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ROBUST ORDERING OF TWO INCOME DISTRIBUTIONS BY MEANS OF POVERTY INDICES

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Abstract

We present a wide class of empirical poverty indices including the most proposed in the literature. By the modern theory of empirical processes indexed by functions, we establish a uniform functional central limit theorem for this class of poverty estimators. From this result, we derive a generalized Wald-type test for comparing in a robust manner two income distributions by using a large spectrum of

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poverty indices. We conduct a simulation experiment which supports well our asymptotic results. The methodology of test is then applied to a real data set for analyzing poverty in four areas in Sénégal.

1. Introduction

Research on poverty measurement and analysis has been considerably developed during the last decades. Several approaches based on the income distribution, have been investigated for either estimating the degree of poverty in a given population or making comparisons between two different populations. An important question is whether one can say, with statistical confidence, that a distribution A dominates another distribution B in terms of poverty. In other words, one often asks for conditions under which two income distributions can be ranked in a non-ambiguous manner by a large spectrum of poverty indices.

There are many authors who were interested in this topic. For instance, Atkinson [1], Foster and Shorrocks [5, 6] Dardanoni and Forcina [2], Zheng [20], Davidson and Duclos [3] and Davidson and Duclos [4] are well-known references. But most of the methods utilized in these papers are based on the Lorenz or stochastic dominance curves or some transformations therefrom. In this context, testing poverty orderings requires defining a link between these curves and poverty indices.

Conversely, in this paper, we propose a method that deals directly with the poverty indices themselves and exploit their covariance structure in order to test for robust poverty rankings between two distributions. Our methodology provides a unified approach that is both applicable to the additively separable poverty measures as well as to the non-additively separable ones, as the Sen-like poverty indices.

To make this approach possible, we deal with a functional representation of the poverty indices. Indeed, when G denotes the income distribution function of the population of interest, and $z > 0$ represents the poverty line (i.e., the income level which separates the poor from the non poor), then a poverty-index can be theoretically defined as:

$$J(w,\ f) = \int_0^z w[G(y),\ G(z)]\,f(y,\ z)\,dG(y), \tag{1}$$

where *w* and *f* are specific bi-variate real functions satisfying some regularity conditions due to the normative properties desirable on a poverty-index. In fact, *w* is interpreted as a weighting function, whereas *f* represents a kernel measuring the individual poverty gaps relatively to the threshold *z*. In other words, $f(y,\ z)$ is the contribution of an individual with income *y* to the global poverty of the population. The choice of these two functions depends on the desirable properties that one wish to be satisfied by the poverty-index $J(w,\ f)$. Each choice of *w* and *f* leads to a particular formula which evaluates poverty in some way. Here are some examples of classical poverty indices encompassed by (1):

- The index proposed by Sen [12] can be written in the form (1). It is defined as

$$SEN = \int_0^z 2\left(1 - \frac{G(y)}{G(z)}\right)\left(\frac{z-y}{z}\right)dG(y),$$

with

$$w[G(y),\ G(z)] = 2\left(1 - \frac{G(y)}{G(z)}\right) \text{ and } f(y,\ z) = \frac{z-y}{z}.$$

- This index has been generalized by Kakwani [9] into

$$KAK(k) = \int_0^z (k+1)\left(1 - \frac{G(y)}{G(z)}\right)^k\left(\frac{z-y}{z}\right)dG(y), \quad k \geq 0,$$

with

$$w[G(y),\ G(z)] = (k+1)\left(1 - \frac{G(y)}{G(z)}\right)^k \text{ and } f(y,\ z) = \frac{z-y}{z}.$$

- The measure introduced by Shorrocks [13] is also a modification of the Sen index. It is defined as

$$SHO = \int_0^z 2(1 - G(y))\left(\frac{z-y}{z}\right)dG(y),$$

with

$$w[G(y), G(z)] = 2(1 - G(y)) \quad \text{and} \quad f(y, z) = \frac{z - y}{z}.$$

• Foster et al. [7] proposed an important class of additively separable indices defined as

$$FGT(\alpha) = \int_0^z \left(\frac{z - y}{z}\right)^{\alpha} dG(y), \quad \alpha \geq 0,$$

with

$$w[G(y), G(z)] = 1, \quad \text{and} \quad f(y, z) = \left(\frac{z - y}{z}\right)^{\alpha}.$$

• Watts [17] has also proposed a measure defined as

$$W = \int_0^z \log\left(\frac{y}{z}\right) dG(y),$$

with

$$w[G(y), G(z)] = 1, \quad \text{and} \quad f(y, z) = \log\left(\frac{y}{z}\right).$$

For a large survey of the literature on poverty measurement, we refer the interested reader to Zheng [19]. Our concern in the present paper is to study the asymptotic properties of a class of estimators, say $J_n(w, f)$, for the functional index $J(w, f)$, where w and f belong to suitable classes of functions. Making use of the modern theory of weak convergence of empirical processes indexed by functions, we establish a uniform central limit theorem for J_n, which represents a wide class of empirical poverty measures containing the most currently used in empirical studies. From these results, we derive a generalized Wald-type test for establishing a robust poverty ordering between two income distributions.

The remainder of the paper is organized as follows: In Section 2, we present our asymptotic results which are the almost sure consistency and a functional central limit theorem for the class of estimators J_n. In Section 3, we describe a Wald-type procedure for testing robust poverty orderings between two income distributions by using multiple poverty indices. In Section 4, we give a simulation experiment and apply the methodology of test to the 1994-senegalese data for analyzing poverty in four different areas. Finally, Section 5 concludes the paper, and the Appendix (Section 6) contains the simulation results and the proof of Theorem 2.2.

2. Asymptotic Results

Let $Y_1, \ldots, Y_n$ be an independent and identically distributed random sample of a random variable Y representing the income of an individual in a given population. Our functional $J(w, f)$ will be defined on the class of functions $\mathcal{W} \times \mathcal{F}$ such that

$$\mathcal{W} = \{w : [0, 1] \times [0, 1] \to \mathbb{R}_+,\ w \text{ continuous and } u \mapsto w(u, \cdot) \text{ is non-increasing}\}$$

and

$$\mathcal{F} = \{f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+,\ f \text{ continuous and } y \mapsto f(y, \cdot) \text{ is non-increasing}\}.$$

To construct an estimator for the functional $J(w, f)$, $(w, f) \in \mathcal{W} \times \mathcal{F}$, we denote the empirical distribution function associated with the sample $Y_1, \ldots, Y_n$ by $G_n(y) = n^{-1}\sum_{j=1}^{n} \mathbb{I}(Y_j \le y)$, where $\mathbb{I}(A)$ represents the indicator function of a set $A \subset \mathbb{R}$.

