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THE STABILITY OF SOLITARY WAVES ON A VISCOUS LIQUID OF FINITE DEPTH

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Abstract

The stability of unidirectional propagation of long waves on a viscous liquid, in a channel of finite depth *H*, satisfying the viscous counterpart of Korteweg-de Vries (KdV) equation whose solutions represent solitary waves, is explored. It is shown that the solutions of such solitary waves on a viscous liquid are stable. The proof, given here, depends on two nonlinear functionals which are invariant in time for the solutions. It is further shown that the viscous KdV equation possesses a variational principle. Also, considered here are two nonlinear invariants for the regularized equation, namely, the viscous Benjamin, Bona and Mahony (BBM) equation which has smoother mathematical properties. It has been proved that, in the absence of viscosity, both KdV and BBM equations share the same two nonlinear invariants, but in the presence of viscosity, these two equations have different nonlinear invariants.

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1. Introduction

In 1844, Russell [1] first observed a new form of water wave which is now commonly known as solitary waves. In his paper [1], he remarked that the waves individually appear to be stable. He stated that when a solitary wave propagates along a uniform canal, it displays a remarkable property of permanence. This, of course, immediately gives an observer a great degree of confidence that these waves are stable. Since Russell's initial observation, there have been many experiments performed that show, beyond any doubt, that solitary waves in water are indeed stable. Benjamin [2] was amongst the first researchers that established, with some degree of analytical rigor, that solitary waves are stable. He commented that '... a rigorous demonstration would be exceedingly difficult owing to the complications of the exact nonlinear problems'.

Benjamin [2] demonstrated that solitary wave of the Korteweg-de Vries (KdV) equation, which in 1895 was originally derived by Korteweg and de Vries [3] as an approximation for long water waves of small but finite amplitude, are stable. His stability criterion, for KdV equations, hinged on two nonlinear functionals which he showed are invariant with time. The KdV equation he considered is

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.1}$$

where for water wave problems is nondimensionalized according to $u = 3\eta/2H$, $x = \sqrt{6}x^*/H$ and $t = (6g/H)^{1/2}t^*$, where η is the vertical displacement of the free surface, H is the undisturbed water depth and x^*, t^* are dimensional distance and time, respectively. Benjamin [2] considered nonperiod solution of (1.1) defined for $-\infty < x < \infty$, $t \geq 0$ and found solitary-wave solutions by considering

$$u = \Phi(\bar{x}) \quad \text{with} \quad \bar{x} = x - (1 + C)t$$

which represents a steadily traveling wave of speed $1 + C$. Next, by assuming that the function $\Phi(\bar{x})$ and its second derivative vanish as $\bar{x} \to \pm\infty$, he obtained the following ordering differential equation upon substitution of

$\Phi(\bar{x})$ into (1.1) which integrated at once to yield

$$\Phi'' + \frac{1}{2}\Phi^2 = C\Phi. \tag{1.2}$$

It is well known that if $C > 0$ (1.2) admits the solution

$$\Phi = 3C\operatorname{sech}^2\left[\frac{1}{2}C^{\frac{1}{2}}(\bar{x} + a)\right] \tag{1.3}$$

which vanishes at infinity, with *a* being a constant.

The main purpose of this paper is to establish the stability of the viscous counterpart of the KdV equation, namely,

$$u_t + u_x + uu_x + u_{xxx} = \nu u_{xx} \tag{1.4}$$

for where $\nu > 0$. However, as was noted by Benjamin [2], in order to establish solitary-wave solution, Φ, arising in the similar manner to that of equation (1.2), (see the proof of Theorem 2 below) it is necessary to obtain solutions u of (1.4) that in some definite sense are initially close to Φ. This task can only be achieved if the existence and regularity of the viscous KdV equation can be established for all values of $\nu > 0$. This in turn implies that solutions of (1.4) must be guaranteed to exist for the chosen class of initial waveforms, having smoothness properties that can be used in the stability analysis. One possible path for assuming the regularity of solutions is to consider the C^∞ solutions which converge to zero, together with all their derivatives, as $x \to \pm\infty$. However, this is too restricted an assumption and the general applicability of the stability criterion will be subject.

