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APPLICATIONS OF DIFFERENTIAL SUBORDINATION TO THE GENERALIZED DZIOK-SRIVASTAVA CONVOLUTION OPERATOR

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Abstract

In this article, we investigate some inclusion properties of the generalized Dziok-Srivastava convolution operator on some class of analytic functions. Using the principle of differential subordination, we extended the work done by Xu et al. [6] to the τ-valent type and calculated some sharp results.

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1. Introduction and Preliminaries

Let $\mathcal{A}(\tau)$ denote the class of analytic functions in a unit disc U of the form

$$f(z) = z^{\tau} + \sum_{\kappa=1}^{\infty} \tau^{\kappa} a_{\kappa} z^{\tau+\kappa}, \quad \tau \in \mathbb{N}, \quad z \neq \frac{1}{\tau}, \quad \tau|z| < 1 \qquad (1.1)$$

and $H[a, \kappa]$ denote the class of functions analytic in U of the form

$$f(z) = a + a_{\kappa} z^{\tau+\kappa} + a_{\kappa+1} z^{\tau+\kappa+1} + \cdots, \quad a \in \mathbb{R}.$$

The Hadamard product $\chi_1 * \chi_2$ of functions χ_ι is defined by

$$(\chi_1 * \chi_2)(z) = \sum_{\kappa=0}^{\infty} a_{\kappa,1} a_{\kappa,2} z^{\kappa},$$

where $\chi_\iota(z) = \sum_{\kappa=0}^{\infty} a_{\kappa,\iota} z^{\kappa}$ is analytic in U.

The function $f(z) \in \mathcal{A}(\tau)$ was defined in [1] as

$$\Theta(z) * \frac{z^{\tau}}{1 - \tau z}, \quad z \neq \frac{1}{\tau}, \quad \tau|z| < 1, \quad \tau \in \mathbb{N},$$

where $\Theta(z) = \sum_{\kappa=0}^{\infty} a_{\kappa} z^{\tau+\kappa}$, $a_0 = 1$.

A function $f(z)$ in $A(\tau)$ is said to be τ-*valently starlike of order* α $(0 \leq \alpha < \tau)$ if

$$\Re e\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha, \qquad (1.2)$$

also function $f(z)$ in $A(\tau)$ is said to be τ-*valently convex of order* α $(0 \leq \alpha < \tau)$ if

$$\Re e\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha. \qquad (1.3)$$

We denote the class of functions which is τ-valently starlike of order α

and τ-valently convex of order α by $\mathcal{S}(\tau, \alpha)$ and $\mathcal{C}(\tau, \alpha)$, respectively. Some properties of functions in classes $\mathcal{S}(\tau, \alpha)$ and $\mathcal{C}(\tau, \alpha)$ had been studied extensively, see [22, p. 188] and their references.

Remark 1.1. Since (1.1) is equivalent to

$$g(z) = z^{\tau} + \sum_{\kappa=\tau+1}^{\infty} a_{\kappa} z^{\kappa},$$

τ-valently convexity implies τ-valently starlike.

The well known generalized Pochhammer symbol denoted by $(\eta)_{\kappa}$ is defined by

$$(\eta)_{\kappa} = \begin{cases} \kappa! & (\eta = 1), \\ 1 & (\kappa = 0;\ \eta \in \mathbb{C}\backslash 0), \\ \eta(\eta+1)\cdots(\eta+\kappa-1) & (\kappa \in \mathbb{N};\ \eta \in \mathbb{C}). \end{cases} \tag{1.4}$$

Let

$$\mathcal{K}_{a,\tau}(z) = \frac{z^{\tau}}{(1-\tau z)^{a}} = z^{\tau} + \sum_{\kappa=1}^{\infty} \frac{(a)_{\kappa}}{(1)_{\kappa}} \tau^{\kappa} z^{\tau+\kappa}. \tag{1.5}$$

The Hadamard product of $\mathcal{K}_{a,\tau}$ with $f \in A(\tau)$ denoted by $\mathcal{K}_{a,\tau} * f$ is defined by $(\mathcal{K}_{a,\tau} * f)(z) = \sum_{\kappa=0}^{\infty} \frac{(a)_{\kappa}}{(1)_{\kappa}} a_{\kappa} z^{\tau+\kappa}$.