Then, a direct estimator for $J(w, f)$ is given by

$$J_n(w, f) = \int_0^z w[G_n(y), G_n(z)] f(y, z) dG_n(y),$$

which is obtained by simply substituting the empirical distribution function G_n for G in (1). Let $Y_{1,n} \le \cdots \le Y_{n,n}$ denote the order statistics corresponding to the sample $Y_1, \ldots, Y_n$. Then $J_n(w, f)$ can be rewritten as

$$J_n(w, f) = \frac{1}{n}\sum_{j=1}^{n} w[G_n(Y_{j,n}), G_n(z)] f(Y_{j,n}, z)\mathbb{I}(Y_{j,n} \leq z). \tag{2}$$

Now, we can establish the consistency of $J_n(w, f)$ for $J(w, f)$, for any couple $(w, f) \in \mathcal{W} \times \mathcal{F}$. Let us introduce some notations. Denote by P the common probability law of the $Y_j's$, and by $\mathbb{P}_n$ its associated empirical measure; that is, $\mathbb{P}_n = n^{-1}\sum_{j=1}^{n}\delta_{Y_j}$, where δ_y represents the Dirac mass at point y. Then, the empirical process associated with the sample $Y_1, ..., Y_n$ is defined as $\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P)$. For any measurable real-valued function φ, define

$$P\varphi = \int \varphi dP =: \mathbb{E}\varphi(Y), \quad \mathbb{P}_n\varphi = \frac{1}{n}\sum_{j=1}^{n}\varphi(Y_j), \quad \mathbb{G}_n(\varphi) = \sqrt{n}(\mathbb{P}_n - P)\varphi,$$

where "$\mathbb{E}$" denotes the expectation.

For any functions $w \in \mathcal{W}$, $f \in \mathcal{F}$, define the sequence of real-valued functions $(h_{w,f,G_n})_{n\geq 1}$ as

$$h_{w,f,G_n}(y) = w[G_n(y), G_n(z)] f(y, z)\mathbb{I}(y \leq z), \quad y \geq 0.$$

It is clear that $h_{w,f,G_n}(y)$ converges almost surely to the real-valued function

$$h_{w,f,G}(y) = w[G(y), G(z)] f(y, z)\mathbb{I}(y \leq z),$$

for all $y \geq 0$, and for any $(w, f) \in \mathcal{W} \times \mathcal{F}$. From these notations, one can write

$$J_n(w, f) = \frac{1}{n}\sum_{j=1}^{n} h_{w,f,G_n}(Y_j) = \mathbb{P}_n h_{w,f,G_n}$$

and

$$J(w, f) = \int_0^{\infty} h_{w,f,G}(y)dG(y) = \mathbb{E}h_{w,f,G}(Y) = Ph_{w,f,G}.$$

Proposition 2.1. *For any couple of functions* $(w, f) \in \mathcal{W} \times \mathcal{F}$, *one has with probability* 1,

$$J_n(w, f) \to J(w, f), \quad n \to \infty.$$

Proof. Given $(w, f) \in \mathcal{F} \times \mathcal{W}$, one can write

$$J_n(w, f) - J(w, f) = \mathbb{P}_n[h_{w, f, G_n} - h_{w, f, G}] + [\mathbb{P}_n h_{w, f, G} - Ph_{w, f, G}].$$

Then, to prove that $J_n(w, f)$ converges almost surely to $J(w, f)$, we first observe that by the strong law of large numbers and the continuity of the function *w*, the sequence of functions $(h_{w, f, G_n})_{n \geq 1}$ converges almost surely and uniformly to $h_{w, f, G}$. That is, for large value of *n*, the supremum norm of the quantity $h_{w, f, G_n} - h_{w, f, G}$ may be bounded up by ε, for any $\varepsilon > 0$. It follows from this, that $\mathbb{P}_n[h_{w, f, G_n} - h_{w, f, G}]$ converges almost surely to 0, as $n \to \infty$.

Next, since $Ph_{w, f, G} = \mathbb{E}h_{w, f, G}(Y) = J(w, f) < \infty$, the same argument of the strong law of large numbers enables us to conclude that $\mathbb{P}_n h_{w, f, G} - Ph_{w, f, G}$ converges almost surely to 0, as $n \to \infty$, and hence $J_n(w, f)$ converges almost surely to $J(w, f)$, for any functions *w* and *f*. □

To establish our uniform central limit theorem, we observe that, by definition, the function $h_{w, f, G}$ is truncated above *z*. Then, the monotonicity condition in the definition of the classes $\mathcal{W}$ and $\mathcal{F}$ implies that the function $h_{w, f, G}$ is bounded on $\mathbb{R}^+$ for all $(w, f) \in \mathcal{W} \times \mathcal{F}$, and hence is *P*-square integrable. In the sequel, we also need the following condition:

(A) $\mathcal{W}$ and $\mathcal{F}$ are pointwise measurable classes of functions. That is, they contain each one a countable subclass $\mathcal{G}$ such that for all $g \in \mathcal{G}$, there exists a sequence $\{g_m\}_{m \geq 1} \subset \mathcal{G}$, with $g_m(y) \to g(y)$ for every *y*.

By this condition, one has

$$\sup_{w \in \mathcal{W},\, f \in \mathcal{F}} | h_{w,\, f,\, G}(y) - Ph_{w,\, f,\, G} | < \infty, \text{ for all } y \geq 0.$$

Now, we are ready to state our main theorem, where $l^{\infty}(\mathcal{W} \times \mathcal{F})$ denotes the set of all real-valued, bounded functions defined on $\mathcal{W} \times \mathcal{F}$.

Theorem 2.2. *Let z be a positive real number and* $G(y)$ *be a continuous cumulative distribution function. If the functions in* $\mathcal{W}$ *are differentiable, with continuous first-order partial derivatives, and condition* (A) *holds, then the process* $\{\sqrt{n}[J_n(w, f) - J(w, f)] : w \in \mathcal{W},\, f \in \mathcal{F}\}$ *converges weakly in* $l^{\infty}(\mathcal{W} \times \mathcal{F})$ *to a zero-mean Gaussian process with covariance function defined, for any* $(w, f), (\tilde{w}, \tilde{f}) \in \mathcal{W} \times \mathcal{F}$, *as*

$$\begin{aligned} &\Sigma[(w, f); (\tilde{w}, \tilde{f})] \\ &= \int_0^z w[G(y), G(z)] f(y, z) \tilde{w}[G(y), G(z)] \tilde{f}(y, z) dG(y) \\ &\quad - \int_0^z w[G(y), G(z)] f(y, z) dG(y) \int_0^z \tilde{w}[G(y), G(z)] \tilde{f}(y, z) dG(y) \\ &\quad + \int_0^z \int_0^z a_1(x, y) [G(x) \wedge G(y) - G(x)G(y)] dG(x) dG(y) \\ &\quad + [1 - G(z)] \int_0^z \int_0^z a_2(x, y) G(y) dG(x) dG(y) \\ &\quad + [1 - G(z)] \int_0^z \int_0^z a_3(x, y) G(x) dG(x) dG(y) \\ &\quad + G(z)[1 - G(z)] \int_0^z \int_0^z a_4(x, y) dG(x) dG(y), \end{aligned}$$

where

$$a_1(x, y) = \frac{\partial w}{\partial u}[G(x), G(z)] f(x, z) \frac{\partial \tilde{w}}{\partial u}[G(y), G(z)] \tilde{f}(y, z),$$

$$a_2(x, y) = \frac{\partial w}{\partial u}[G(x), G(z)] f(x, z) \frac{\partial \tilde{w}}{\partial v}[G(y), G(z)] \tilde{f}(y, z),$$

$$a_3(x, y) = \frac{\partial w}{\partial v}[G(x), G(z)] f(x, z) \frac{\partial \tilde{w}}{\partial u}[G(y), G(z)] \tilde{f}(y, z),$$

$$a_4(x, y) = \frac{\partial w}{\partial v}[G(x), G(z)] f(x, z) \frac{\partial \tilde{w}}{\partial v}[G(y), G(z)] \tilde{f}(y, z).$$

