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SOME COMMON FIXED POINT THEOREMS FOR NON-SELF MAPPINGS (ψ, φ)-WEAK CONTRACTIVE CONDITIONS OF INTEGRAL TYPE IN SYMMETRIC SPACES

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Abstract

We prove some common fixed point theorems for non-self mappings (ψ, φ)-weak contractive conditions of integral type in symmetric spaces by changing the contractive conditions which generalize the results of Kutbi et al. [14].

1. Introduction

The celebrated Banach contraction principle is indeed the most

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fundamental result of metrical fixed point theory, which states that a contraction mapping of a complete metric space into itself has a unique fixed point. This theorem is very efficiently utilized to establish the existence of solutions of non-linear Volterra integral equations, Fredholm integral equations, and nonlinear integrodifferential equations in Banach spaces besides supporting the convergence of algorithms in computational mathematics.

The concept of weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997 wherein authors introduced the following notion for mappings defined on a Hilbert space *H*.

Consider the following set of real functions:

$$\Phi = \{\varphi : [0, \infty) \to [0, \infty) : \varphi \text{ is lower semi-continuous and } \varphi^{-1}(\{0\}) = \{0\}\}. \tag{1}$$

A mapping $T : X \to X$ is called a φ-*weak contraction* if there exists a function $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in X. \tag{2}$$

However, the main fixed point theorem for a self-mapping satisfying (ψ, φ)-weak contractive conditions contained in Dutta and Choudhury [2] is given below, but, before that, we consider the following set of real functions:

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi \text{ is continuous nondecreasing } \psi^{-1}(\{0\}) = \{0\}\}. \tag{3}$$

Theorem 1.1 (See [14])**.** *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a self-mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{4}$$

for some $\psi \in \Psi$ *and* $\varphi \in \Phi$ *and all* $x, y \in X$. *Then T has a unique fixed point in X.*

The object of this paper is to prove some integral type common fixed point theorems for two pairs of nonself weakly compatible mappings satisfying generalized (ψ, φ)-contractive conditions by using the common limit range property in symmetric spaces.

2. Preliminaries

A common fixed point result generally involves conditions on commutativity, continuity, and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. Hence, one is always required to improve one or more of these conditions in order to prove a new common fixed point theorem. It can be observed that in the case of two mappings $A, S : X \to X$, where (X, d) is metric space (or symmetric space), one can consider the following classes of mappings for the existence and uniqueness of common fixed points:

$$d(Ax, Ay) \leq F(m(x, y)), \tag{5}$$

where F is some function and $m(x, y)$ is the maximum of one of the sets. Thus,

$$M^{5}_{A,S}(x, y)$$

$$= \{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\},$$

$$M^{4}_{A,S}(x, y)$$

$$= \left\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\right\},$$

$$M^{3}_{A,S}(x, y)$$

$$= \left\{d(Sx, Sy), \frac{1}{2}(d(Sx, Ax) + d(Sy, Ay)), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\right\}, \tag{6}$$

A further possible generalization is to consider four mappings instead of two and ascertain analogous common fixed point theorems. In the case of four mappings $A, B, S, T : X \to X$, where (X, d) is metric space (or symmetric space), the corresponding sets take the form

$$M^5_{A, B, S, T}(x, y)$$
$$= [\{d^2(Sx, Ty), d^2(Sx, Ax), d(By, Ty), d(Sx, Ty),$$
$$d(Sx, By), d(Sx, Ty), d(By, Ty), d^2(By, Ty)\}]^{\frac{1}{2}},$$

$$M^4_{A, B, S, T}(x, y)$$
$$= \left[\left\{d^2(Sx, Ty), d^2(Sx, Ax), d(By, Ty), d(Sx, Ty), d(Sx, By),\right.\right.$$
$$\left.\left.\frac{1}{2}(d(Sx, Ty) + d(By, Ty)), d^2(By, Ty)\right\}\right]^{\frac{1}{2}},$$

$$M^3_{A, B, S, T}(x, y)$$
$$= \left[\left\{d^2(Sx, Ty), d^2(Sx, Ax), d(By, Ty), \frac{1}{2}(d(Sx, Ty) + d(Sx, By)),\right.\right.$$
$$\left.\left.\frac{1}{2}(d(Sx, Ty) + d(By, Ty)), d^2(By, Ty)\right\}\right]^{\frac{1}{2}}. \tag{7}$$