If ϱ and ρ are analytic in U, then ϱ is said to be *subordinate* to ρ, written as $\varrho \prec \rho$ or $\varrho(z) \prec \rho(z)$, when ϱ is univalent in U, $\varrho(0) = \rho(0)$ and $\varrho(U) \subset \rho(U)$.

Let $\varphi(a, c; z)$ be the incomplete beta function defined by

$$\varphi(a, c; z) = z + \sum_{\kappa=1}^{\infty} \frac{(a)_{\kappa}}{(c)_{\kappa}} z^{\kappa} \quad (z \in U;\ c \notin \mathbb{Z}_0^{-}).$$

In 1999, Dziok-Srivastava [2] (see also [3]) defined the generalized hypergeometric function as

$$\varphi(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z + \sum_{\kappa=1}^{\infty} \frac{(\alpha_1)_\kappa \cdots (\alpha_q)_\kappa}{(\beta_1)_\kappa \cdots (\beta_s)_\kappa \kappa!} z^\kappa, \tag{1.6}$$

where $q, s \in \mathbb{N}$ with $q \leq s+1$ and $\alpha_\imath \neq 0$, $\beta_\ell \notin \mathbb{Z}_0^-$ for all $\imath = 1, 2, \dots, q$ and $\ell = 1, 2, \dots, s$. Xu et al. [6] defined the τ-valent function of the type (1.6) as follows:

$$\begin{aligned}\varphi(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z^\tau) &= \mathcal{H}_\tau(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \\ &= \mathcal{H}_\tau(\alpha_\imath; \beta_\ell; z) \quad (\imath = 1, \dots, q,\ \ell = 1, \dots, s) \\ &= z^\tau + \sum_{\kappa=1}^{\infty} \frac{(\alpha_1)_\kappa \cdots (\alpha_q)_\kappa}{(\beta_1)_\kappa \cdots (\beta_s)_\kappa} \frac{z^{\tau+\kappa}}{\kappa!} \quad (z \in U). \end{aligned} \tag{1.7}$$

Xu et al. [6] used the linear Dziok-Srivastava operator $\mathcal{H}_1(\alpha_\imath; \beta_\ell; z)$ convoluted with $f(z)$ in the class $A(1)$, as well as the function $h(z) \in \mathcal{H}[1, \kappa]$ which is convex and univalent in the unit disc U, with $\Re e\{h(z)\} > 0$ to define three subclasses of normalized analytic function as follow:

$$\mathcal{P}_{\alpha_\imath; \beta_\ell}(h, \lambda)$$

$$:= \left\{ f \in A(1) : \frac{z(\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)'(z) + \lambda z^2 (\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)''(z)}{(1-\lambda)(\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)(z) + \lambda z (\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)'(z)} \prec h(z) \right\}$$

$$(z \in U), \quad (\lambda \in [0, 1];\ h \in \mathcal{H}[1, \kappa]), \tag{1.8}$$

$$\mathcal{T}_{\alpha_\imath; \beta_\ell}(h, \alpha)$$

$$:= \left\{ f \in A(1) : \frac{(\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)(z)}{z} + \alpha (\mathcal{H}_1(\alpha_\imath; \beta_\ell) f)'(z) \prec h(z) \right\}$$

$$(z \in U), \quad (\alpha \geq 0;\ h \in \mathcal{H}[1, \kappa]) \tag{1.9}$$

and

$$\mathcal{R}_{\alpha_\imath;\,\beta_\ell}(h,\,\alpha)$$

$$:= \{f \in A(1) : (\mathcal{H}_1(\alpha_\imath;\,\beta_\ell)f)'(z) + \alpha z(\mathcal{H}_1(\alpha_\imath;\,\beta_\ell)f)''(z) \prec h(z)\}$$

$$(z \in U), \quad (\alpha \geq 0;\, h \in \mathcal{H}[1,\,\kappa]). \tag{1.10}$$

Previously, many authors defined subclasses of normalized analytic functions similar to (1.8)-(1.10). They used specific values of $\alpha_\imath$ and β_ℓ, where $\imath = 1, 2, ..., s,$ $\ell = 1, 2, ..., q.$ Example of such authors is Özkan and Altıntaş [4] who used

- $(\mathcal{K}_{a,1} * f)(z) = \mathcal{L}(a,\,1)f(z) = \mathcal{H}_1(a;\,1)f(z),$

and also Trojnar-Spelina [5] who used

- $\mathcal{L}(\alpha_1,\,\beta_1)f(z) = \mathcal{H}_1(\alpha_1,\,1;\,\beta_1)f(z),$

where $\mathcal{L}(a,\,c)f$ is Carlson-Shaffer linear operator on $A(1).$ For more details of Carlson-Shaffer operator, see [7]. The authors [4-6] established some inclusion relationships of their classes of functions. Some other interesting subclasses of analytic function were studied recently using the generalized Dziok-Srivastava operator and for more details, see [8-18].