Proof. We only give a sketch here. The proof itself is postponed to the last section. For any couple of functions $(w, f) \in \mathcal{W} \times \mathcal{F}$, one can decompose

$$\sqrt{n}[J_n(w, f) - J(w, f)] = \mathbb{G}_n(h_{w, f, G_n} - h_{w, f, G}) + \mathbb{G}_n(h_{w, f, G}) + \mathbb{W}_n(w, f), \tag{3}$$

where $\mathbb{W}_n(w, f) = \sqrt{n}(Ph_{w, f, G_n} - Ph_{w, f, G})$. Now, it remains to study the asymptotic behavior of the three terms in the right-hand side of (3). Beginning with the second term, one shows that the class of functions $\mathcal{H}_{z, G} = \{y \mapsto h_{w, f, G}(y) : w \in \mathcal{W}, f \in \mathcal{F}\}$ is P-Donsker, which readily implies that the empirical process $\{\mathbb{G}_n(h_{w, f, G}) : w \in \mathcal{W}, f \in \mathcal{F}\}$ converges weakly in $l^{\infty}(\mathcal{H}_{z, G})$ to a zero-mean Gaussian process $\mathbb{G}$, with an explicit covariance function. Next, one proves that the first term $\mathbb{G}_n(h_{w, f, G_n} - h_{w, f, G})$ tends in probability to zero, uniformly in $\mathcal{W} \times \mathcal{F}$, as n tends to infinity. Finally, making use of the functional delta-method, we establish that the third term $\mathbb{W}_n(w, f)$ converges weakly in $l^{\infty}(\mathcal{W} \times \mathcal{F})$ to a centered Gaussian limit process $\mathbb{W}$, with a well-defined covariance structure too. We complete the proof by showing that the joint process $(\mathbb{G}_n, \mathbb{W}_n)$ converges in distribution to a Gaussian limit process $(\mathbb{G}, \mathbb{W})$ which, by the continuous mapping theorem, implies that the process $(\mathbb{G}_n + \mathbb{W}_n)$ converges weakly to $\mathbb{G} + \mathbb{W}$. Now, observe that the cross covariance of the two processes $\mathbb{G}_n$ and $\mathbb{W}_n$ is nil. Indeed, for any $(w, f) \in \mathcal{W} \times \mathcal{F}$, one has

$$\begin{aligned}
\operatorname{cov}(\mathbb{G}_n(h_{w,f,G}), \mathbb{W}_n(w, f)) &= n \operatorname{cov}\left(\frac{1}{n}\sum_{j=1}^{n} h_{w,f,G}(Y_j), Ph_{w,f,G_n}\right) \\
&= n \operatorname{cov}\left(\frac{1}{n}\sum_{j=1}^{n} h_{w,f,G}(Y_j), \mathbb{E}h_{w,f,G_n}(Y)\right) \\
&= \sum_{j=1}^{n} \operatorname{cov}(h_{w,f,G}(Y_j), \mathbb{E}h_{w,f,G_n}(Y)) \\
&= 0.
\end{aligned}$$

By combining this and the asymptotic gaussianity of the process $(\mathbb{G}_n, \mathbb{W}_n)$, we can infer that the processes $\mathbb{G}_n$ and $\mathbb{W}_n$ are asymptotically independent and hence, the covariance kernel Σ is obtained by adding the covariance functions of the two processes. □

Remark 1. An immediate consequence of Theorem 2.2 is the convergence of the finite marginal distributions of the process $J_n(w, f)$ suitably normalized and centered. That is, for any integer $d \geq 2$ and for any functions $(w_1, f_1), \ldots, (w_d, f_d) \in \mathcal{W} \times \mathcal{F}$, the vector $(J_n(w_1, f_1), \ldots, J_n(w_d, f_d))$ is asymptotically Gaussian, with covariance matrix $\Sigma = (\Sigma_{k,l})_{1 \leq k, l \leq d}$, where $\Sigma_{k,l} = \Sigma[(w_k, f_k); (w_l, f_l)]$. Further the matrix Σ can be consistently estimated, allowing us to undertake non-parametric statistical inference. Indeed, for any $1 \leq k, l \leq d$, a natural estimator of $\Sigma_{k,l}$ is

$$\begin{aligned}
\hat{\Sigma}_{k,l} = \frac{1}{n}\sum_{j=1}^{q} w_k\left(\frac{j}{n}, \frac{q}{n}\right) f_k(Y_{j,n}, z) w_l\left(\frac{j}{n}, \frac{q}{n}\right) f_l(Y_{j,n}, z) \\
- \frac{1}{n^2}\sum_{i=1}^{q}\sum_{j=1}^{q} w_k\left(\frac{i}{n}, \frac{q}{n}\right) f_k(Y_{i,n}, z) w_l\left(\frac{j}{n}, \frac{q}{n}\right) f_l(Y_{j,n}, z)
\end{aligned}$$

$$+\frac{1}{n^2}\sum_{i=1}^{q}\sum_{j=1}^{q} a_1(Y_{i,n}, Y_{j,n})\left(\frac{i}{n} \wedge \frac{j}{n} - \frac{i}{n}\frac{j}{n}\right)$$

$$+\left(1-\frac{q}{n}\right)\frac{1}{n^2}\sum_{i=1}^{q}\sum_{j=1}^{q} a_2(Y_{i,n}, Y_{j,n})\frac{j}{n}$$

$$+\left(1-\frac{q}{n}\right)\frac{1}{n^2}\sum_{i=1}^{q}\sum_{j=1}^{q} a_3(Y_{i,n}, Y_{j,n})\frac{i}{n}$$

$$+\frac{q}{n}\left(1-\frac{q}{n}\right)\frac{1}{n^2}\sum_{i=1}^{q}\sum_{j=1}^{q} a_4(Y_{i,n}, Y_{j,n}), \tag{4}$$

where q is the number of poor (individuals whose incomes are less than z) in the sample, $a_r,\ r = 1, 2, 3, 4$ are real-valued functions defined in Theorem 2.2, and $Y_{1,n} \leq \cdots \leq Y_{n,n}$ are the order statistics associated with the sample $Y_1, ..., Y_n$.

