\tau} H(t_1), & t_1 - \tau \leq t < t_1, \\[2ex] 0, & t < t_1 - \tau. \end{cases}$$

Clearly, $H(t) \geq 0.$ Define

$$u(t) = \int_{0}^{\infty} H(t - s\tau)\Delta s, \quad t \geq t_1.$$

It is obvious that $u(t) - u(t-\tau) = H(t),$ $0 < u(t) \leq (\omega - \alpha)t$ and $\lim_{t\to\infty} u(t)/t = 0.$

Let X denote the Banach space $l_{\infty}^{t_1}$ of all real sequences $x = \{x(t)\}_{t=t_1}^{\infty}$ with the norm $\| x \| = \sup_{t \geq t_1} | x(t) |.$ Define a set Ω by

$$\Omega = \{x(t) \in X : 0 \le x(t) \le u(t),\ t \ge t_1\}$$

and an operator T on Ω by

$$(Tx)(t) = \begin{cases} x(t-\tau) + \int_t^\infty F(s,\ \alpha(s-\sigma) + x(s-\delta))\Delta s, & t \ge t_1 + \mu, \\ \dfrac{tu(t)(Tx)(t_1+\mu)}{(t_1+\mu)u(t_1+\mu)} + u(t)\left(1 - \dfrac{t}{t_1+\mu}\right), & t_1 \le t < t_1 + \mu, \end{cases}$$

where $\mu = \max\{\tau, \sigma\}$. For $x \in \Omega$, we have

$$0 \le (Tx)(t) \le u(t-\tau) + \int_t^\infty F(s,\ \alpha(s-\sigma) + (\omega-\alpha)s)\Delta s$$

$$= u(t-\tau) + H(t) = u(t).$$

That is, $T\Omega \subset \Omega$.

Define a series of sequences $\{y^{(k)}(t)\}$, $k = 0, 1, 2, \ldots$ as follows:

$$y^{(0)}(t) = u(t), \qquad t \ge t_1,$$

$$y^{(k)}(t) = (Ty^{(k-1)})(t), \quad t \ge t_1,\ k = 1, 2, \ldots.$$

By induction, we can prove that

$$0 < y^{(k)}(t) \le y^{(k-1)}(t) \le u(t), \quad t \ge t_1,\ k = 1, 2, \ldots.$$

Then there exists $y(t) \in \Omega$ such that $\lim_{k\to\infty} y^{(k)}(t) = y(t),\ t \ge t_1$. Clearly, $y(t) > 0$ for $t \ge t_1$ and $y(t) = (Ty)(t)$. It is a positive solution of (12). The proof is complete.

Theorem 2. *Equation* (1) *has A-type positive solution if and only if*

$$\int_{t_0}^\infty sF(s,\ \omega(s-\sigma))\Delta s < \infty \ \textit{for some}\ \omega > 0. \tag{10}$$

Proof. If (10) holds, then it follows from Lemma 3 that

$$\sum_{i=0}^\infty \left(\int_{t_1+i\tau}^\infty F(s,\ \omega(s-\sigma))\Delta s\right) < \infty.$$

Then $u(t)$ defined in the proof of Lemma 4 is bounded. Since $y(t) \le u(t)$

for $t \geq t_1$, the solution $y(t)$ of (9) is bounded. It is clear that $\alpha t + y(t)$ is an A-type solution of equation (1).

Conversely, let $x(t)$ be an A-type positive solution of equation (1). Then $x(t)$ can be expressed in the form (2). Thus, there exist two numbers $t_1 \geq t_0$ and $0 < \omega < \alpha$ such that $|\beta(t)| \leq L$ and $x(t) \geq \omega t$ for $t \geq t_1$. In view of Lemma 1, we have $\lim_{t \to \infty} z(t) = \alpha\tau$ and obtain

$$z(t) = \alpha\tau + \int_t^\infty F(s, x(s - \delta))\Delta s \geq \alpha\tau + \int_t^\infty F(s, \omega(s - \sigma))\Delta s.$$

That is,

$$x(t) \geq x(t - \tau) + \alpha\tau + \int_t^\infty F(s, \omega(s - \sigma))\Delta s.$$

By induction, we can prove that

$$\begin{aligned} \alpha(t_1 + k\tau) + L &\geq x(t_1 + k\tau) \\ &\geq x(t_1) + k\alpha\tau + \sum_{i=1}^{k} \int_{t_1 + i\tau}^\infty F(t, \omega(t - \sigma))\Delta t, \quad t \geq 0. \end{aligned}$$

Thus, we have

$$\sum_{i=1}^{k} \int_{t_1 + i\tau}^\infty F(t, \omega(t - \sigma))\Delta t \leq \alpha\tau + L - x(t_1).$$

Letting $k \to \infty$,

$$\sum_{i=1}^{k} \int_{t_1 + i\tau}^\infty F(t, \omega(t - \sigma))\Delta t < \infty.$$

In view of Lemma 3, we see that (10) holds. The proof is complete.