Motivated by all these, we extend the work done by Xu et al. [6] using the linear Dziok-Srivastava operator $\mathcal{H}_1(\alpha_\imath;\,\beta_\ell;\,z)$ convoluted with the function *f* of the class $A(1)$ to $\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell;\,z)$ convoluting with *f* of the class $A(\tau)$ with $h(z)$ of class $\mathcal{H}[a,\,\kappa]$ starlike in the unit disc *U*. We define the generalized form of the analytic subclass as follows: a function $f \in A(\tau)$ is said to be in the *class* $\mathcal{P}_{\alpha_\imath;\,\beta_\ell}(h,\,\lambda;\,\tau)$ if

$$\frac{z[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)f]^{(\tau)}(z) + \lambda z^2[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)f]^{(\tau+1)}(z)}{(1-\lambda)[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)f]^{(\tau-1)}(z) + \lambda z[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)f]^{(\tau)}(z)} \prec h(z),$$

$$\lambda \in [0,\,1],\;\; h \in \mathcal{H}[0,\,\kappa],\;\; (z \in U). \tag{1.11}$$

The function $f \in A(\tau)$ is said to be in the *class* $\mathcal{T}_{\alpha_\imath;\, \beta_\ell}(h, \alpha; \tau)$ if

$$(1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell)f]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell)f]^{(\tau)}(z) \prec h(z),$$

$$\alpha \geq 0, \; h \in \mathcal{H}[a, \kappa], \; (z \in U). \tag{1.12}$$

The function $f \in A(\tau)$ is said to be in the *class* $\mathcal{R}_{\alpha_\imath;\, \beta_\ell}(h, \alpha; \tau)$ if

$$[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell)f]^{(\tau)}(z) + \alpha z[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell)f]^{(\tau+1)}(z) \prec h(z),$$

$$\alpha \geq 0, \; h \in \mathcal{H}[a, \kappa], \; (z \in U). \tag{1.13}$$

We review some known preliminary results needed to establish our proof.

Theorem 1.1 [20, 19]**.** *If* Θ *convex is in the unit disc U with* $\Re e\{\gamma\} \geq 0$, $\gamma \neq 0$, $\Theta(0) = a$, $a \neq 0$ *and there exists* $\vartheta \in H[a, \kappa]$ *which satisfies*

$$\vartheta(z) + \frac{z\vartheta'(z)}{\gamma} \prec \Theta(z), \tag{1.14}$$

then

$$\vartheta(z) \prec \sigma(z) \prec \Theta(z),$$

where

$$\sigma(z) = \gamma\kappa^{-1}z^{\left(\frac{-\gamma}{\kappa}\right)}\int_0^z \Theta(\xi)\xi^{\left(\frac{\gamma}{\kappa}\right)-1}d\xi. \tag{1.15}$$

The function σ *is convex and the best* (a, κ)*-dominant.*

Theorem 1.2 [20, 21]**.** *Let* ϑ *be starlike in U with* $\vartheta(0) = 0$, *if* $\Theta \in H[a, n]$, $a \neq 0$ *satisfies*

$$z\Theta'(z) \prec \vartheta(z),$$

then

$$\Theta(z) \prec \Upsilon(z) = a + \frac{1}{\kappa}\int_0^z \vartheta(\xi)\xi^{-1}d\xi.$$

The function Υ *is convex and the best* (a, κ)*-dominant.*

The objective of this article is to investigate some inclusion properties of the generalized Dziok-Srivastava convolution operator on some class of analytic functions and using the principle of differential subordination, extended the work done by Xu et al. [6] to the τ-valent type and calculated some sharp results.

2. Inclusion Relationships

In this section, we use the classes defined in (1.11)-(1.13) to establish some inclusion properties. We used same technique as in Xu et al. [6] to prove our results.

