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SOME EXTREMUM PROBLEMS RELATED TO MORLEY'S THEOREM

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Abstract

The paper considers three extremum problems related to Morley's theorem. One of the problems deals with searching which triangle with given circumradius has its Morley triangle of maximal area. The second one is devoted to the research of maximal ratio of the area of Morley triangle to the area of its original triangle. The third problem considers a question similar to the previous one for a triangle where one of its angles is given. In the paper, it is proved that in the first two cases the answer is an equilateral triangle and in the third one the answer depends on the value of the given angle.

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1. Introduction

Morley's theorem states that the three points of intersection of the adjacent angle trisectors of any triangle form an equilateral triangle [1-3] (Figure 1). The equilateral triangle XYZ is called the *Morley triangle* of the original triangle ABC.

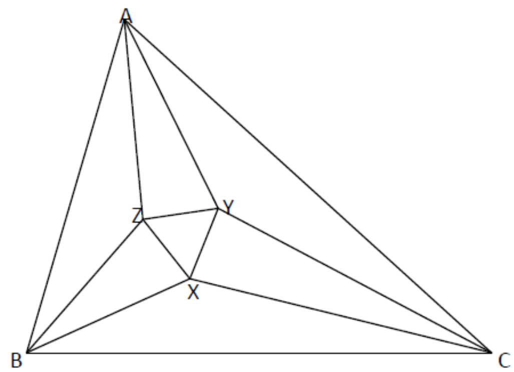


Figure 1

Morley triangle has sides

$$XY = YZ = ZX = 8R\sin\frac{\alpha}{3}\sin\frac{\beta}{3}\sin\frac{\gamma}{3},$$

where α, β, γ denote the angles of the original triangle ABC, R is its circumradius [4]. Then the area S_1 of the Morley triangle is

$$S_1 = 16\sqrt{3}R^2\sin^2\frac{\alpha}{3}\sin^2\frac{\beta}{3}\sin^2\frac{\gamma}{3}. \tag{1}$$

In this note, we will find the answers to the following three problems:

1. How can one characterize the triangle with given circumradius R having Morley triangle of the largest area?

2. How can one characterize the original triangle for which the ratio between the area of its Morley triangle and the area of the original triangle itself is maximal?

3. How can one characterize the original triangle for which the ratio between the area of its Morley triangle and the area of the original triangle itself is maximal when one of its angles is given?

2. The Morley Triangle of the Largest Area

In order to answer to the first question, we will prove the following theorem.

Theorem 1. *The function* $f(x, y) = \sin\frac{x}{3}\sin\frac{y}{3}\sin\left(\frac{\pi - x - y}{3}\right),\ 0 \le x, y,$ $x + y \le \pi,$ *receives the maximal value at the point* $(\pi/3, \pi/3).$

Proof. The function $f(x, y)$ is continuous on compact set $Q = \{(x, y) \in R^2 \mid 0 \le x, y, x + y \le \pi\},$ on the boundary of the set $f(x, y) = 0$ and in the interior $f(x, y) > 0.$ So $f(x, y)$ receives the maximal value in the interior of Q. In order to find the maximum of $f(x, y),$ we solve the system of equations (for $0 \le x, y, x + y \le \pi$):

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$$

(this is a necessary condition for a function $f(x, y)$ to have an extremum at some point). Finding the partial derivatives of $f(x, y)$ leads to the following system:

$$\begin{cases} \cos\dfrac{x}{3}\sin\left(\dfrac{\pi - x - y}{3}\right) - \sin\dfrac{x}{3}\cos\left(\dfrac{\pi - x - y}{3}\right) = 0 \\ \cos\dfrac{y}{3}\sin\left(\dfrac{\pi - x - y}{3}\right) - \sin\dfrac{y}{3}\cos\left(\dfrac{\pi - x - y}{3}\right) = 0 \end{cases}$$

or

$$\begin{cases} \sin\left(\dfrac{\pi - 2x - y}{3}\right) = 0 \\ \sin\left(\dfrac{\pi - x - 2y}{3}\right) = 0. \end{cases}$$

One can easily check that the last system of equations has only one solution $x = y = \pi/3$ in the interior of Q.

Therefore, $f(x, y)$ receives the maximal value at the point $(\pi/3, \pi/3)$ only, $f_{\max} = \sin^3 \frac{\pi}{9}$.

From this and from (1), one can easily come to the following conclusion:

Conclusion 1. Among all triangles inscribed in a circle of given radius R, the equilateral triangle has Morley triangle of the largest area $S_{1\max} = 16\sqrt{3}R^2 \sin^6 \frac{\pi}{9}$.

3. The Maximal Ratio of the Area of Morley Triangle to the Area of its Original Triangle

The area S of a triangle with given angles α, β, γ and with circumradius R is equal to $S = 2R^2 \sin\alpha \sin\beta \sin\gamma$. Then

$$\frac{S_1}{S} = 8\sqrt{3} \cdot \frac{\sin^2 \frac{\alpha}{3} \sin^2 \frac{\beta}{3} \sin^2 \frac{\gamma}{3}}{\sin\alpha \sin\beta \sin\gamma}. \tag{2}$$

In order to find the maximal value of (2), we will prove the next theorem.