As in Benjamin [2], we notice that there exists a stronger mathematical foundation with regard to the Benjamin, Bona and Mahony [4], commonly referred to as the BBM equation,

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.5}$$

which is a rational alternative to equation (1.1) for a long-wave and having almost the same solitary-wave solutions as that of (1.3). Benjamin et al. [4]

showed that solutions of (1.5) have, in general, better smoothness properties compare with (1.1). Similarly, Sajjadi and Smith [5] showed the same properties hold for the viscous counterpart of equation (1.5), namely

$$u_t + u_x + uu_x - u_{xxt} = \nu u_{xx} \tag{1.6}$$

for $\nu > 0$.

In this paper, we therefore establish the stability and invariants of equation (1.6) subject to the initial wave form $u(x, 0) = G(x)$, such that $G \in C^2(\mathbb{R}) \cap W_2'(\mathbb{R})$, for $-\infty < x < \infty$, $t \geq 0$.

2. Variational Principle

In Benjamin's stability theory [2] use was made of the nonlinear functionals which he showed to be invariant with time. He followed the work of Miura et al. [6] who had shown, for Korteweg-de Vries equation, there are infinitely many C^∞ solutions exist that, together with all their derivatives, vanish as $x \to \pm\infty$. Apparently the two invariants used in his study were first noticed in 1877 by Boussinesq [7].

Multiplying equation (1.4) by $2u$, we may cast it in the form

$$(u^2)_t + \left[u^2 + \frac{2}{3}u^3 + 2u(u_{xx} - \nu u_x) - (1-\nu)u_x^2\right]_x = 0.$$

Assuming the terms in the square brackets all vanish as $x \to \pm\infty$, then it is easy to see that

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2 dx = 0$$

and hence

$$V(u) = \int_{-\infty}^{\infty} u^2 dx = \text{const.}$$

Next, consider the identity $2u_x H_x - u^2 H = 0$, where

$$H \equiv u_t + u_x + uu_x + u_{xxx} - \nu u_{xx} = 0.$$

This equation may be re-arranged in the following form:

$$\left(u_x^2 - \frac{1}{3}u^3\right)_t$$

$$+\left[-\frac{1}{3}u^3 - \frac{1}{4}u^4 + (1+\nu)u_x^2 + 2u_x u_{xxx} - u_{xx}^2 + 2uu_x^2 - u^2 u_{xx} - \nu u_x\right]_x = 0.$$

Once again, assuming the group of terms in the square brackets vanish as $x \to \pm\infty$, we arrive at the conclusion that

$$M(u) = \int_{-\infty}^{\infty}\left(u_x^2 - \frac{1}{3}u^3\right)dx = \text{const.}$$

Thus, the conditional variational problem

$$\delta M = 0 \quad \text{for } V \text{ fixed}$$

is interestingly enough the same as that of Benjamin [2] and can be solved by the viscous solitary wave Φ. This means that, the Euler-Lagrange equation for this problem is simply the viscous solitary-wave equation (1.7), where the speed of propagation c relative to critical velocity appears as the Lagrange multiplier.

Note that, similar to Benjamin's investigation [2], the statement made in the previous paragraph is the same principle originally proposed by Lax [8], Lemma 1.2 on page 475 of his paper, extended to include the viscous dissipative term on the right-hand side of equation (1.6). Thus, $E(u)$ and $I(u)$ are two nonlinear functionals with, $E \neq I$, and $I(u)$ that are invariants for the viscous KdV equation, then the variational problem

$$\delta E = 0 \quad \text{for } I \text{ fixed}$$

can be solved for the viscous solitary wave. This means that Φ will satisfy the Euler-Lagrange equation

$$\nabla E(\Phi) = \mathscr{C}\nabla I(\Phi)$$

provided the multiplier $\mathscr{C}$ is suitably chosen, as we shall demonstrate this for viscous solitary waves in an open channel (see Section 3).

The above principle implies that if u is close to Φ and if $M(\Phi)$ is the absolute minimum for given $V(u) = V(\Phi)$, such that

$$\Delta M(u, \Phi) = M(u) - M(\Phi) \geq 0$$

then both $V(u)$ and $\Delta M(u, \Phi)$ will be independent of time. This means that if initially a solution is close to Φ, then its solution will be subject to the constraint that the positive functional $\Delta M(u, \Phi)$ remains the same small value as that was initially given.

3. Variational Principle for Viscous Solitary Waves in an Open Channel

To demonstrate the forgoing theory, we shall now focus our attention to the classical problem of a solitary wave [9, Section 252] but extended to a case of viscous liquids.