In this case, (5) is usually replaced by

$$d(Ax, By) \leq F(m(x, y)), \tag{8}$$

where $m(x, y)$ is the maximum of one of the M-sets.

A symmetric on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let

$B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if, and only if, for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

The difference of a symmetric and a metric comes from the triangle inequality. Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point theorems some additional axioms are required. The following axioms, which are available in Wilson [3], Aliouche [4], and Imdad et al. [5], are relevant to this presentation.

From now on symmetric space will be denoted by (X, d) whereas a nonempty arbitrary set will be denoted by Y.

(W_3) Given $\{x_n\}$, and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$ [3].

(W_4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $d(y_n, x) = 0$ [3].

(HE) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $d(x_n, y_n) = 0$ [4].

$(1\mathrm{C})$ A symmetric d is said to be 1-*continuous* if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, where $\{x_n\}$ is a sequence in X and $x, y \in X$ [6].

(CC) A symmetric d is said to be *continuous* if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$, where $\{x_n\}$ and $\{y_n\}$ are sequences in X and $x, y \in X$ [6].

Here, it is observed that $(\mathrm{CC}) \Rightarrow (1\mathrm{C})$, $(\mathrm{W}_4) \Rightarrow (\mathrm{W}_3)$, $(1\mathrm{C}) \Rightarrow (\mathrm{W}_3)$

but the converse implications are not true. In general, all other possible implications amongst (W_3), $(1\mathrm{C})$, and (HE) are not true. However, (CC) implies all the remaining four conditions, namely, (W_3), (W_4), (HE), and $(1\mathrm{C})$.

Definition 2.1 (See [14])**.** Let (A, S) be a pair of self-mappings defined on a nonempty set *X* equipped with a symmetric *d*. Then the mappings *A* and *S* are said to be

(1) *commuting* if $ASx = SAx$ for all $x \in X$,

(2) *compatible* [7] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in *Y* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$,

(3) *noncompatible* [8] if there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_x = \lim_{n\to\infty} Sx_n$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent,

(4) *weakly compatible* [9] if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$, for some $x \in X$,

(5) *satisfying* the property $(E.A)$ [10] if there exists a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Any pair of compatible as well as noncompatible self-mappings satisfies the property $(E.A)$ but a pair of mappings satisfying the property $(E.A)$ needs not be noncompatible.

Definition 2.2 (See [11])**.** Let *Y* be an arbitrary set and let *X* be a nonempty set equipped with symmetric *d*. Then the pairs (A, S) and (B, T) of mappings from *Y* into *X* are said to *share* the common property $(E.A)$, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \tag{9}$$

for some $z \in X$.

Definition 2.3 (See [12]). Let Y be an arbitrary set and let X be a nonempty set equipped with symmetric d. Then the pairs (A, S) of mappings from Y into X is said to have the *common limit range property* with respect to the mappings S (denoted by $(CLR_S))$ if there exist two sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \tag{10}$$

for some $z \in S(Y)$.

Definition 2.4 (See [14]). Let Y be an arbitrary set and let X be a nonempty set equipped with symmetric d. Then the pairs (A, S) and (B, T) of mappings from Y into X is said to have the *common limit range property* with respect to the mappings S and T, (denoted by $(CLR_{ST}))$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \tag{11}$$

for some $z \in S(Y) \cap T(Y)$.

Remark 2.1 (See [14]). It is clear that (CLR_{ST}) property implies the common property $(E.A)$ but the converse is not true.