Theorem 2. *Function* $g(x, y) = \dfrac{\sin^2 x \sin^2 y \sin^2(\pi/3 - x - y)}{\sin 3x \sin 3y \sin(\pi - 3x - 3y)}$, $x, y > 0$, $x + y < \pi/3$, *receives the maximal value at the point* $(\pi/9, \pi/9)$.

Proof. The continuous function $g(x, y)$ is defined in open set with boundary Q, $Q = \{x, y \mid x, y > 0, x + y < \pi/3\}$ and $\lim_{(x, y)\to(a, b)} g(x, y) = 0$

for each point $(a, b) \in Q$ (one can easily check it). Also, $g(x, y) > 0$ in the set. Therefore, $g(x, y)$ receives the maximal value in it.

Using the formula $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi$, we obtain that

$$g(x, y) = \frac{\sin x}{3 - 4\sin^2 x} \cdot \frac{\sin y}{3 - 4\sin^2 y} \cdot \frac{\sin(\pi/3 - x - y)}{3 - 4\sin^2(\pi/3 - x - y)}.$$

In order to find the maximum of $g(x, y)$, we first have to solve the system of equations:

$$\begin{cases} \dfrac{\partial g}{\partial x} = 0 \\ \dfrac{\partial g}{\partial y} = 0. \end{cases}$$

This system implies that

$$\begin{cases} \dfrac{3\cos x + 4\sin^2 x\cos x}{3\sin x - 4\sin^3 x} \\ = \dfrac{3\cos(\pi/3 - x - y) + 4\sin^2(\pi/3 - x - y)\cos(\pi/3 - x - y)}{3\sin(\pi/3 - x - y) - 4\sin^3(\pi/3 - x - y)} \\ \\ \dfrac{3\cos y + 4\sin^2 y\cos y}{3\sin y - 4\sin^3 y} \\ = \dfrac{3\cos(\pi/3 - x - y) + 4\sin^2(\pi/3 - x - y)\cos(\pi/3 - x - y)}{3\sin(\pi/3 - x - y) - 4\sin^3(\pi/3 - x - y)}. \end{cases} \tag{3}$$

Let $u(x) = \dfrac{3\cos x + 4\sin^2 x\cos x}{3\sin x - 4\sin^3 x}$. Then the system (3) is equivalent to $\begin{cases} u(x) = u(z) \\ u(y) = u(z) \end{cases}$, where $z = \pi/3 - x - y$ (reminder : $0 < x, y, z < \pi/3$).

In order to prove $x = y = z$, we find the derivative of $u(x)$:

$$u'(x) = \frac{-32\sin^4 x + 48\sin^2 x - 9}{(3\sin x - 4\sin^3 x)^2} = \frac{-32(\sin^2 x - t_1)(\sin^2 x - t_2)}{(3\sin x - 4\sin^3 x)^2},$$

where $(t_1 \approx 0.22,\ t_2 \approx 1.28)$. Thus, $u'(x) < 0$ in $(0, \arcsin\sqrt{t_1})$ and $u'(x) > 0$ in $(\arcsin\sqrt{t_1}, \pi/3)$. Then continuous function $u(x)$ decreases in $(0, \arcsin\sqrt{t_1})$ and it increases in $(\arcsin\sqrt{t_1}, \pi/3)$. Suppose that the system (3) has solution (x_0, y_0). Then at least two of three values $x_0, y_0, z_0 = \pi/3 - x_0 - y_0$ are together in $(0, \arcsin\sqrt{t_1}]$ or in $[\arcsin\sqrt{t_1}, \pi/3)$. So, in any case, two of these three values x_0, y_0, z_0 are equal to each other, i.e., at least two angles of triangle *ABC* are equal to each other. We want to prove that triangle *ABC* is equilateral.

Without loss of generality, we have to consider two cases:

Case 1. $x_0 = y_0$. In order to prove that in this case $x_0 = y_0 = z_0 = \pi/9$ let us consider the function

$$w(x) = \frac{\sin^4 x \sin^2(\pi/3 - 2x)}{\sin^2 3x \sin(\pi - 6x)} \tag{4}$$

obtained from $g(x, y)$ by substituting x instead of y $(0 < x < \pi/6)$. We prove that $w(x)$ receives the maximal value when $x = \pi/9$.

A necessary condition for a function $w(x)$ to have an extremum at some point is

$$w'(x) = 0. \tag{5}$$

From (5), we obtain that

$$2\cot x - 3\cot 3x - 2\cot(\pi/3 - 2x) + 3\cot(\pi - 6x) = 0. \tag{6}$$

Using one of the numerical methods for solving equations, one can check that (6) has only one solution $x = \pi/9$ in $(0, \pi/6)$.