Theorem 2.1. *Let* $\mathcal{P}_{\alpha_\imath;\,\beta_\ell}(h, \lambda;\, \tau)$ *be as defined in* (1.11). *Then*

$$f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda;\, \tau+1) \Leftrightarrow g(z) \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, 0;\, \tau), \tag{2.1}$$

where

$$g(z) = \lambda z f^{(\tau+1)}(z) + (1-\lambda) f^{(\tau)}(z) \tag{2.2}$$

and

$$f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda;\, \tau+1)$$

$$\Rightarrow \Psi = \lambda f^{(\tau)} + (1-\lambda)\int_0^z \xi^{-1} f^{(\tau)}(\xi)d\xi \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, 1;\, \tau). \tag{2.3}$$

Proof. If $f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda;\, \tau+1),$ then there exists h starlike in U with $h(0) = 0$ such that the subordination (1.11) holds. Let $g(z) = \lambda z f^{(\tau+1)}(z) + (1-\lambda) f^{(\tau)}(z).$ Then

$$\frac{z[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell) f]^{(\tau+1)}(z) + \lambda z^2[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell) f]^{(\tau+2)}(z)}{(1-\lambda)[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell) f]^{(\tau)}(z) + \lambda z[\mathcal{H}_\tau(\alpha_\imath;\, \beta_\ell) f]^{(\tau+1)}(z)}$$

$$= \frac{[\varphi(\alpha_1, \ldots, \alpha_q;\, \beta_1, \ldots, \beta_s;\, z^\tau)]^{(\tau)} * zg'(z)}{[\varphi(\alpha_1, \ldots, \alpha_q;\, \beta_1, \ldots, \beta_s;\, z^\tau)]^{(\tau)} * g(z)}$$

$$= \frac{z[[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)]^{(\tau)}g]'(z)}{([\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)]^{(\tau)}g)(z)}.$$

This shows that when $f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,\lambda;\,\tau+1)$, then $g(z) = \lambda z f^{(\tau+1)}(z) + (1-\lambda) f^{(\tau)}(z) \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,0;\,\tau)$. The converse also holds.

Assume $f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,\lambda;\,\tau+1)$. Then we have shown that $g(z) = \lambda z f^{(\tau+1)}(z) + (1-\lambda) f^{(\tau)}(z) \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,0;\,\tau)$. We also note from (2.3) that $z\Psi'(z) = g(z),\ z \in U$ that means

$$\Psi \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,1;\,\tau) \Leftrightarrow z\Psi' \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,0;\,\tau)$$

and hence $\Psi \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h,\,\lambda;\,\tau)$. □

Theorem 2.2. *Let* $\mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,\lambda;\,\tau)$ *be as defined in* (1.12) *with h convex in U*, $\alpha = \tau!$ *and* $\Re e\{h(z)\} > 0$. *Then*

$$f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,\alpha;\,\tau) \Rightarrow f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,0;\,\tau-1) \tag{2.4}$$

and

$$\mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,\alpha;\,\tau) \subset \mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,\delta;\,\tau-1). \tag{2.5}$$

Proof. Let $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h,\,\alpha;\,\tau)$ and $h(0) = \tau!$. Then subordination (1.12) holds. Let

$$q(z) = \frac{[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)]^{(\tau-1)}(z)}{z}. \tag{2.6}$$

Simple calculation on (2.6) gives

$$q(z) + \alpha z q'(z) = (1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)\,f]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath;\,\beta_\ell)]^{(\tau)}(z).$$

Assume $\gamma = \dfrac{1}{\alpha},\ \alpha > 0$. Then, by Theorem 1.1,

$$q(z) \prec h(z) \quad (z \in U),$$

which shows that $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, 0; \tau - 1)$ and this concludes the first part of Theorem 2.2. To show the proof of the second part of Theorem 2.2. If $\delta = 0$, then the proof is obvious. Assume $\delta \neq 0$ and $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau)$. Then there exists a function $h(U)$ such that

$$(1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau)}(z) \in h(U).$$

Now

$$(1-\delta)\frac{[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau-1)}(z)}{z} + \delta[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau)}(z)$$

$$= \left(1 - \frac{\delta}{\alpha}\right)\frac{[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau-1)}(z)}{z}$$

$$+ \frac{\delta}{\alpha}\left((1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau)}(z)\right).$$

Since $\dfrac{\delta}{\alpha} < 1$ and $h(U)$ is convex, we conclude that

$$(1-\delta)\frac{[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau-1)}(z)}{z} + \delta[\mathcal{H}_\tau(\alpha_\imath;\beta_\ell)f]^{(\tau)}(z) \in h(U),$$

as h is convex univalent in U, then $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \delta; \tau - 1)$. This ends the proof. □