Theorem 3. *Equation* (1) *has a B-type positive solution if*

$$\int_{t_0}^\infty F(t, \lambda(t - \sigma))\Delta t < \infty \text{ and } \int_{t_0}^\infty tF(t, \omega(t - \sigma))\Delta t = \infty \tag{11}$$

for some $0 < \omega < \lambda$. *Conversely, if equation* (1) *has a B-type positive solution, then* (11) *holds for some* $0 < \lambda < \omega$.

Proof. Let $\alpha \in (\omega, \lambda)$ be a constant. In view of Lemma 4, equation (2) has a positive solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t = 0$. It is clear that $\alpha t + y(t)$ is a solution of equation (1). Suppose that $y(t)$ is bounded. As in the proof of Theorem 2, we see that (10) holds. This is a contradiction. Thus, $\alpha t + y(t)$ is a B-type solution of (1).

Let $x(t) = \lambda t + \theta(t)$ be a B-type of equation (1), where $\lambda > 0$ is a constant and $\theta(t)$ is an unbounded real sequence satisfying $\lim_{t\to\infty} \theta(t)/t = 0$. Then $\lim_{t\to\infty} x(t)/t = \lambda$ and $\lim_{t\to\infty} z(t) = \lambda\tau > 0$ by Lemma 1. In view of Lemma 2, we know that $\theta(t)$ is eventually positive. Thus, there is a number $t_1 \geq t_0$ such that $\theta(t-\delta) > 0$ for $t \geq t_1$. We have

$$\begin{aligned} z(t_1) - \lambda\tau &= \int_{t_1}^{\infty} F(t,\, x(t-\delta))\Delta t \\ &= \int_{t_1}^{\infty} F(t,\, \lambda(t-\sigma) + \theta(t-\delta))\Delta t \\ &\geq \int_{t_1}^{\infty} F(t,\, \lambda(t-\sigma))\Delta t. \end{aligned}$$

On the other hand, $\theta(t)$ satisfies the equation

$$(\theta(t) - \theta(t-\delta))^{\Delta} + F(t, \lambda(t-\sigma) + \theta(t-\delta)) = 0.$$

Note that $\lim_{t\to\infty} z(t) = \lambda\tau$, and we have $\lim_{t\to\infty}(\theta(t) - \theta(t-\delta)) = 0$. Summing the above equation from t to ∞, we obtain

$$\theta(t) = \theta(t-\delta) + \int_{t}^{\infty} F(s,\, \lambda(s-\sigma) + \theta(s-\delta))\Delta s.$$

Let $\omega > \lambda$ be a constant. Since $\lim_{t\to\infty} \theta(t)/t = 0$, there is a number $t_1 \geq t_0$ such that $\theta(t-\delta) < (\omega - \lambda)(t-\sigma)$ for $t \geq t_1$. Thus, we have

$$\begin{aligned} \theta(t) &\leq \theta(t-\delta) + \int_{t}^{\infty} F(s,\, \omega(s-\sigma))\Delta s \\ &\leq \theta(t-k\tau) + \sum_{i=1}^{k-1} \int_{t-i\tau}^{\infty} F(s,\, \omega(s-\sigma))\Delta s \end{aligned}$$

$$\leq M + \sum_{i=1}^{k} \int_{t_1 + i\tau}^{\infty} F(s, \omega(s - \sigma))\Delta s,$$

where $t_1 \leq t - k\tau \leq t_1 + \tau$ and $M = \max\{\theta(t) : t_1 \leq t \leq t_1 + \tau\}$. Since $\theta(t)$ is unbounded, we have

$$\sum_{i=1}^{\infty} \left(\int_{t_1 + i\tau}^{\infty} F(s, \omega(s - \sigma))\Delta s \right) = \infty.$$

By Lemma 3, we know that (11) holds. The proof is complete.