We first note that solitary waves of permanent form do exist for a wide range of amplitudes but not exceeding the maximum value which is about 0.8 times the depth of the undisturbed liquid.

Following the recent work of Sajjadi [10] on viscous potential flows for water waves we will consider the propagation of a solitary wave on a viscous liquid in an open-channel. Referring to Figure 1, and take the (x, y)-coordinates as shown, we shall assume the free surface is represented by $y = \eta(x, t)$, such that η takes the value H asymptotically for $x \to \pm\infty$. We shall further assume the motion is started from rest and is irrotational, such that $\nabla \times u = 0$, and thus the velocity vector may be expressed in terms of a velocity potential $\Phi(x, y, t)$, such that $u = \nabla\Phi(x, y, t)$. Therefore, the motion satisfies Laplaces equation $\nabla^2\Phi = 0$, and we shall assume further that $|\nabla\Phi| \to 0$ for $x \to \pm\infty$. The boundary conditions are, at the bottom of the channel

$$\Phi_y = 0 \quad \text{at} \quad y = 0. \tag{3.1}$$

At the free surface, the kinematic and dynamic conditions are given, respectively,

$$\eta_t + \Phi_x \eta_x - \Phi_y = 0 \quad \text{at} \quad y = \eta \tag{3.2}$$

and

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + g(\eta + H) - 2\nu \Phi_{yy} = 0 \quad \text{at} \quad y = \eta, \tag{3.3}$$

where ν is the kinematic viscosity of the liquid and g is the acceleration due to gravity. The last term in (3.3) arises from the fact that for incompressible viscous potential flows the viscous contribution to the stress does not always vanish [10, 11].

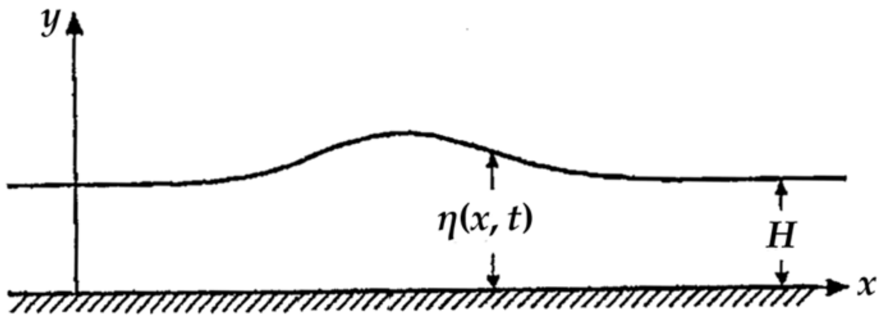

Figure 1. Schematic diagram of a solitary wave on a viscous liquid.

For a solitary wave propagating with velocity c we have $\partial/\partial t \equiv -c\partial/\partial x$ and surface boundary conditions (3.2) and (3.3) reduce to

$$\Phi_x \eta_x - \Phi_y = c\eta_x \quad \text{at} \quad y = \eta \tag{3.4}$$

and

$$\frac{1}{2} |\nabla \Phi|^2 + g(\eta - H) - 2\nu \Phi_{yy} = c\Phi_x \quad \text{at} \quad y = \eta. \tag{3.5}$$

Following Benjamin [2], we assert the total energy for an arbitrary motion that vanishes sufficiently rapidly is $x \to \pm\infty$ is given by

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^{\eta} \{ |\nabla \Phi|^2 - \nu \Phi_y^2 \} dy dx + \frac{1}{2} \int_{-\infty}^{\infty} g(\eta - H)^2 dx. \tag{3.6}$$

Note that the first integral in (3.6) represents the total kinetic energy and the second integral is the potential energy relative to the state of the liquid at rest. Furthermore, the total horizontal momentum (or the impulse of the motion [2]) maybe expressed as

$$I = \int_{-\infty}^{\infty}\int_{0}^{\eta}\Phi_x dydx = -\int_{-\infty}^{\infty}\Phi(x, \eta, t)\eta_x dx. \tag{3.7}$$

Proposition 1. *The variational principle for a solitary wave on a viscous liquid is given by*

$$\delta E = 0 \quad \textit{for } V \textit{ fixed},$$

where δE *represents the first variation of the integral E which arises from weak variations of both dependent variables* Φ *and* η.