Case 2. $x_0 = z_0$. We have $y_0 = \pi/3 - 2x_0$ and then this case reduces to the previous one.

From Theorem 2, we can conclude that the ratio (2) receives maximal value only for $\alpha = \beta = \gamma = \pi/3$, i.e., when the original triangle *ABC* is equilateral. The maximal value of (2) is $\frac{64}{3} \cdot \sin^6 \frac{\pi}{9} = 0.034\ldots$. That means that the area of Morley triangle is less than 3.5% of the area of the original triangle.

4. The Maximal Ratio of the Area of Morley Triangle to the Area of its Original Triangle When One Angle of the Original Triangle is Given

It may seem that the answers obtained in Theorems 1 and 2 were obvious in advance from symmetry considerations. We want to note here that the symmetry consideration does not always leads to the correct result. Indeed, let us consider the following problem: how can one characterize the triangle with given angle *A* such that it has maximal ratio of the area of its Morley triangle to the area of the original triangle itself? It seems from symmetry considerations that it must be the isosceles triangle. But it is not correct, in general. As it will be shown below, it depends on the value of angle *A*.

In order to explore this problem, we have to consider the function

$$g(x) = \frac{\sin^2 \frac{x}{3} \sin^2 \frac{\alpha}{3} \sin^2\left(\frac{\pi - x - \alpha}{3}\right)}{\sin x \sin \alpha \sin(\pi - x - \alpha)},$$

where α is a given angle $(\alpha \in (0, \pi))$ and $x \in (0, \pi - \alpha)$ (see Theorem 2). The function $g(x)$ is continuous and positive in $(0, \pi - \alpha)$. It may be easily checked that $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to \pi - \alpha} g(x) = 0$. Therefore, $g(x)$ receives the maximal value in $(0, \pi - \alpha)$.

In order to find extremum of $g(x)$, we must differentiate $g(x)$ and then solve the equation $g'(x) = 0$.

This leads us to the equation

$$2\sin x\sin(x+\alpha)\sin\left(\frac{\pi-2x-\alpha}{3}\right)$$

$$=3\sin\frac{x}{3}\sin\left(\frac{\pi-x-\alpha}{3}\right)\sin(2x+\alpha). \tag{7}$$

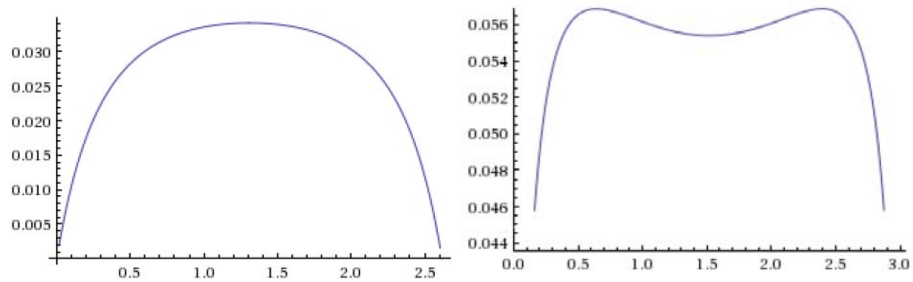
One of the obvious solutions of (7) is $x=\frac{\pi-\alpha}{2}$, that corresponds to isosceles triangle. But this does not generally mean that for this triangle $g(x)$ receives its maximal value. So, using one of the numerical methods for solving equations, it is possible to obtain that there exists a critical value α' of α, $0.21475221<\alpha'<0.21475222$, i.e., $12.3^\circ<\alpha'<12.4^\circ$ such that

(1) For $\alpha\geq\alpha'$, equation (7) has only one solution $x_0=\frac{\pi-\alpha}{2}$ in $(0,\pi-\alpha)$ and in this case the isosceles triangle with angles α, x_0, x_0 corresponds to the solution of the problem.

(2) For $\alpha<\alpha'$, equation (7) has two other solutions x_1 and $\pi-\alpha-x_1$ in $(0,\pi-\alpha)$ except for x_0. These solutions are symmetric with respect to the point x_0 and they correspond to the same triangle with angles $\alpha, x_1, \pi-\alpha-x_1$, that is not an isosceles triangle. One can check that in this case, x_0 is the point of local minimum for function $g(x)$, whereas the two other solutions are the points of its maximum.

The two cases above are illustrated in Figure 2 (a), (b) for $\alpha=30^\circ$ (a) and $\alpha=6^\circ$ (b). In the first case $(\alpha\geq\alpha')$ the maximal ratio (approximately 0.02857) between the area of Morley triangle and the area of its original triangle is attached for the isosceles triangle with the angles 30°, 75°, 75°.

In the second case $(\alpha < \alpha')$ the maximal ratio (approximately 0.00918) is attached for $x \approx 36.75^{\circ}$, i.e., for the triangle with angles 6°, 36.75°, 137.25° and for the isosceles triangle $(x = 87^{\circ})$ the ratio has a local minimum (approximately 0.00894).



(a) (b)

Figure 2

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