Definition 2.5 (See [14]). Two families of self-mappings $\{A_i\}_{i=1}^{m}$ and $\{S_k\}_{k=1}^{n}$ are said to be *pairwise commuting* if

(1) $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, ..., m\}$,

(2) $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, ..., n\}$,

(3) $A_iS_k = S_kA_i$ for all $i \in \{1, 2, ..., m\}$ and $k \in \{1, 2, ..., n\}$.

3. Results

Now we state and prove our main results for four mappings employing the common limit range property in symmetric spaces. Firstly, we prove the following lemma.

Lemma 3.1 (See [14]). *Let* (X, d) *be a symmetric space wherein* d *satisfies the conditions* (CC) *whereas* Y *is an arbitrary nonempty set with* A, B, S *and* $T : Y \to X$. *Suppose that*

(1) *the pair* (A, S) $($*or* $(B, T))$ *satisfies the* (CLR_S) $($*or* $(CLR_T))$ *property*,

(2) $A(Y) \subset T(Y)$,

(3) $T(Y)$ $($*or* $S(Y))$ *is a closed subset of* X,

(4) $\{By_n\}$ *converges for every sequence* $\{y_n\}$ *in* Y *whenever* $\{Ty_n\}$ *converges* (*or* $\{Ax_n\}$ *converges for every sequence* $\{x_n\}$ *in* Y *whenever* $\{Sx_n\}$ *converges*),

(5) *there exists* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that for all* $x, y \in Y$, *we have*

$$\psi\left(\int_0^{d(Ax, By)} \phi(t)\,dt\right) \leq \psi\left(\int_0^{m(x, y)} \phi(t)\,dt\right) - \varphi\left(\int_0^{m(x, y)} \phi(t)\,dt\right) \qquad \text{(a)}$$

where

$$m(x, y) = \max[\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\}]$$

and $\phi : [0, \infty) \to [0, \infty]$ *is a Lebesgue-integrable mapping which is summable and nonnegative such that*

$$\int_0^{\varepsilon} \phi(t)\,dt > 0, \qquad \text{(b)}$$

for all $\varepsilon > 0$.

Then the pairs (A, S) *and* (B, T) *satisfy the* (CLR_{ST}) *property*.

Lemma 3.2. *Let* (X, d) *be a symmetric space wherein* d *satisfies the conditions* (CC) *whereas* Y *is an arbitrary nonempty set with* A, B, S *and*

$T : Y \to X$. *Suppose that*

(1) *the pair* (A, S) $($*or* $(B, T))$ *satisfies the* (CLR_S) $($*or* $(CLR_T))$ *property*,

(2) $A(Y) \subset T(Y)$,

(3) $T(Y)$ $($*or* $S(Y))$ *is a closed subset of* X,

(4) $\{By_n\}$ *converges for every sequence* $\{y_n\}$ *in* Y *whenever* $\{Ty_n\}$ *converges* $($*or* $\{Ax_n\}$ *converges for every sequence* $\{x_n\}$ *in* Y *whenever* $\{Sx_n\}$ *converges*$)$,

(5) *there exists* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that for all* $x, y \in Y$, *we have*

$$\psi\left(\int_0^{d(Ax, By)} \phi(t)dt\right) \leq \psi\left(\int_0^{m(x, y)} \phi(t)dt\right) - \varphi\left(\int_0^{m(x, y)} \phi(t)dt\right), \quad (12)$$

where

$$m(x, y) = \max M^5_{A, B, S, T}(x, y) \quad (13)$$

and $\phi : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping which is summable and nonnegative such that*

$$\int_0^{\varepsilon} \phi(t)dt > 0, \quad (14)$$

for all $\varepsilon > 0$.

Then the pairs (A, S) *and* (B, T) *satisfy the* (CLR_{ST}) *property*.