3. Testing Procedure

Instead of working separately with single poverty indices or considering a particular class of poverty measures like the additive indices or the non-additive ones, our approach consider an arbitrary poverty-index vector $J = (J^1, ..., J^d),\ \ d \in \mathbb{N}^*$, and check for whether a poverty ranking can be established between two distributions by using simultaneously all components of J. Let A and B denote two income population distributions, J_A and J_B represent respectively their corresponding poverty-index vectors. There are several ways for testing poverty orderings. For instance, Kaur et al. [10] and Howes [8] utilized an approach which is based on a t-ratio statistic. While Davidson and Duclos [4] developed an empirical likelihood approach which computes the test statistic as the difference between the unconstrained and the constrained empirical likelihood functions. But it turns that all these procedures depend on a finite grid of points and do not make use of the covariance structure of the different points in the grid. As a result, they are lacking in power relative to tests that deal with this covariance structure. In this paper, we follow the latter idea, and make use of the asymptotic covariance function of the poverty indices given in Theorem 2.2 to establish a robust poverty ranking between two independent distributions. We shall employ the generalized Wald method, described for instance in Kodde and Palm [11] and Wolak [18], and applied in Zheng [21] for testing poverty dominance. Namely, we focus on the following two testing problems:

$$H_0 : J_A = J_B \text{ versus } H_1 : J_A \geq J_B$$

and

$$H_0 : J_A \geq J_B \text{ versus } H_1 : J_A \ngeq J_B,$$

where the notation $J_A \geq J_B$ means that each component of J_A is greater or equal to the corresponding component for J_B, with at least one strict inequality. That is distribution *B* dominates distribution *A*.

Assume that two independent samples of size n_A and n_B are, respectively, drawn from the populations *A* and *B*. Denote by $\hat{J}_A$ and $\hat{J}_B$ the estimators of the theoretical poverty-index vectors J_A and J_B from these samples. By Theorem 2.2, $\hat{J}_A$ and $\hat{J}_B$ are asymptotically normally distributed. That is, for n_A and n_B large enough,

$$\sqrt{n_A}(\hat{J}_A - J_A) \sim N(0, \Sigma_A)$$

and

$$\sqrt{n_B}(\hat{J}_B - J_B) \sim N(0, \Sigma_B).$$

Let $\Delta\hat{J} = \hat{J}_A - \hat{J}_B$. Then $\Delta\hat{J}$ is asymptotically normally distributed with zero-mean and covariance matrix Σ which, by independence of the two samples, decomposes into

$$\Sigma = \frac{1}{n_A}\Sigma_A + \frac{1}{n_B}\Sigma_B.$$

This suggests considering as an estimator of Σ, the random matrix

$$\hat{\Sigma} = \frac{1}{n_A}\hat{\Sigma}_A + \frac{1}{n_B}\hat{\Sigma}_B.$$

The crucial step for the generalized Wald method is the computation of the following quadratic problem:

$$\min_{v \geq 0} (\Delta\hat{J} - v)' \hat{\Sigma}^{-1} (\Delta\hat{J} - v).$$

Indeed, if $\tilde{v}$ is a solution of this quadratic problem, then the test statistics are given by

$$T = (\Delta\hat{J})' \hat{\Sigma}^{-1} (\Delta\hat{J}) - (\Delta\hat{J} - \tilde{v})' \hat{\Sigma}^{-1} (\Delta\hat{J} - \tilde{v}),$$

for the null hypothesis of equivalence $(H_0 : J_A = J_B)$ against the alternative of dominance $(H_1 : J_A \geq J_B)$, and

$$S = (\Delta\hat{J} - \tilde{v})' \hat{\Sigma}^{-1} (\Delta\hat{J} - \tilde{v}),$$

for the null hypothesis of dominance $(H_0 : J_A \geq J_B)$ against the general alternative of nondominance $(H_1 : J_A \ngeq J_B)$.

It has been proved (see, e.g., Kodde and Palm [11], Wolak [18) that, under H_0, these statistics are, each one, asymptotically distributed as a weighted sum of χ^2 variables, with degree of freedom $df = 0, \ldots, d$. To conclude the test, we must compare the observed values for the statistic T or S to the lower and upper bounds q_l and q_u for the critical value of the test, given a significance level $0 < \alpha < 1$. A table for these lower and upper bounds is available in Kodde and Palm [11]. q_l is obtained, with a degree of freedom $df = 1$, and satisfies

$$\alpha = \frac{1}{2}\mathbb{P}(\chi^2(1) \geq q_l).$$

While q_u, solves the general equation, with a degree of freedom $df = d$, i.e.,

$$\alpha = \frac{1}{2}\mathbb{P}(\chi^2(d-1) \geq q_u) + \frac{1}{2}\mathbb{P}(\chi^2(d) \geq q_u).$$

Finally, the decision rule is the same for the two tests. If the value of T (or S) is less than q_l, we do not reject H_0. If the value of T (or S) is greater than q_u, we reject H_0. If the value of T (or S) is between q_l and q_u, the test is inconclusive and Monte Carlo simulations are needed to complete the inference (see, e.g., Wolak [18]).

4. Applications

4.1. Simulation experiment

Since our theoretical results hold only asymptotically, it is important to know the minimum sample size from which they are applicable. We are particularly concerned with the asymptotic normality. For this we generate, using R software, random data sets from three classical distribution models: Exponential, Pareto and Uniform $(0, 1)$. We also consider four well-known poverty indices: the FGT measures with parameters $\alpha = 1, 2$, the Sen index and the Shorrocks one's. For each of these poverty indices represented by J, we denote $J_{n,i}$ its estimate from the ith generated sample, and compute the p-value (p_i) of the normality test for this sample i,

$$p_i = 2(1 - \Phi(z_{n,i})), \quad z_{n,i} = \left|\frac{\sqrt{n}(J_{n,i} - J)}{\hat{\sigma}}\right|,$$

where Φ is the standard normal distribution and $\hat{\sigma} = \Sigma_{k,k}^{1/2}$ is extracted from the covariance matrix Σ. After $B = 1000$ replications, we report the mean p-value for each index and for different sample sizes, $n = 50, 100, 200, 500, 1000$. The poverty line z is fixed as an α-quantile to keep in the range of the generated data. For lack of space, we shall only present the results for three values of α: 0.2, 0.5, 0.8. The results are shown in Tables 2, 3, 4 in the Appendix, where all the *p*-values are largely greater than the nominal level of 5%. That is, our poverty-index estimators are asymptotically Gaussian. Further, this property is true for relatively small sample of size $n = 50$ or 100. We also have satisfactory results for higher sample sizes of order $n = 500$ or 1000.

4.2. Illustration with real data sets

Now we apply our results to analyzing poverty in Senegal. We use income data sets available in ESAM 1 database which is a Living Standard Survey on households conducted by the "Agence Nationale de la Statistique et de la Démographie (ANSD)" in 1994. The data consist of a sample of $n = 3278$ households for all over the country of Senegal, with a weight allocated to each area. For lack of space, we restrict our analysis on four areas (or regions): Dakar (DK), Ziguinchor (ZG), Saint-Louis (SL) and Kolda (KD). Considering the income per capita as a measure of well-being, we compare these four areas using the generalized Wald test described in Section 3. The poverty line was fixed at $z = 143080$ FCFA at that period, corresponding to an income of US $1 per day, for each individual.