Theorem 2.3. *Let* $\mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ *be as defined in* (1.13) *with h convex in U,* $a = \tau!$ *and* $\Re e\{h(z)\} > 0$. *Then*

$$f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau) \Rightarrow f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, 0; \tau - 1) \tag{2.7}$$

and

$$\mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau) \subset \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \delta; \tau - 1). \tag{2.8}$$

Proof. Let $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau)$, assume

$$q(z) = [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z). \tag{2.9}$$

Then

$$q(z) + z\alpha q'(z) = [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \alpha z[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)}(z)$$

and by Theorem 1.1 with $\gamma = \frac{1}{\alpha}$, $\alpha > 0$,

$$q(z) \prec h(z).$$

To show the proof of the second part of Theorem 2.2. If $\delta = 0$, then the proof is obvious. Assume $\delta \neq 0$ and $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau)$. Then there exists a function $h(U)$ such that

$$[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \alpha[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)}(z) \in h(U).$$

Now

$$\begin{aligned} &[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \delta[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)}(z) \\ &= \left(1 - \frac{\delta}{\alpha}\right)[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) \\ &\quad + \frac{\delta}{\alpha}([\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \alpha[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)}(z)). \end{aligned}$$

Since $\frac{\delta}{\alpha} < 1$ and $h(U)$ is convex, we conclude that

$$[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \delta[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)}(z) \in h(U),$$

as h is convex univalent in U, then $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \delta; \tau - 1)$. This ends the proof. □

Theorem 2.4. *Let* $\mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ *be as defined in* (1.12) *with h convex* *in* U, $a = r!$ *and* $\Re e\{h(z)\} > 0$. *Then*

$$f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau) \Leftrightarrow zf'(z) \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, 0; \tau - 1) \tag{2.10}$$

and

$$\mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau) \subset \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau). \tag{2.11}$$

Proof. Using the fact that

$$(1-\alpha)\frac{([\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau-1)} * (zf^{(\tau)}))(z)}{z}$$

$$+ \alpha([\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau)} * (f^{(\tau)} + zf^{(\tau+1)}))(z)$$

$$= [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \alpha([\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau-1)})(z),$$

the first part of the theorem is proved.

Now, letting

$$q(z) = (1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z),$$

with the assumption that $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau)$, we have

$$q(z) + zq'(z) = [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau)}(z) + \alpha z([\mathcal{H}_\tau(\alpha_\imath; \beta_\ell) f]^{(\tau+1)})(z),$$

then, by Theorem 1.1, with $\gamma = 1$ implies

$$q(z) \prec h(z) \quad (z \in U).$$

This shows that $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \alpha; \tau)$. □

3. Applications of Differential Subordinations

3.1. Sharp results

We establish some sharp results of Theorems 2.1-2.4.

Theorem 3.1. *Let* $\mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ *be as defined in* (1.12) *with h starlike* *in* $U,\ h(0) = 0,$ *if*

$$f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda;\ \tau + 1), \tag{3.1}$$

then

$$q(z) = ([\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g)(z) \prec v(z) = r! \exp\left[\int_0^z h(\xi)\xi^{-1} d\xi\right],$$

where $g(z)$ *is* (2.2). *The function v is the best* $(\tau!,\ \kappa)$*-dominant*.

Proof. Let $f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}(h, \lambda;\ \tau + 1).$ Then $q(0) = ([\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g)(0)$ $= \tau! \neq 0$ and

$$\frac{[[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g]'(z)}{([\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g)(z)} \prec h(z),$$

so $q(z) \neq 0$ in U. Let $\psi(z) = \log q(z).$ Then $\psi(z) \in \mathcal{H}[\log q(z),\ \kappa]$ and

$$z\psi'(z) = z\frac{q'(z)}{q(z)} = \frac{[[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g]'(z)}{([\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g)(z)}.$$

Then, by Theorem 1.1, Theorem 3.1 is proved. □

Theorem 3.2. *Let* $\mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \lambda;\ \tau)$ *be as defined in* (1.12) *with h convex in* $U,\ a = r!$ *and* $\Re e\{h(z)\} > 0.$ *Then*

$$f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \lambda;\ \tau) \Rightarrow q(z) = \frac{[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau-1)}(z)}{z}$$

$$\prec v(z) = z\left(\frac{-\gamma}{\kappa}\right)\gamma\kappa\int_0^z h(\xi)\xi^{\left(\frac{\gamma}{\kappa}\right)-1} d\xi. \tag{3.2}$$

The function v is convex and the best $(\tau!,\ \kappa)$*-dominant*.