Proof. First, we show that the conclusion of this theorem holds for first case. Since the pair (A, S) enjoy the (CLR_S) property; therefore there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \quad (15)$$

for some $z \in S(Y)$. Since $A(Y) \subset T(Y)$, hence for each sequence $\{x_n\}$ there exists a sequence $\{y_n\}$ in Y such that $Ax_n = Ty_n$.

Therefore, by closedness of $T(Y)$,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z \tag{16}$$

for some $z \in T(Y)$ and in all $z \in S(Y) \cap T(Y)$. Thus, in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. Since by (4), $\{By_n\}$ converges, in all we need to show that $\{By_n\} \to z$ as $n \to \infty$. Assume this contrary, we get $\{By_n\} \to t(\neq z)$ as $n \to \infty$. Now, using inequality (12) with $x = x_n$, $y = y_n$, we have

$$\psi\left(\int_0^{d(Ax_n, By_n)} \phi(t)dt\right) \leq \psi\left(\int_0^{m(x_n, y_n)} \phi(t)dt\right) - \varphi\left(\int_0^{m(x_n, y_n)} \phi(t)dt\right), \tag{17}$$

where

$$m(x_n, y_n) = \max[\{d^2(Sx_n, Ty_n), d^2(Sx_n, Ax_n), d(By_n, Ty_n), d(Sx_n, Ty_n),$$

$$d(Sx_n, By_n), d(Sx_n, Ty_n), d(By_n, Ty_n), d^2(By_n, Ty_n)\}]^{\frac{1}{2}}. \tag{18}$$

Taking limit as $n \to \infty$ and using property (CC) in inequality (17), we get

$$\lim_{n\to\infty} \psi\left(\int_0^{d(Ax_n, By_n)} \phi(t)dt\right)$$

$$\leq \lim_{n\to\infty} \psi\left(\int_0^{m(x_n, y_n)} \phi(t)dt\right) - \lim_{n\to\infty} \varphi\left(\int_0^{m(x_n, y_n)} \phi(t)dt\right), \tag{19}$$

that is,

$$\psi\left(\int_0^{d(z,t)} \phi(t)dt\right) = \psi\left(\lim_{n\to\infty}\int_0^{d(Ax_n, By_n)} \phi(t)dt\right)$$

$$\leq \psi\left(\lim_{n\to\infty}\int_0^{m(x_n, y_n)} \phi(t)dt\right) - \varphi\left(\lim_{n\to\infty}\int_0^{m(x_n, y_n)} \phi(t)dt\right),$$

(20)

where

$$\lim_{n\to\infty} m(x_n,\ y_n)$$

$$= [\max\{d^2(z,\ z),\ d^2(z,\ z),\ d(t,\ z),\ d(z,\ z),\ d(z,\ t),\ d(z,\ z),$$

$$d(t,\ z),\ d^2(z,\ t)\}]^{\frac{1}{2}}$$

$$= [\max\{0,\ 0,\ d(z,\ t),\ 0,\ d(z,\ t),\ 0,\ d(z,\ t),\ d^2(z,\ t)\}]^{\frac{1}{2}}$$

$$= d(z,\ t). \tag{21}$$

Hence inequality (20) implies

$$\psi\left(\int_0^{d(z,t)} \phi(t)dt\right) \leq \psi\left(\int_0^{d(z,t)} \phi(t)dt\right) - \varphi\left(\int_0^{d(z,t)} \phi(t)dt\right); \tag{22}$$

that is, $\varphi\left(\int_0^{d(z,t)} \phi(t)dt\right) \leq 0$ and so $\varphi\left(\int_0^{d(z,t)} \phi(t)dt\right) = 0$ and, by the property of the function φ, we have $d(z,\ t) = 0$ or equivalently $z = t,$ which contradicts the hypothesis $t \neq z.$ Hence both the pairs $(A,\ S)$ and $(B,\ T)$ satisfy the (CLR_{ST}) property.