Proof. Applying the Greens theorem and making use of (3.1) and noting that $\nabla^2\Phi = 0,$ we obtain from (3.6),

$$\delta E = \int_{-\infty}^{\infty}\left[(\Phi_y - \Phi_x\eta_x)\delta\Phi_s + \left\{\frac{1}{2}|\nabla\Phi_s|^2 + g(\eta - H) - 2\nu(\Phi_{yy})_s\right\}\delta\eta\right]dx, \tag{3.8}$$

where the subscript s denotes evaluation at the free surface. In a similar manner, from (3.7), we get

$$\delta I = \int_{-\infty}^{\infty}\{-\eta_x\delta\Phi_s + (\Phi_x)_s\delta\eta\}dx. \tag{3.9}$$

Now, for E to be stationary subject to the condition that I remains constant then from (3.8) and (3.9), we have

$$\Phi_y - \Phi_x\eta_x = c(-\eta_x) \tag{3.10}$$

and

$$\frac{1}{2}|\nabla\Phi_s|^2 + g(\eta - H) - 2\nu(\Phi_{yy})_s = c(\Phi_x)_s, \tag{3.11}$$

where c, the Lagrange multiplier, is a constant. We see at once that (3.10) and (3.11) are exactly the surface kinematic and dynamic boundary conditions to be satisfied for a solitary wave propagating with speed c on a viscous liquid. This completes the proof of the proposition. □

4. Principles of Stability

We shall now consider the stability of equation (1.6) subject to the initial condition $u(x, 0) = G(x)$. In order to establish the stability of viscous solitary wave solution Φ, we need to consider the aggregate of solutions u of equation (1.6) which are initially close enough to Φ.

The concept of stability was originally established by Lyapunov [12]. He stated that the stability of the particular motion $\Phi\{x - (1 + C)t\}$ needs to be compared with a general class of motions $u = u(x, t)$ which will evolve from the imposed initial conditions that is close to the initial condition for Φ. This suggests that the purpose of stability if u at $t = 0$ is made to be close to Φ initially, then u will remain close for all $t > 0$.

To prove the stability of equation (1.6), we first state two lemmas followed by two theorems, which when they are proved, the principle of stability will be established.

Lemma 1. *For equation* (1.6) *the functionals*

$$E(u) = \int_{-\infty}^{+\infty} (u^2 + u_x^2)\, dx \tag{4.1}$$

and

$$F(u) = \int_{-\infty}^{+\infty} \left(u^2 + \frac{1}{3}u^3 + \frac{1}{2}\nu u_x^2\right) dx \tag{4.2}$$

are invariant in time.

Lemma 2. *For a solution u of the partial differential equation* (1.1) *and their corresponding invariants, from Theorems* 1 *and* 2, *the following inequalities*:

$$\sup|u| \le \frac{1}{\sqrt{2}}\|u\|$$

and

$$F(u) \le \left(1 + \frac{1}{3}\sup|u| + \frac{1}{2}\nu\right)E(u) \tag{4.3}$$

hold.

Theorem 1. *For equation* (1.6), *the invariant functionals* (4.1) *and* (4.2) *exist, well defined, and bounded*.

Theorem 2. *Solutions of equation* (1.6) *are stable given that the initial wave form* $u(x, 0) = G(x) \in C^2[\mathbb{R} \times W_2^1(\mathbb{R})]$, *where stability is with respect to the considered metrics* $d_I(u, \Phi) = \|u(x) - \Phi(x)\|$ *and* $d_{II} = \inf_\xi \|u(x - \xi) - \Phi(x)\|$, *where* $\xi \in \mathbb{R}$ *and* $\|u\|$ *is the expected norm of the Sobolev space* $W_2^1(\mathbb{R})$ *namely*

$$\|u\| = \int_{-\infty}^{+\infty}(u^2 + u_x^2)^{\frac{1}{2}}dx.$$

Proof of Lemma 1. We begin by taking equation (1.6) and multiplying it through by $2u$. Thus, we obtain

$$(u^2 + u_x^2)_t + \left(u^2 + \frac{2}{3}u^3 - 2uu_{xt} - 2\nu uu_x\right)_x + 2\nu u_x^2 = 0. \tag{4.4}$$