To implement the test, we consider $d = 4$ classical poverty indices among the most currently used in practice: the two Foster-Greer-Thorbecke measures for $\alpha = 1, 2$ (FGT(1), FGT(2)), the Sen index (SEN) and the Shorrocks index (SHO). We define the vector

$$J = (FGT(1), FGT(2), SEN, SHO),$$

which is estimated by

$$\hat{J} = (\widehat{FGT}(1), \widehat{FGT}(2), \widehat{SEN}, \widehat{SHO}),$$

where each component is calculated from random samples drawn, respectively, from the four areas, via the general formula (2) in Section 2.

For any pair of areas, we compute the two statistics $T = (\Delta\hat{J})'\hat{\Sigma}^{-1}(\Delta\hat{J}) - (\Delta\hat{J} - \tilde{v})'\hat{\Sigma}^{-1}(\Delta\hat{J} - \tilde{v})$, and $S = (\Delta\hat{J} - \tilde{v})'\hat{\Sigma}^{-1}(\Delta\hat{J} - \tilde{v})$, and report the results in Table 1 below. The vector $\tilde{v}$ was obtained by applying the command *constrOptim* of R software.

Table 1. Values of the statistics *T* and *S* for any pair of areas

Areas	*T*	*S*
(DK, ZG)	19.41	47.32
(DK, SL)	52.80	95.26
(DK, KD)	85.85	194.43
(ZG, SL)	11.58	0.76
(ZG, KD)	32.91	24.19
(SL, KD)	17.10	21.25

Each line in Table 1 corresponds to a pair of areas and contains two value statistics. The first statistic *T* is for the null hypothesis that the two areas are equivalent in terms of poverty. While the second statistic *S* is for the null hypothesis that the second area dominates the first. Comparing these values with the lower bound $q_l = 2.706$ and the upper bound $q_u = 8.761$ obtained by choosing a significance level $\alpha = 5\%$ (see, e.g., Kodde and Palm [11]), we observe that Kolda is dominated by all the other three regions because the null hypothesis is rejected for each test. That is, Kolda is the poorest area. We can also say that Dakar dominates the other regions, as the values of two statistics lead to rejection of the null hypotheses. Finally, we can conclude by the following ordering: Dakar dominates Ziguinchor, which dominates Saint-Louis. Kolda is at the bottom.

5. Conclusion

In the present paper, we have established a uniform central limit theorem for a large class of empirical poverty indices containing the most proposed in the literature. From this result, we have constructed a Wald-type test for establishing a robust poverty ordering between two independent populations. The test was also applied to four areas in Sénégal for comparing their poverty levels.

Nevertheless, our methodology applies only to two independent populations and does not take into account the dependence of the data, as it is the case of the panel data. This issue will be handled in a forthcoming paper.

6. Appendix

6.1. Simulation results

Table 2. Simulation results for exponential distribution of parameter $\lambda = 1$

	n / Index	50	100	200	500	1000
$\alpha = 0.2$	FGT(1)	0.51	0.43	0.48	0.48	0.48
	FGT(2)	0.43	0.46	0.53	0.48	0.46
	SEN	0.53	0.51	0.57	0.60	0.55
	SHO	0.46	0.41	0.46	0.49	0.47
$\alpha = 0.5$	FGT(1)	0.52	0.46	0.53	0.48	0.49
	FGT(2)	0.46	0.46	0.48	0.47	0.59
	SEN	0.57	0.63	0.63	0.56	0.60
	SHO	0.53	0.57	0.57	0.54	0.58
$\alpha = 0.8$	FGT(1)	0.49	0.50	0.46	0.49	0.50
	FGT(2)	0.52	0.43	0.53	0.51	0.56
	SEN	0.65	0.68	0.67	0.67	0.66
	SHO	0.72	0.71	0.74	0.73	0.70

Table 3. Simulation results for uniform $(0, 1)$ distribution

	n / Index	50	100	200	500	1000
$\alpha = 0.2$	FGT(1)	0.46	0.47	0.46	0.48	0.49
	FGT(2)	0.49	0.50	0.51	0.46	0.48
	SEN	0.54	0.60	0.58	0.58	0.57
	SHO	0.55	0.52	0.54	0.52	0.56
$\alpha = 0.5$	FGT(1)	0.52	0.51	0.53	0.47	0.51
	FGT(2)	0.43	0.50	0.54	0.49	0.53
	SEN	0.64	0.57	0.62	0.60	0.60
	SHO	0.57	0.62	0.61	0.60	0.60
$\alpha = 0.8$	FGT(1)	0.52	0.52	0.50	0.50	0.48
	FGT(2)	0.48	0.48	0.45	0.47	0.51
	SEN	0.66	0.69	0.68	0.67	0.68
	SHO	0.68	0.73	0.69	0.68	0.68

Table 4. Simulation results for Pareto distribution with parameters $\gamma = 0.3$ and $c = 1$

	n / Index	50	100	200	500	1000
$\alpha = 0.2$	FGT(1)	0.50	0.50	0.52	0.48	0.49
	FGT(2)	0.48	0.51	0.48	0.52	0.48
	SEN	0.27	0.28	0.25	0.26	0.27
	SHO	0.34	0.35	0.37	0.30	0.29
$\alpha = 0.5$	FGT(1)	0.47	0.49	0.49	0.49	0.51
	FGT(2)	0.46	0.51	0.50	0.54	0.53
	SEN	0.37	0.34	0.34	0.33	0.34
	SHO	0.23	0.20	0.21	0.26	0.27
$\alpha = 0.8$	FGT(1)	0.48	0.49	0.48	0.50	0.60
	FGT(2)	0.52	0.50	0.51	0.50	0.50
	SEN	0.11	0.15	0.18	0.16	0.14
	SHO	0.22	0.18	0.22	0.23	0.27

6.2. Proof of Theorem 2.2

First, recall the definition of the classes of functions $\mathcal{W}$ and $\mathcal{F}$

$$\mathcal{W} = \{w : [0, 1] \times [0, 1] \to \mathbb{R}_+,\ w \text{ continuous, and } u \mapsto w(u, \cdot)$$
$$\text{is non-increasing}\},$$
$$\mathcal{F} = \{f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+,\ f \text{ continuous, and } y \mapsto f(y, \cdot)$$
$$\text{is non-increasing}\}.$$

Next, introduce the class of functions

$$\mathcal{K} = \{k : \mathbb{R} \to [0, 1] \text{ increasing}\}.$$

For $z > 0$ fixed, $w \in \mathcal{W}$, $f \in \mathcal{F}$, $k \in \mathcal{K}$ define the real-valued function

$$h_{w, f, k}(y) = w[k(y), k(z)] f(y, z) \mathbb{I}(y < z), \text{ for all } y \in \mathbb{R}_+$$

and let $\mathcal{H}_z$ be the class of functions defined as

$$\mathcal{H}_z = \{y \mapsto h_{w, f, k}(y) : w \in \mathcal{W},\ f \in \mathcal{F},\ k \in \mathcal{K}\}.$$