In the second case, it is similar to the first case. So, in order to avoid repetition, the details of the proof are omitted. This completes the proof.

Theorem 3.1 (See [14]). *Let $(X,\ d)$ be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y is an arbitrary nonempty set with $A,\ B,\ S,\ T : Y \to X$ which satisfy the inequalities* (*a*) *and* (*b*) *of Lemma* 3.1. *Suppose that the pairs $(A,\ S)$* and *$(B,\ T)$ satisfy the (CLR_{ST}) property*. *Then $(A,\ S)$ and $(B,\ T)$ have a coincidence point each*. *Moreover, if $Y = X,$ then A, B, S and T have a unique common fixed point provided both the pairs $(A,\ S)$ and $(B,\ T)$ are weakly compatible*.

Theorem 3.2. *Let $(X,\ d)$ be a symmetric space wherein d satisfies the* *conditions* $(1C)$ *and* (HE) *whereas* Y *is an arbitrary nonempty set with* $A, B, S, T : Y \to X$, *which satisfy the inequalities* (12) *and* (14) *of Lemma* 3.2. *Suppose that the pairs* (A, S) *and* (B, T) *satisfy the* (CLR_{ST}) *property. Then* (A, S) *and* (B, T) *have a coincidence point each. Moreover, if* $Y = X$, *then* A, B, S *and* T *have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. Since the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} d(Ax_n, z) = \lim_{n\to\infty} d(Sx_n, z) = \lim_{n\to\infty} d(By_n, z) = \lim_{n\to\infty} d(Ty_n, z) = 0, \tag{23}$$

for some $z \in S(Y) \cap T(Y)$. It follows from $z \in S(Y)$ that there exists a point $w \in Y$ such that $Sw = z$. We assert that $Aw = z$. If not, then, using inequality (12) with $x = w$ and $y = y_n$, one obtains

$$\psi\left(\int_0^{d(Aw, By_n)} \phi(t)dt\right) \le \psi\left(\int_0^{m(w, y_n)} \phi(t)dt\right) - \varphi\left(\int_0^{m(w, y_n)} \phi(t)dt\right), \tag{24}$$

where

$$m(w, y_n) = \max[\{d^2(Sw, Ty_n), d^2(Sw, Aw), d(By_n, Ty_n), d(Sw, Ty_n),$$

$$d(Sw, By_n), d(Sw, Ty_n), d(By_n, Ty_n), d^2(By_n, Ty_n)\}]^{\frac{1}{2}}. \tag{25}$$

Taking limit as $n \to \infty$ and using properties $(1C)$ and (HE) in inequality (24), one obtains

$$\lim_{n\to\infty} \psi\left(\int_0^{d(Aw, By_n)} \phi(t)dt\right)$$

$$\le \lim_{n\to\infty} \psi\left(\int_0^{m(w, y_n)} \phi(t)dt\right) - \lim_{n\to\infty} \varphi\left(\int_0^{m(w, y_n)} \phi(t)dt\right), \tag{26}$$

that is,

$$\psi\left(\int_0^{d(Aw,\,z)}\phi(t)\,dt\right) = \psi\left(\lim_{n\to\infty}\int_0^{d(Aw,\,By_n)}\phi(t)\,dt\right)$$

$$\leq \psi\left(\lim_{n\to\infty}\int_0^{m(w,\,y_n)}\phi(t)\,dt\right) - \varphi\left(\lim_{n\to\infty}\int_0^{m(w,\,y_n)}\phi(t)\,dt\right), \tag{27}$$

where

$$\lim_{n\to\infty} m(w,\ y_n) = [\max\{d^2(z,\ z),\ d^2(z,\ Aw),\ d(z,\ z),\ d(z,\ z),\ d(z,\ z),$$