Theorem 4. *Equation* (1) *has a C-type positive solution if and only if*

$$\int_{t_0}^{\infty} tF(t, c)\Delta t < \infty \text{ for some } c > 0. \tag{12}$$

Proof. Let (12) hold. It follows by Lemma 3 that

$$\sum_{i=0}^{\infty} \left(\int_{t_0 + i\tau}^{\infty} F(t, c)\Delta t \right) < \infty.$$

Then there exists a number $t \geq t_0$ such that

$$\sum_{i=0}^{\infty} \left(\int_{t + i\tau}^{\infty} F(h, c)\Delta h \right) \leq c \text{ for } t \geq t_1.$$

Set

$$H(t) = \begin{cases} \int_{t}^{\infty} F(s, c)\Delta s, & t \geq t_1, \\ \dfrac{t - t_1 + \tau}{\tau} H(t_1), & t_1 - \tau \leq t < t_1, \\ 0, & t < t_1 - \tau. \end{cases}$$

Clearly, $H(t) \geq 0$. Define

$$u(t) = \int_{0}^{\infty} H(t - i\tau)\Delta t, \quad t \geq t_1.$$

It is obvious that $u(t) - u(t - \tau) = H(t)$, $M < u(t) \leq c$ for $t \geq t_1 + \tau$, where $M = \int_{t_1 + \tau}^{\infty} F(t, c)\Delta t$. Let $y(t) = u(t) - M$. Then we have

$$y(t) = y(t-\tau) + H(t) = y(t) + \int_t^\infty F(t,\ c)\Delta t, \text{ for } t \geq t_1 + \tau.$$

Let X denote the Banach space $l_\infty^{t_2}$ of all real sequences $x = \{x(t)\}_{t=t_2}^\infty$ with the norm $\| x \| = \sup_{t \geq t_2} | x(t) |$, where $t_2 = t_1 + \tau$. Define a set Ω by

$$\Omega = \{x(t) \in X : 0 \leq x(t) \leq u(t),\ t \geq t_2\}$$

and an operator T on Ω by

$$(Tx)(t) = \begin{cases} x(t-\tau) + \displaystyle\int_t^\infty F(s,\ M + x(s-\delta))\Delta s, & t \geq t_2 + \sigma, \\ \dfrac{ty(t)(Tx)(t_2+\delta)}{(t_1+\sigma)u(t_2+\delta)} + y(t)\left(1 - \dfrac{t}{(t_1+\sigma)}\right), & t_2 \leq t < t_2 + \sigma. \end{cases}$$

It is clear that $T\Omega \subset \Omega$.

Define a series of sequences $\{x^{(k)}(t)\}$, $k = 0, 1, 2, \ldots$ as follows:

$$x^{(0)}(t) = y(t), \qquad t \geq t_2,$$

$$x^{(k)}(t) = (Tx^{(k-1)})(t), \quad t \geq t_1,\ k = 1, 2, \ldots .$$

By induction, we can prove that

$$0 < x^{(k)}(t) \leq x^{(k-1)}(t) \leq y(t), \quad t \geq t_2,\ k = 1, 2, \ldots .$$

Then there exists $x(t) \in \Omega$ such that $\lim_{k\to\infty} x^{(k)}(t) = x(t)$, $t \geq t_2$. Clearly, $x(t) > 0$ for $t \geq t_2$ and $x(t) = (Tx)(t)$. $M + x(t)$ is a positive solution of (1). Since $M + x(t) \leq y(t) + M = u(t) \leq c$ for $t \geq t_2$, $M + x(t)$ is a C-type positive solution of (1).

Conversely, let $x(t)$ be a C-type positive solution of equation (1). Then $\lim_{t\to\infty}(x(t) - x(t-\tau)) = 0$ and there is a number $t_1 \geq t_0$ such that $x(t-\delta) > 0$ for $t \geq t_1$. Thus we obtain

$$x(t) = x(t-\tau) + \int_t^\infty F(s,\ x(s-\delta))\Delta s, \quad t \geq t_1.$$

Let $m = \{x(t) : t_1 - \tau \le t \le t_1\}$. We have $x(t) \ge m$ for $t \ge t_1$. Thus, we obtain

$$x(t) \ge x(t-\tau) + \int_t^\infty F(s, m)\Delta s, \quad t \ge t_1 + \sigma.$$