According to the sketch given at the end of the statement of Theorem 2.2, we split the proof into four parts. In the first, we establish the Donsker property for the class $\mathcal{H}_z$, and derive from this, that the empirical process $\{\mathbb{G}_n(h_{w, f, G}) : w \in \mathcal{W},\ f \in \mathcal{F}\}$ converges weakly to a limit Gaussian process $\mathbb{G}(h_{w, f, G})$. In the second, we show that

$$\sup_{(w, f) \in \mathcal{W} \times \mathcal{F}} |\mathbb{G}_n(h_{w, f, G_n} - h_{w, f, G})| \to_p 0, \quad n \to \infty, \tag{5}$$

where "$\to_p$" denotes the convergence in probability. In the third part, we prove the weak convergence of the process $\mathbb{W}_n(w, f)$ to a zero-mean Gaussian process $\mathbb{W}(w, f)$ in $l^\infty(\mathcal{W} \times \mathcal{F})$. Finally, in the last part, we prove that the joint process $(\mathbb{G}_n, \mathbb{W}_n)$ converges weakly to $(\mathbb{G}, \mathbb{W})$ which is a zero-mean Gaussian process.

6.3. Part I

Recall that P is the common probability law of the $Y_j' s$ and G stands for its cumulative distribution function. We have to prove that the class of functions $\mathcal{H}_z$ is P-Donsker. This will be done if we prove that the bracketing integral

$$J_{[\ ]}(\infty, \mathcal{H}_z, L_2(P)) = \int_0^{\infty} \sqrt{\log N_{[\ ]}(\varepsilon, \mathcal{H}_z, L_2(P))}\, d\varepsilon$$

is finite, where $N_{[\ ]}(\cdot)$ denotes the bracketing number. Before proving this, observe that the elements of $\mathcal{H}_z$ are continuous and increasing functions, bounded on $\mathbb{R}_+$ by $w[k(0), k(z)] f(0, z)$, for every $(w, f, k) \in \mathcal{W} \times \mathcal{F} \times \mathcal{K}.$ By assumption (A), the classes of functions $\mathcal{W}$ and $\mathcal{F}$ are pointwise measurable. Further, Lemma 2.2 of [14] entails that the δ-entropy, relatively to the supremum norm, of the class of increasing functions $\mathcal{K}$ is finite for any $\delta > 0.$ That is, the class $\mathcal{K}$ is totally bounded relatively to the supremum norm, and hence is pointwise measurable. This enables us to take the supremum over the set $\mathcal{W} \times \mathcal{F} \times \mathcal{K}$ as equal to the supremum over a countable subset $\mathcal{G}_0 \subset \mathcal{W} \times \mathcal{F} \times \mathcal{K}.$ Since for $z > 0$ fixed, the quantity $w[k(0), k(z)] f(0, z)$ is finite for any $(w, f, k) \in \mathcal{W} \times \mathcal{F} \times \mathcal{K},$ we may define the constant function

$$H(y) = \sup_{(w, f, k) \in \mathcal{W} \times \mathcal{F} \times \mathcal{K}} w[k(0), k(z)] f(0, z), \quad \forall y \in \mathbb{R}_+,$$

as an envelope function for the class $\mathcal{H}_z.$ Then $\mathcal{H}_z$ is uniformly bounded by $H(y),$ and we may assume without loss of generality that $H(y) \equiv 1.$ Thus, $\mathcal{H}_z$ is a subset of the class of monotone functions defined on $\mathbb{R}$ with values in $[0, 1].$ It follows from [15, Theorem 2.7.5, p. 159] that for all $\varepsilon > 0,$

$$\log N_{[\ ]}(\varepsilon, \mathcal{H}_z, L_2(P)) < C\varepsilon^{-1}, \tag{6}$$

where C is a positive constant.

From the fact that the elements of $\mathcal{H}_z$ take their values in $[0, 1]$, for $\varepsilon > 1$ the number of ε-brackets needed to cover $\mathcal{H}_z$ is just 1. Then $J_{[\ ]}(\infty, \mathcal{H}_z, L_2(P))$ would be finite if

$$\int_0^1 \sqrt{\log N_{[\ ]}(\varepsilon, \mathcal{H}_z, L_2(P))}\, d\varepsilon < \infty.$$

Now, integrating both sides of (6), one obtains

$$\int_0^1 \sqrt{\log N_{[\ ]}(\varepsilon, \mathcal{H}_z, L_2(P))}\, d\varepsilon < \sqrt{C}\int_0^1 \varepsilon^{-1/2} d\varepsilon = 2\sqrt{C} < \infty.$$

That is, $J_{[\ ]}(\infty, \mathcal{H}_z, L_2(P))$ is finite and the class $\mathcal{H}_z$ is P-Donsker. In particular, for $k = G$ (the distribution function associated with the probability law P), the class $\mathcal{H}_z$ restricts to

$$\mathcal{H}_{z,G} = \{h_{w, f, G} : w \in \mathcal{W},\ f \in \mathcal{F}\},$$

which may be identified to $\mathcal{W} \times \mathcal{F}$. Since $\mathcal{H}_{z,G} \subset \mathcal{H}_z$ is P-Donsker, so is the class $\mathcal{W} \times \mathcal{F}$. Then it follows that the empirical process $\{\mathbb{G}_n(h_{w, f, G}) : w \in \mathcal{W},\ f \in \mathcal{F}\}$ converges weakly in $l^{\infty}(\mathcal{H}_{z,G})$ to a tight limit process $\mathbb{G}$, which is a zero-mean Gaussian process with covariance function defined, for all (w, f) and $(\tilde{w}, \tilde{f})$, by

$$\begin{aligned}
&\operatorname{cov}(\mathbb{G}(h_{w, f, G}), \mathbb{G}(h_{\tilde{w}, \tilde{f}, G}))\\
&= Ph_{w, f, G}h_{\tilde{w}, \tilde{f}, G} - Ph_{w, f, G}Ph_{\tilde{w}, \tilde{f}, G}\\
&= \int_0^z w[G(y), G(z)]\, f(y, z)\tilde{w}[G(y), G(z)]\, \tilde{f}(y, z)\, dG(y)\\
&\quad - \int_0^z w[G(y), G(z)]\, f(y, z)\, dG(y)\int_0^z \tilde{w}[G(y), G(z)]\, \tilde{f}(y, z)\, dG(y).
\end{aligned}$$