Integrating (4.4) with respect to x and noting, using the prior assumption on u and u_x, that all of the terms in the middle bracket vanish as $x \to \pm\infty$ we obtain

$$\frac{d}{dt}\left[\int_{-\infty}^{+\infty}(u^2 + u_x^2)dx\right] + 2\nu\int_{-\infty}^{+\infty}u_x^2 dx = 0.$$

Since the term $2\nu u_x^2$ is positive definite, it follows that

$$\int_{-\infty}^{+\infty} u_x^2 dx \leq \left(\int_{-\infty}^{+\infty} u_x dx\right)\left(\int_{-\infty}^{+\infty} u_x dx\right),$$

however it is clear that, using the assumption that the solution $u(x, t)$ vanishes at $x \to \pm\infty$, that $\int_{-\infty}^{+\infty} u_x dx$ vanishes. Hence, we obtain

$$\frac{d}{dt}\left[\int_{-\infty}^{+\infty}(u^2 + u_x^2)dx\right] = 0.$$

This yields the result that states

$$E(u) = \int_{-\infty}^{+\infty}(u^2 + u_x^2)dx$$

is invariant in time, and this completes the proof of establishing for the first invariant $E(u)$.

To prove the second invariant (4.2), we first define

$$\omega(x, t) = \int_{-\infty}^{+\infty} u_t(s, t)ds$$

which allows equation (1.6) to be rewritten as

$$\omega_x - \omega_{xxx} + \left(u + \frac{1}{2}u^2\right)_x - \nu u_{xx} = 0.$$

Multiply through by ω and integrating with respect to x we obtain

$$\int_{-\infty}^{+\infty}\left[\omega\omega_x - \omega\omega_{xxx} + \omega\left(u + \frac{1}{2}u^2\right)_x - \nu\omega u_{xx}\right]dx = 0. \tag{4.5}$$

The first term in the integrand can be written as $\frac{1}{2}[\omega^2]_x$ and thus we may argue that since u and $u_{xt} = \omega_{xx}$ vanishes as $x \to \pm\infty$ so must ω. Hence, equation (4.5) becomes

$$\int_{-\infty}^{+\infty}\left[-\omega\omega_{xxx} + \omega\left(u + \frac{1}{2}u^2\right)_x - \nu\omega u_{xx}\right]dx = 0.$$

Similarly the second term in the integrand of the integral (4.5), i.e. $\omega_x\omega_{xx}$, also vanishes as $x \to \pm\infty$. Therefore, an integration by parts gives

$$\int_{-\infty}^{+\infty} \omega\left(u + \frac{1}{2}u^2\right)_x - \nu\omega u_{xx} dx = 0.$$

A second integration by parts then yields

$$\int_{-\infty}^{+\infty} \omega_x\left(u + \frac{1}{2}u^2\right) + \nu\omega_x u_x dx = 0. \tag{4.6}$$

But since $\omega_x = u_t$ equation (4.6) becomes

$$\int_{-\infty}^{+\infty} \left(u + \frac{1}{2}u^2 + \nu u_x\right) u_t dx = 0$$

which may be cast as

$$\frac{1}{2}\frac{d}{dt}\left[\int_{-\infty}^{+\infty} \left(u^2 + \frac{1}{3}u^3 + \frac{1}{2}\nu u_x^2\right) dx\right] = 0.$$

Hence we obtain

$$F(u) = \int_{-\infty}^{+\infty} \left(u^2 + \frac{1}{3}u^3 + \frac{1}{2}\nu u_x^2\right) dx \tag{4.7}$$

is invariant in time.

Proof of Lemma 2. Before proving the individual inequalities, we recall that from Parseval's theorem we may rewrite our norm as

$$\| u \| = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} (1 + \omega^2) | \hat{u} |^2 d\omega}, \tag{4.8}$$

where the usual Fourier transform notation is utilized for $\hat{u}$.

Proof. (i) Firstly, it is clear that

$$\sup| u | \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \| \hat{u} \| d\omega,$$

and secondly, by a simple algebraic modification and making use of the Cauchy-Schwartz inequality it follows that

$$\int_{-\infty}^{+\infty} \| \hat{u} \| d\omega = \int_{-\infty}^{+\infty} \| \hat{u} \| \sqrt{\frac{1+\omega^2}{1+\omega^2}} d\omega$$

$$\leq \left(\sqrt{\int_{-\infty}^{+\infty} (1+\omega^2) \| \hat{u} \|^2 d\omega} \right) \left(\sqrt{\int_{-\infty}^{+\infty} \frac{1}{1+\omega^2} d\omega} \right).$$

Using a standard inverse trigonometric substitution, it can be shown that the value for the last integral is equal to π, hence we obtain

$$\sup | u | \leq \sqrt{\pi} \sqrt{\int_{-\infty}^{+\infty} (1+\omega^2) \| \hat{u} \|^2 d\omega}. \tag{4.9}$$

Thus, from (4.8) and (4.9) it follows that

$$\sup | u | \leq \frac{1}{\sqrt{2}} \| u \|$$

and this completes the proof of establishing the first identity.