$$d(z,\ z),\ d(z,\ z),\ d^2(z,\ z)\}]^{\frac{1}{2}}$$

$$= [\max\{0,\ d^2(z,\ Aw),\ 0,\ 0,\ 0,\ 0,\ 0,\ d^2(z,\ z)\}]^{\frac{1}{2}}$$

$$= d(z,\ Aw). \tag{28}$$

From inequality (27), one gets

$$\psi\left(\int_0^{d(Aw,\,z)}\phi(t)\,dt\right) \leq \psi\left(\int_0^{d(Aw,\,z)}\phi(t)\,dt\right) - \varphi\left(\int_0^{d(Aw,\,z)}\phi(t)\,dt\right) \tag{29}$$

so that $\varphi\left(\int_0^{d(Aw,\,z)}\phi(t)\,dt\right) = 0;$ that is, $d(Aw,\ z) = 0.$ Hence $Aw = Sw = z$ which shows that w is a coincidence point of the pair $(A,\ S)$.

Also $z \in T(Y)$, there exists a point $v \in Y$ such that $Tv = z$. We assert that $Bv = z$. If not, then, using inequality (12) with $x = w,\ y = v,$ we have

$$\psi\left(\int_0^{d(z,\,Bv)}\phi(t)\,dt\right) = \psi\left(\int_0^{d(Aw,\,Bv)}\phi(t)\,dt\right)$$

$$\leq \psi\left(\int_0^{m(w,\,v)}\phi(t)\,dt\right) - \varphi\left(\int_0^{m(w,\,v)}\phi(t)\,dt\right), \tag{30}$$

where

$$m(w,\ v) = [\max\{d^2(Sw,\ Tv),\ d^2(Sw,\ Aw),\ d(Bv,\ Tv),\ d(Sw,\ Tv),$$

$$d(Sw,\ Bv),\ d(Sw,\ Tv),\ d(Bv,\ Tv),\ d^2(Bv,\ Tv)\}]^{\frac{1}{2}}$$

$$= [\max\{d^2(z,\ z),\ d^2(z,\ z),\ d(Bv,\ z),\ d(z,\ z),\ d(z,\ Bv),\ d(z,\ z),$$

$$d(Bv,\ z),\ d^2(Bv,\ z)\}]^{\frac{1}{2}}$$

$$= [\max\{0,\ 0,\ d(Bv,\ z),\ 0,\ d(z,\ Bv),\ 0,\ d(Bv,\ z),\ d^2(Bv,\ z)\}]^{\frac{1}{2}}$$

$$= d(z,\ Bv). \tag{31}$$

Hence inequality (30) implies

$$\psi\left(\int_0^{d(z,Bv)} \phi(t)dt\right) \le \psi\left(\int_0^{d(z,Bv)} \phi(t)dt\right) - \varphi\left(\int_0^{d(z,Bv)} \phi(t)dt\right) \tag{32}$$

so that $\varphi\left(\int_0^{d(z,Bv)} \phi(t)dt\right) = 0$; that is, $d(z,\ Bv) = 0.$ Therefore $z = Bv = Tv$ which shows that v is a coincidence point of the pair $(B,\ T)$.

Consider $Y = X$. Since the pair $(A,\ S)$ is weakly compatible and $Aw = Sw$, hence $Az = ASw = SAw = SZ$. Now we assert that z is a common fixed point of the pair $(A,\ S)$. To accomplish this, using inequality (12) with $x = z$ and $y = v$, one gets

$$\psi\left(\int_0^{d(Az,z)} \phi(t)dt\right) = \psi\left(\lim_{n\to\infty} \int_0^{d(Az,Bv)} \phi(t)dt\right)$$

$$\le \psi\left(\lim_{n\to\infty} \int_0^{m(z,v)} \phi(t)dt\right) - \varphi\left(\lim_{n\to\infty} \int_0^{m(z,v)} \phi(t)dt\right), \tag{33}$$