6.4. Part II

For establishing (5), we first remark that for any $(w, f) \in \mathcal{W} \times \mathcal{F}$, the functions $h_{w,\,f,\,G}$ and $h_{w,\,f,\,G_n}$ are elements of $\mathcal{H}_z$, which is shown to be P-Donsker according to the preview part. Since $h_{w,\,f,\,G}$ and $h_{w,\,f,\,G_n}$ are bounded, they are in $L_2(P) = L_2(G)$. Now, one has

$$\int_0^{\infty} [h_{w,\,f,\,G_n}(y) - h_{w,\,f,\,G}(y)]^2 dG(y)$$

$$\leq \sup_{y \leq z} [h_{w,\,f,\,G_n}(y) - h_{w,\,f,\,G}(y)]^2$$

$$\leq \sup_{y \leq z} |\, w(G_n(y),\, G_n(z)) - w(G(y),\, G(z)) \,|^2 f^2(y,\, z)$$

$$\leq\; f^2(0,\, z) \sup_{y \leq z} |\, w(G_n(y),\, G_n(z)) - w(G(y),\, G(z)) \,|^2,$$

which tends almost surely to 0, as $n \to \infty$, by continuity of the function w and the fact that the empirical distribution function $G_n(y)$ converges almost surely to $G(y)$ for all $y \in \mathbb{R}$. Thus, as n tends to infinity, $\int_0^{\infty} [h_{w,\,f,\,G_n}(y) - h_{w,\,f,\,G}(y)]^2 dG(y)$ converges almost surely and hence in probability to zero. It follows from Lemma 19.24 of [16] that $\mathbb{G}_n(h_{w,\,f,\,G_n} - h_{w,\,f,\,G}) \to_p 0$, $n \to \infty$ which, by the continuous mapping theorem, implies that

$$\sup_{(w,\,f) \in \mathcal{W} \times \mathcal{F}} |\, \mathbb{G}_n(h_{w,\,f,\,G_n} - h_{w,\,f,\,G}) \,| \to_p 0, \quad n \to \infty.$$

This establishes the second part of our proof.

6.5. Part III

For any given functions $(w,\, f) \in \mathcal{W} \times \mathcal{F}$, we define on the class $\mathcal{K} = \{k : \mathbb{R} \to [0,\, 1],\ \text{increasing}\}$ the following operator:

$$\phi_{w,\,f} : k \mapsto \phi_{w,\,f}(k) = \int_0^{z} w[k(y),\, k(z)]\, f(y,\, z) dP(y) = Ph_{w,\,f,\,k}.$$

Recall that $\frac{\partial}{\partial u}\xi(a, b)$ and $\frac{\partial}{\partial v}\xi(a, b)$ are the partial derivatives of a differentiable function $\xi(u, v)$ with respect to its first and second arguments, taken at $(u, v) = (a, b)$. Let

$$\mathcal{K}' = \{k \in \mathcal{K},\ k \text{ continuous}\}.$$

For all $k \in \mathcal{K}$, and $s_t \in \mathcal{K}$ such that $k + ts_t \in \mathcal{K}$ and $s_t \to s \in \mathcal{K}'$, as $t \to 0$, one has by a first-order Taylor expansion of w, for some functions ζ and π defined on $\mathbb{R}$, with values in $(0, 1)$:

$$\frac{\phi_{w,f}(k + ts_t) - \phi_{w,f}(k)}{t}$$

$$= \int_0^z s_t(y)\frac{\partial}{\partial u}w[k(y) + t\pi(t)s_t(y),\ k(z) + t\zeta(t)s_t(z)]\,f(y, z)dG(y)$$

$$+ \int_0^z s_t(z)\frac{\partial}{\partial v}w[k(y) + t\pi(t)s_t(y),\ k(z) + t\zeta(t)s_t(z)]\,f(y, z)dG(y)$$

$$=: I_t + II_t.$$

Now, we have to show that as $t \to 0$,

$$I_t \to I = \int_0^z s(y)\frac{\partial}{\partial u}w[k(y),\ k(z)]\,f(y, z)dG(y),$$

$$II_t \to II = \int_0^z s(z)\frac{\partial}{\partial v}w[k(y),\ k(z)]\,f(y, z)dG(y).$$

We only establish the first result as the other can be handled with the same techniques. By assumption (A) the function w and its first-order partial derivatives are bounded on $(0, z]$ and one has:

$$|\,I_t - I\,| \le \sup_{y \le z}\Big|\, s_t(y)\frac{\partial}{\partial u}w[k(y) + t\pi(t)s_t(y),\ k(z) + t\zeta(t)s_t(z)]$$

$$- s(y)\frac{\partial}{\partial u}w[k(y),\ k(z)]\Big|\, f(y, z)\int_0^z dG(y).$$

Adding and subtracting appropriate terms and observing that both k and s are bounded by 1, one has:

$$| I_t - I | \leq \sup_{y \leq z} | s_t(y) - s(y) |$$

$$\times \sup_{y \leq z} \left\{ \left| \frac{\partial}{\partial u} w[k(y) + t\pi(t) s_t(y),\ k(z) + t\zeta(t) s_t(z)] \right| f(y,\ z) \right\}$$

$$+ \sup_{y \leq z} \left\{ \left| \frac{\partial}{\partial u} w[k(y) + t\pi(t) s_t(y),\ k(z) + t\zeta(t) s_t(z)] \right. \right.$$

$$\left. \left. - \frac{\partial}{\partial u} w[k(y),\ k(z)] \right| f(y,\ z) \right\}.$$

The fact that $s_t \to s,$ as $t \to 0$ entails that $| s_t(y) - s(y) | \to 0,$ as $t \to 0.$ Consequently, the first term in the right-hand side of the above inequality tends to 0, as t tends to 0. The second term also tends to 0, as t goes to 0. This is due to the continuity of w and its first-order partial derivatives. It results from above that, ϕ is Hadamard-differentiable at $k \in \mathcal{K},$ tangentially to $\mathcal{K}',$ with derivative $\phi'_{w,\,f}[k],$ given for all $s \in \mathcal{K}$ by

$$\phi'_{w,\,f}[k](s) = \int_0^z \left\{ s(y) \frac{\partial}{\partial u} w[k(y),\ k(z)] + s(z) \frac{\partial}{\partial v} w[k(y),\ k(z)] \right\} f(y,\ z) dG(y).$$

Since $\sqrt{n}[G_n - G]$ converges weakly to $\mathbb{B} \circ G,$ where $\mathbb{B}$ stands for the standard Brownian bridge, it follows from the functional delta-method (see, e.g., [16]) that $\sqrt{n}[\phi_{w,\,f}(G_n) - \phi_{w,\,f}(G)] = \sqrt{n}[Ph_{w,\,f,\,G_n} - Ph_{w,\,f,\,G}]$ $= \mathbb{W}_n(w,\ f)$ converges in distribution to the Gaussian variable

$$\phi'_{w,\,f}[G](\mathbb{B} \circ G)$$

$$= \int_0^z \left\{ \mathbb{B} \circ G(y) \frac{\partial}{\partial u} w[G(y), G(z)] + \mathbb{B} \circ G(z) \frac{\partial}{\partial v} w[G(y), G(z)] \right\} f(y,\ z) dG(y)$$