(ii) Using the definition of the invariants (4.1) and (4.2), we can see at once that

$$F(u) = \int_{-\infty}^{+\infty} \left(u^2 + \frac{1}{3} u^3 + \frac{1}{2} \nu u_x^2 \right) dx$$

$$= \int_{-\infty}^{+\infty} \left(u^2 + \frac{1}{3} u^3 \right) dx + \int_{-\infty}^{+\infty} \frac{1}{2} \nu u_x^2 dx$$

$$\leq \left(1 + \frac{1}{3} \sup | u | \right) \int_{-\infty}^{+\infty} u^2 dx + \int_{-\infty}^{+\infty} \left(\frac{1}{2} \nu u_x^2 \right) dx$$

$$\leq \left(1 + \frac{1}{3} \sup | u | \right) \int_{-\infty}^{+\infty} (u^2 + u_x^2) dx + \int_{-\infty}^{+\infty} \left(\frac{1}{2} \nu (u^2 + u_x^2) dx \right)$$

$$= \left(1 + \frac{1}{3} \sup | u | + \frac{1}{2} \nu \right) E(u).$$

Hence, this completes the proof of the second inequality by showing that

$$F(u) \leq \left(1 + \frac{1}{3}\sup|u| + \frac{1}{2}\nu\right)E(u). \tag{4.10}$$

Proof of Theorem 1. We recall that the existence theory for $E|(u)|$ for the initial waveform $u(x, 0) = G(x) \in C^2[\mathbb{R} \times W_2^1(\mathbb{R})]$ has already been established [5]. Moreover in [5], it was proved that as $x \to \pm\infty$ that $E(u)$ exists and is equal to $E(G) = \| G \|_{1,2}^2$.

Now, applying the inequalities from Lemma 1 the existence of $F(u)$ becomes apparent. Thus, it follows that if

$$E(u) < \infty,$$

(which was previously shown [5]), then

$$\sup|u| \leq \frac{1}{\sqrt{2}}\| u \|. \tag{4.11}$$

The inequality (4.11) shows that $\sup|u|$ is bounded on $\mathbb{R}$, and thus from (4.10) we obtain

$$F(u) \leq \left(1 + \frac{1}{3}\sup|u| + \frac{1}{2}\nu\right)E(u).$$

Hence, from (4.10) and (4.11) the existence and boundness of $F(u)$ is proved, and this completes the proof of Theorem 1.

Proof of Theorem 2. We now focus our attention to the class of viscous solitary-wave solutions which may be obtained by making the following change of variables

$$u = \Phi(\bar{x}),$$

where

$$\bar{x} = x - (1 + c)t.$$

This, of course, represents a wave traveling with velocity $1 + c.$ Making the assumption that $\Phi(\bar{x})$ and second derivative of Φ vanish at infinity then substituting the above chain of variables into (1.6) and integrating once we obtain

$$2(1 + c)\Phi'' - 2c\Phi + \Phi^2 = 2\nu\Phi'. \tag{4.12}$$

We refer to the above equation, whose solution vanishes at infinity, as the viscous solitary-wave solution.

We note that, according to the Lyapunov definition [4], Φ is stable with respect to two different metrics d_I and d_{II} provided the following condition holds. That is to say, given any small number $\varepsilon > 0$ then there exists a corresponding number $\delta_\varepsilon > 0$ such that the initial condition

$$[d_I(u, \Phi)]_{t=0} \leq \delta_\varepsilon. \tag{4.13}$$

This then ensures that

$$d_{II}(u, \Phi) \leq \varepsilon$$

for all $t \geq 0,$ where Φ is the viscous solitary-wave solution and u is any bounded solution to (1.6).