where

$$m(z, v) = [\max\{d^2(Sz, Tv), d^2(Sz, Az), d(Bv, Tv), d(Sz, Tv),$$

$$d(Sz, Bv), d(Sz, Tv), d(Bv, Tv), d^2(Bv, Tv)\}]^{\frac{1}{2}}$$

$$= [\max\{d^2(Az, z), d^2(Az, Az), d(z, z), d(Az, z), d(Az, z),$$

$$d(Sz, z), d(z, z), d^2(z, z)\}]^{\frac{1}{2}}$$

$$= d(Az, z). \tag{34}$$

From inequality (33), we have

$$\psi\left(\int_0^{d(Az, z)} \phi(t)dt\right) \leq \psi\left(\int_0^{d(Az, z)} \phi(t)dt\right) - \varphi\left(\int_0^{d(Az, z)} \phi(t)dt\right) \tag{35}$$

so that $\varphi\left(\int_0^{d(Az, z)} \phi(t)dt\right) = 0;$ that is, $d(Az, z) = 0.$ Hence $Az = Sz = z$ which shows that z is a common fixed point of the pair (A, S).

Also the pair (B, T) is weakly compatible and $Bv = Tv$, hence $Bz = BTv = TBv = Tz.$ Using inequality (12) with $x = w$ and $y = z,$ we have

$$\psi\left(\int_0^{d(z, Bz)} \phi(t)dt\right) = \psi\left(\int_0^{d(Aw, Bz)} \phi(t)dt\right)$$

$$\leq \psi\left(\lim_{n\to\infty} \int_0^{m(w, z)} \phi(t)dt\right) - \varphi\left(\int_0^{m(w, z)} \phi(t)dt\right),$$

where

$$m(w, z) = [\max\{d^2(Sw, Tz), d^2(Sw, Aw), d(Bz, Tz), d(Sw, Tz),$$

$$d(Sw, Bz), d(Sw, Tz), d(Bz, Tz), d^2(Bz, Tz)\}]^{\frac{1}{2}}$$

$$= [\max\{d^2(z, Bz), d^2(z, z), d(Bz, Bz), d(z, Bz), d(z, Bz),$$

$$d(z, Tz),\ d(Bz, Bz),\ d^2(Bz, Bz)\}]^{\frac{1}{2}}$$

$$= d(z, Bz). \tag{36}$$

From inequality (36), we have

$$\psi\left(\int_0^{d(z, Bv)} \phi(t)dt\right) \leq \psi\left(\int_0^{d(z, Bv)} \phi(t)dt\right) - \varphi\left(\int_0^{d(z, Bv)} \phi(t)dt\right) \tag{37}$$

so that $\varphi\left(\int_0^{d(z, Bz)} \phi(t)dt\right) = 0$; that is, $d(z, Bz) = 0$. Hence $Bz = Tz = z$ which shows that z is a common fixed point of the pair (B, T).

Hence z is a common fixed point of the pairs (A, S) and (B, T). Moreover, if $Y = X$, then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$. This concludes the proof.

Example 3.1. Consider $X = Y = [4, 13)$ equipped with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in X$ which also satisfies $(1C)$ and (HE). Define the mappings A, B, S and T by

$$Ax = \begin{cases} 4, & \text{if } x \in \{4\} \cup (7, 13), \\ 7, & \text{if } x \in (4, 7]; \end{cases}$$

$$Bx = \begin{cases} 4, & \text{if } x \in \{4\} \cup (7, 13), \\ 6, & \text{if } x \in (4, 7]; \end{cases}$$

$$Sx = \begin{cases} 4, & \text{if } x \in 4, \\ 8, & \text{if } x \in (4, 7], \\ \dfrac{5x+1}{10}, & \text{if } x \in (7, 13); \end{cases}$$

$$Tx = \begin{cases} 4, & \text{if } x = 4, \\ 10, & \text{if } x \in (4, 7], \\ x - 5, & \text{if } x \in (7, 13); \end{cases}$$