$$=: \mathbb{W}(w,\ f).$$

Since the class of functions $\mathcal{W} \times \mathcal{F}$ is shown to be Donsker according to Part I, we can infer that the process $\{\mathbb{W}_n(w,\ f) : w \in \mathcal{W},\ f \in \mathcal{F}\}$ converges in distribution to $\mathbb{W}(w, f)$ which is a zero-mean Gaussian process, with covariance kernel given, for all (w, f) and $(\tilde{w}, \tilde{f})$, by

$$\operatorname{cov}(\mathbb{W}(w, f), \mathbb{W}(\tilde{w}, \tilde{f})) = \int_0^z\int_0^z a_1(x, y)[G(x) \wedge G(y) - G(x)G(y)]dG(x)dG(y)$$

$$+ [1 - G(z)]\int_0^z\int_0^z a_2(x, y)G(y)dG(x)dG(y)$$

$$+ [1 - G(z)]\int_0^z\int_0^z a_3(x, y)G(x)dG(x)dG(y)$$

$$+ G(z)[1 - G(z)]\int_0^z\int_0^z a_4(x, y)dG(x)dG(y),$$

where

$$a_1(x, y) = \frac{\partial w}{\partial u}[G(x), G(z)]\, f(x, z)\frac{\partial \tilde{w}}{\partial u}[G(y), G(z)]\, \tilde{f}(y, z),$$

$$a_2(x, y) = \frac{\partial w}{\partial u}[G(x), G(z)]\, f(x, z)\frac{\partial \tilde{w}}{\partial v}[G(y), G(z)]\, \tilde{f}(y, z),$$

$$a_3(x, y) = \frac{\partial w}{\partial v}[G(x), G(z)]\, f(x, z)\frac{\partial \tilde{w}}{\partial u}[G(y), G(z)]\, \tilde{f}(y, z),$$

$$a_4(x, y) = \frac{\partial w}{\partial v}[G(x), G(z)]\, f(x, z)\frac{\partial \tilde{w}}{\partial v}[G(y), G(z)]\, \tilde{f}(y, z).$$

6.6. Part IV

Here we show that the couple of processes $(\mathbb{G}_n, \mathbb{W}_n)$ converges weakly to joint process $(\mathbb{G}, \mathbb{W})$ which is a zero-mean Gaussian process. To this end, we show that it is tight and that its finite marginal distributions converge to those of a Gaussian process.

The tightness follows immediately from Parts I and III, where it is proved that $\mathbb{G}_n$ converges weakly to a tight Gaussian process $\mathbb{G} \in l^{\infty}(\mathcal{H}_{z,G})$, and $\mathbb{W}_n$ converges weakly to a tight Gaussian process $\mathbb{W} \in l^{\infty}(\mathcal{W} \times \mathcal{F})$.

For the study of the finite dimensional distributions, we have to show that for all $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_\ell \in \mathbb{R}$ and $(w_l, f_l), (\tilde{w}_i, \tilde{f}_i) \in \mathcal{W} \times \mathcal{F}$, $l = 1, ..., m$, $i = 1, ..., \ell$, the linear combination

$$\sum_{l=1}^{m} \alpha_l \mathbb{G}_n(h_{w_l, f_l, G}) + \sum_{i=1}^{\ell} \beta_i \mathbb{W}_n(\tilde{w}_i, \tilde{f}_i) \tag{7}$$

is asymptotically Gaussian. For this, we make use of the asymptotic linearity of the two processes $\mathbb{G}_n$ and $\mathbb{W}_n$. For larger values of n, the latter can be expressed in terms of the former. Indeed for all $(w, f) \in \mathcal{W} \times \mathcal{F}$ denote by $L_{w, f}$ the Hadamard derivative of $\phi_{w, f}$ at G; that is $L_{w, f} = \phi'_{w, f}[G]$. Then for larger values of n one has

$$\mathbb{W}_n(w, f) = \sqrt{n}(Ph_{w, f, G_n} - Ph_{w, f, G}) = L_{w, f}(\sqrt{n}[G_n - G]) + o_P(1).$$

Since $G_n(\cdot) = n^{-1}\sum_{j=1}^{n} \mathbb{I}(Y_j \leq \cdot) = n^{-1}\sum_{j=1}^{n} \mathbb{I}_{[Y_j, \infty)}(\cdot)$, using the linearity of $L_{w, f}$, we obtain for n large enough that

$$\begin{aligned}\mathbb{W}_n(w, f) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [L_{w, f}(\mathbb{I}_{[Y_j, \infty)}(\cdot)) - L_{w, f}(G)] + o_P(1) \\ &= \mathbb{G}_n(L_{w, f}(\mathbb{I}_{[Y_j, \infty)}(\cdot)) + o_P(1).\end{aligned}$$

Combining this with the linearity of $\mathbb{G}_n$ we obtain, for all $\alpha_1, ..., \alpha_m$, $\beta_1, ..., \beta_\ell \in \mathbb{R}$ and $(w_l, f_l), (\tilde{w}_i, \tilde{f}_i) \in \mathcal{W} \times \mathcal{F}$, $l = 1, ..., m$, $i = 1, ..., \ell$, that

$$\begin{aligned}&\sum_{l=1}^{m} \alpha_l \mathbb{G}_n(h_{w_l, f_l, G}) + \sum_{i=1}^{\ell} \beta_i \mathbb{W}_n(\tilde{w}_i, \tilde{f}_i) \\ &= \sum_{l=1}^{m} \alpha_l \mathbb{G}_n(h_{w_l, f_l, G}) + \sum_{i=1}^{\ell} \beta_i \mathbb{G}_n(L_{\tilde{w}_i, \tilde{f}_i}(\mathbb{I}_{[Y_j, \infty)}(\cdot)) + o_P(1) \\ &= \mathbb{G}_n\left(\sum_{l=1}^{m} \alpha_l h_{w_l, f_l, G} + \sum_{i=1}^{\ell} \beta_i L_{\tilde{w}_i, \tilde{f}_i}(\mathbb{I}_{[Y_j, \infty)}(\cdot)\right) + o_P(1).\end{aligned}$$

Recall that $\mathbb{G}_n$ is the empirical process and that the function

$$\sum_{l=1}^{m} \alpha_l h_{w_l, f_l, G} + \sum_{i=1}^{\ell} \beta_i L_{\tilde{w}_i, \tilde{f}_i}(\mathbb{I}_{[Y_j, \infty)}(\cdot))$$

belongs to $L^2(P)$. Then it follows that the random variable defined in (7) is asymptotically Gaussian, and hence the finite marginal distributions of the process $(\mathbb{G}_n, \mathbb{W}_n)$,

$$(\mathbb{G}_n(h_{w_1, f_1, G}), \ldots, \mathbb{G}_n(h_{w_m, f_m, G}), \mathbb{W}_n(\tilde{w}_1, \tilde{f}_1), \ldots, \mathbb{W}_n(\tilde{w}_\ell, \tilde{f}_\ell))$$

are asymptotically Gaussian too. Combining this with the tightness argument enable us to conclude that the joint process $(\mathbb{G}_n, \mathbb{W}_n)$ converges weakly to the process $(\mathbb{G}, \mathbb{W})$ which is Gaussian and centered.

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