Proof. Assume $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}(h, \lambda;\ \tau)$ and $q(z) = \dfrac{[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau-1)}(z)}{z}.$ Then the subordination

$$q(z) + \alpha z q'(z) \prec h(z) \tag{3.3}$$

holds. Applying Theorem 1.1 to (3.3) gives the proof of Theorem 3.2. □

Theorem 3.3. *Let* $\mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ *be as defined in* (1.13) *with h convex in U,* $a = \tau!$ *and* $\Re e\{h(z)\} > 0.$ *Then*

(i)

$$f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau) \Rightarrow q(z) = [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau)} \prec v(z)$$

$$= z\left(\frac{-\gamma}{\kappa}\right)\gamma\kappa \int_0^z h(\xi)\xi^{\left(\frac{\gamma}{\kappa}\right)-1} d\xi, \tag{3.4}$$

(ii)

$$f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau) \Rightarrow q(z) \prec v(z) = z\left(\frac{-\gamma}{\kappa}\right)\gamma\kappa \int_0^z h(\xi)\xi^{\left(\frac{\gamma}{\kappa}\right)-1} d\xi,$$

where

$$q(z) = (1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau-1)}(z)}{z} + \alpha z[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau)}(z).$$

The function v is convex and the best $(\tau!, \kappa)$*-dominant.*

Proof. Let $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ and $q(z) = [\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau)}(z).$ Then

$$q(z) + z\alpha q'(z) \prec h(z) \tag{3.5}$$

and application of Theorem 1.1 to (3.5) yields Theorem 3.3(i).

Let $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}(h, \lambda; \tau)$ and

$$q(z) = (1-\alpha)\frac{[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau-1)}(z)}{z} + \alpha[\mathcal{H}_\tau(\alpha_\imath; \beta_\ell)]^{(\tau)}(z).$$

Then the subordination $q(z) + zq'(z) \prec h(z)$ holds, application of Theorem 1.1 with $\gamma = 1$ concludes the proof of the theorem. □

3.2. Conclusion

We calculated some choices of

$$\mathcal{P}_{\alpha_\imath;\beta_\ell}(h(z), \lambda;\ \tau+1), \quad \mathcal{T}_{\alpha_\imath;\beta_\ell}(h(z), \lambda;\ \tau) \quad \text{and} \quad \mathcal{R}_{\alpha_\imath;\beta_\ell}(h(z), \lambda;\ \tau+1)$$

with their corresponding $v(z)$ for Theorems 3.1-3.3 as follows:

- Let $f \in \mathcal{P}_{\alpha_\imath;\beta_\ell}\left(\frac{z}{1-\tau z}, 0;\ \tau+1\right), \ h \in \mathcal{S}(\tau, \alpha)$. *Then*

$$([\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau)} g)(z) \prec v(z) = \tau!\left(\frac{1}{1-\tau z}\right)^\tau, \quad v \in \mathcal{S}(\tau, \alpha).$$

- Let $f \in \mathcal{T}_{\alpha_\imath;\beta_\ell}\left(\frac{\tau! + (1-2\alpha)(2-\tau z)z}{(1-\tau z)^2}, 1;\ \tau\right), \quad h \in \mathcal{C}(\tau, \alpha), \quad \gamma = 1.$

Then

$$\frac{[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau-1)}(z)}{z} \prec v(z) = \left(\frac{\tau! + (1-2\alpha)z}{(1-\tau z)^2}\right), \quad v \in \mathcal{C}(\tau, \alpha).$$

- Let $f \in \mathcal{R}_{\alpha_\imath;\beta_\ell}\left(\frac{\tau! + (2-\tau z)z}{(1-\tau z)^2}, 1;\ \tau\right), \ h \in \mathcal{C}(\tau, \alpha), \ \gamma = 1$. Then

$$[\mathcal{H}_\tau(\alpha_\imath;\ \beta_\ell)]^{(\tau-1)}(z) \prec v(z) = \left(\frac{\tau! + z}{(1-\tau z)^2}\right), \quad v \in \mathcal{C}(\tau, \alpha).$$

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