We remark that utilizing the above definition directly to prove stability is rather impractical. A better approach is to follow Benjamin et al. [4] that if we could obtain a functional $I(u, \Phi)$ which is invariant with time, where u is any solution to (1.6) and Φ is the viscous solitary-wave solution introduced above, then (4.13) will be satisfied if some finite values α and β exist such that, when $d_I(u, \Phi) \leq A$ at $t = 0,$ then

$$\alpha[d_I(u, \Phi)]^2 \geq I(u, \Phi) \tag{4.13$_A$}$$

and

$$\beta[d_{II}(u, \Phi)]^2 \leq I(u, \Phi).$$

Hence, to prove the stability we simply need to establish the above inequalities, namely to obtain upper and lower bounds for the invariant functional. It is worth noting that by taking $\delta_\varepsilon = \sqrt{\beta/\alpha}\varepsilon$ we can prove that there exists an alternative equivalence definition of stability to that stated above.

However, before proceeding with this approach, we note that we may assume for each $t \geq 0$ the solution u of (1.6) is a function of x, which is an element of the Sobolev functional space $W_2^1(\mathbb{R})$, so is the viscous solitary-wave solution Φ. Thus, we next define the difference $h = u - \Phi$ and note that this function belonging to $W_2^1(\mathbb{R})$ are amongst the class of functions that are square integratable (i.e. in L_2) with derivatives that are also square integratable. Hence, we use the norm

$$\| f \| = \left[\int_{-\infty}^{+\infty}(f^2 + f'^2)dx\right]^{\frac{1}{2}}.$$

Now to establish stability we take the two previously established invariants $E(u)$ and $F(u)$ and write $u = \Phi + h$, with further assumption that $E(u) = E(\Phi)$, and compute the variations

$$\Delta E = E(\Phi + h) - E(\Phi) = \int_{-\infty}^{+\infty}(2\Phi h + h^2 + 2\Phi_x h_x + h_x^2)dx$$

and

$$\begin{aligned}\Delta F &= F(\Phi + h) - F(\Phi) \\ &= \int_{-\infty}^{+\infty}\left(2\Phi h + h^2 + \Phi h^2 + \Phi^2 h + \frac{1}{3}h^3 + v\Phi_x h_x + \frac{1}{2}vh_x^2\right).\end{aligned}$$

However, since $\Delta E = 0$ it follows that $\Delta F = C\Delta E + \Delta F$ and thus, we compute

$$-C\Delta E + \Delta F = -C\int_{-\infty}^{+\infty}(2\Phi h + h^2 + 2\Phi_x h_x + h_x^2)\,dx$$

$$+\int_{-\infty}^{+\infty}\left(2\Phi h + h^2 + \Phi h^2 + \Phi^2 h + \frac{1}{3}h^3 + \nu\Phi_x h_x + \frac{1}{2}\nu h_x^2\right)dx$$

$$= \int_{-\infty}^{+\infty}\Big[h(2\Phi - 2c\Phi + \Phi^2) + h_x(\nu\Phi_x - 2c\Phi_x)$$

$$+ h^2(1 - c + \Phi) + h_x^2\left(\frac{1}{2}\nu - c\right) + \frac{1}{3}h^3\Big]dx. \tag{4.14}$$

Next integration by parts, we get

$$-\int_{-\infty}^{+\infty}\{[(2(1-c)\Phi + \Phi^2)] + (2c\Phi_x + \nu\Phi_x)h_x\}\,dx$$

$$= \int_{-\infty}^{+\infty}[2(1+c)\Phi_{xx} - 2c\Phi + \Phi^2]\,h\,dx + \int_{-\infty}^{+\infty}(-2\Phi_{xx}h + \nu\Phi_x h_x)\,dx.$$

Since Φ satisfies (4.12) the integrals reduce to

$$\int_{-\infty}^{+\infty}\nu\Phi_{xx}h\,dx = \int_{-\infty}^{+\infty}(-2\Phi_{xx}h + \nu\Phi_x h_x)\,dx$$

$$= \int_{-\infty}^{+\infty}-2\Phi_{xx}h\,dx.$$

Hence, (4.14) can be rewritten in the following form:

$$\Delta F = \int_{-\infty}^{+\infty}\left[h^2(c + 1 + \Phi) + h_x^2\left(1 + \frac{1}{2}\nu\right)\right]dx + \int_{-\infty}^{+\infty}\frac{1}{3}h^3\,dx - \int_{-\infty}^{+\infty}2\Phi_{xx}h\,dx.$$

Thus,

$$|\,\Delta F\,| \le \alpha_1\|\,h\,\|^2 + \left|\int_{-\infty}^{+\infty}\frac{1}{3}h^3\,dx\right|,$$

where $\alpha_1 = \max\left(c + 1 + A,\ 1 + \frac{1}{2}v\right)$ with A being an upper bound on $|\,\Phi\,|$.