By induction, we can prove that

$$x(t_1 + \delta + k\tau) \ge x(t_1 + \delta) + k\alpha\tau + \sum_{i=1}^{k} \int_{t_1 + i\tau + \delta}^\infty F(s, m)\Delta s, \quad t \ge t_0.$$

Since $x(t)$ is a C-type solution, there is a number $M > 0$ such that $x(t) \le M$. Hence

$$\sum_{i=1}^{k} \int_{t_1 + i\tau + \delta}^\infty F(s, m)\Delta s \le M.$$

Letting $k \to \infty$,

$$\sum_{i=0}^{\infty} \left(\int_{t_1 + i\tau + \delta}^\infty F(s, m)\Delta s \right) < \infty.$$

In view of Lemma 3, we see that (12) holds. The proof is complete.

Unlike A-type, B-type and C-type positive solutions it is not easy to characterize D-type positive solutions. We only give sufficient conditions for the existence of such solutions.

Theorem 5. *Suppose that* (8) *holds and*

$$\int_{t_0}^\infty F(t, \lambda)\Delta t = \infty \ \textit{for some} \ \omega > \lambda > 0. \tag{13}$$

Then equation (1) *has a D-type positive solution.*

Proof. Let $0 < \lambda < \omega$ be a constant. In view of (8), there exists a number $t_1 \ge t_0$ such that

$$\int_{t+t_1}^\infty F(t, \lambda + \lambda(t - \sigma))\Delta t < \frac{\lambda\tau}{2}, \quad t \ge t_1.$$

Let

$$H(t) = \begin{cases} \int_t^\infty F(s, \lambda + \lambda(s - \sigma))\Delta s, & t \ge t_1, \\ \dfrac{t - t_1 + \tau}{\tau} H(t_1), & t_1 - \tau \le t < t_1, \\ 0, & t < t_1 - \tau. \end{cases}$$

Clearly, $H(t) \ge 0$. Define

$$u(t) = \int_0^\infty H(t - s\tau)\Delta s, \quad t \ge t_1.$$

It is obvious that $u(t) - u(t - \tau) = H(t)$, $0 < u(t) \le \lambda t$ and $\lim_{t\to\infty} u(t)/t = 0$.

Let X denote the Banach space $l_\infty^{t_1}$ of all real sequences $x = \{x(t)\}_{t=t_1}^\infty$ with the norm $\| x \| = \sup_{t \ge t_1} | x(t) |$. Define a set Ω by

$$\Omega = \{x(t) \in X : 0 \le x(t) \le u(t), t \ge t_1\}$$

and an operator T on Ω by

$$(Tx)(t) = \begin{cases} x(t - \tau) + \int_t^\infty F(s, \lambda + x(s - \delta))\Delta s, & t \ge t_1 + \mu, \\ \dfrac{tu(t)(Tx)(t_1 + \mu)}{(t_1 + \mu)u(t_1 + \mu)} + u(t)\left(1 - \dfrac{t}{t_1 + \mu}\right), & t_1 \le t < t_1 + \mu. \end{cases}$$

Clearly, $T\Omega \subset \Omega$.

Define a series of sequences $\{y^{(k)}(t)\}$, $k = 0, 1, 2, \ldots$ as follows:

$$y^{(0)}(t) = u(t), \qquad t \ge t_1,$$

$$y^{(k)}(t) = (Ty^{(k-1)})(t), \qquad t \ge t_1,\ k = 1, 2, \ldots.$$

By induction, we can prove that

$$0 < y^{(k)}(t) \le y^{(k-1)}(t) \le u(t), \quad t \ge t_1,\ k = 1, 2, \ldots.$$

Then there exists $y(t) \in \Omega$ such that $\lim_{k\to\infty} y^{(k)}(t) = y(t)$, $t \geq t_1$. Clearly, $u(t) \geq y(t) > 0$ for $t \geq t_1$ and $y(t) = (Ty)(t)$. It is a positive solution of equation

$$(x(t) - x(t-\tau))^{\Delta} + F(t, \lambda + x(t-\delta)) = 0.$$

Clearly, $\{\lambda + y(t)\}$ is a solution of (1) and satisfies $\lim_{t\to\infty} (\lambda + y(t))/t = 0$. It is similar to the proof of Theorem 4 that contradiction will be obtained. The proof is complete.

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