Next, using (4.3) we obtain that

$$|\Delta F| \leq \alpha_1 \| h \|^2 + \frac{A \| h \|^2}{3\sqrt{2}}.$$

Finally, choosing $\alpha = \alpha_1 + A/(3\sqrt{2})$ and recalling that $d_I(u, \Phi) = \| u - \Phi \| = \| h \|$ it becomes clear that we have satisfied the upper bound for α which is required for first inequality in (4.13_A) with $d_I(u, \Phi) = \| h \|$. Moreover, following a similar development in [2], but with modifications in the algebraic development, a lower bound for β which is required for the second inequality in (4.13_A) can also be obtained.

Thus, we have satisfied (4.13_A) which in turn implies that (4.13) holds. Hence we have proved that solutions u of (1.6) are stable with respect to viscous solitary-wave solution Φ and also with respect to the given metrics d_I and d_{II}.

5. Concluding Remarks

In this paper, we have established that the viscous solitary-wave solution of equation (1.1) is stable. Stability is proved for constant viscosity-like coefficient $\nu > 0$ such that $\nu \in \mathbb{R}$, and further, if the initial perturbations for a given viscous solitary wave are small, then the solution will remain permanently close in shape as it evolves in time.

In our analysis here, we have also discovered the two invariants $V(u)$ and $M(u)$, given by

$$V(u) = \int_{-\infty}^{\infty} u^2 dx = \text{const.}$$

and

$$M(u) = \int_{-\infty}^{\infty} \left(u_x^2 - \frac{1}{3} u^3 \right) dx = \text{const.}$$

are the same for the regular Korteweg-de Vries equation (1.1) and its viscous counterpart, i.e., equation (1.4). However, it should be made clear here that although both BBM and KdV equations (without the viscous term included) have the same two nonlinear invariants, but their viscous counterpart of both equations do have different invariants.

It is interesting to note that any small change in the wave speed will eventually yield to large changes in the position of the wave. Perhaps the conjecture which was made by Gardner et al. [13] and was examined in great detail by Lax [8] can be extended to the viscous counterpart of Kortewig-de Vries equation (1.1). That is to construct an eigenvalue problem similar to

$$\zeta'' + \left(\frac{1}{6}u + \lambda\right)\zeta = 0. \tag{5.1}$$

With $\zeta \in \mathbb{R}$ and bounded as $x \to \pm\infty,$ and show if $u(x, t)$ is the solution of the viscous Korteweg-de Vries, with $\nu \ll 1$ which vanishes as $x \to \pm\infty,$ then any discrete negative eigenvalue λ remains constant as t varies parametrically.

Using a different approach to that of Lax [8], we may solve equation (5.1) for the case where $u = \operatorname{sech}^2 \bar{x}.$ The solution that vanishes as $\bar{x} \to \pm\infty$ may be expressed as

$$\zeta(\bar{x}) = b_1 P_\vartheta^\mu[\tanh(\bar{x})] + b_2 Q_\vartheta^\mu[\tanh(\bar{x})],$$

where P_ϑ^μ and Q_ϑ^μ are associated Legendre functions of the first and the second kind, respectively, b_1 and b_2 are arbitrary constants, and

$$\mu = i\sqrt{\lambda}, \qquad \vartheta = \frac{1}{6}(\sqrt{5} - 3).$$

As $\mu = \sqrt{-\lambda}$ we can see at once that $\lambda < 0,$ which implies that t must vary parametrically.

The conditions that $\zeta \to 0$ as $\bar{x} \to \pm\infty$ give the following set of homogeneous equations (secular equations) for determining λ, namely

$$b_1 P_\vartheta^\mu(1) + b_2 Q_\vartheta^\mu(1) = 0,$$

$$b_1 P_\vartheta^\mu(-1) + b_2 Q_\vartheta^\mu(-1) = 0.$$

In depth exploration of the eigenvalues, λ, satisfying the above set of equations is beyond the scope of this paper, and we will not pursue this any further here.

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