Then $A(X) = \{4, 7\}$, $B(X) = \{4, 6\}$, $T(X) = [4, 10]$ and $S(X) = \left[4, \frac{33}{5}\right)$ $\cup 8$. Now, consider the sequences $x_n = \left\{7 + \frac{1}{n}\right\}$ and $y_n = \{4\}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 4 \in S(X) \cap T(X) \tag{38}$$

that is, both the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Let Lebesgue-integrable $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(t) = e^t$. Take $\psi \in \Psi$ and $\varphi \in \Phi$ given by $\psi(t) = 2t$ and $\varphi(t) = (2/7)t$. In order to check the contractive condition (12), consider the following nine cases:

(i) $x = y = 4$,

(ii) $x = 4,\ y \in (4, 7]$,

(iii) $x = 4,\ y \in (7, 13)$,

(iv) $x \in (4, 7],\ y = 4$,

(v) $x,\ y \in (4, 7]$,

(vi) $x \in (4, 7],\ y \in (7, 13)$,

(vii) $x \in (7, 13),\ y = 4$,

(viii) $x \in (7, 13),\ y \in (4, 7]$,

(ix) $x,\ y \in (7, 13)$.

In the cases (i), (iii), (vii) and (ix), we get that $d(Ax, By) = 0$ and (12) is trivially satisfied.

In the cases (ii) and (viii), we have $d(Ax, By) = 4$ and $m(x, y) = 36$. Now we have

$$\psi\left(\int_0^{d(Ax, By)} \phi(t)dt\right) = \psi\left(\int_0^4 e^t dt\right)$$

$$= \psi(e^4 - 1)$$

$$= 2(e^4 - 1)$$

$$\leq \frac{12}{7}(e^{36} - 1)$$

$$= 2(e^{36} - 1) - \frac{2}{7}(e^{36} - 1)$$

$$= \psi\left(\int_0^{36} e^t dt\right) - \varphi\left(\int_0^{36} e^t dt\right)$$

$$= \psi\left(\int_0^{m(x, y)} \phi(t) dt\right) - \varphi\left(\int_0^{m(x, y)} \phi(t) dt\right).$$

Therefore, we get the fact that (12) holds.

In the case (iv), we get $d(Ax, By) = 9$ and $m(x, y) = 16$. Now we have

$$\psi\left(\int_0^{d(Ax, By)} \phi(t) dt\right) = \psi\left(\int_0^{9} e^t dt\right)$$

$$= \psi(e^9 - 1)$$

$$= 2(e^9 - 1)$$

$$\leq \frac{12}{7}(e^{16} - 1)$$

$$= 2(e^{16} - 1) - \frac{2}{7}(e^{16} - 1)$$

$$= \psi\left(\int_0^{16} e^t dt\right) - \varphi\left(\int_0^{16} e^t dt\right)$$

$$= \psi\left(\int_0^{m(x, y)} \phi(t) dt\right) - \varphi\left(\int_0^{m(x, y)} \phi(t) dt\right).$$

This implies that (12) holds.

In the case (vi), we get $d(Ax, By) = 9$ and $m(x, y) = 36$. Now we have

$$\begin{aligned}
\psi\left(\int_0^{d(Ax, By)} \phi(t)dt\right) &= \psi\left(\int_0^{9} e^t dt\right) \\
&= \psi(e^9 - 1) \\
&= 2(e^9 - 1) \\
&\leq \frac{12}{7}(e^{36} - 1) \\
&= 2(e^{36} - 1) - \frac{2}{7}(e^{36} - 1) \\
&= \psi\left(\int_0^{36} e^t dt\right) - \varphi\left(\int_0^{36} e^t dt\right) \\
&= \psi\left(\int_0^{m(x, y)} \phi(t)dt\right) - \varphi\left(\int_0^{m(x, y)} \phi(t)dt\right).
\end{aligned}$$

This implies that (12) holds.

Finally, in the case (v), we obtain $d(Ax, By) = 1$ and $m(x, y) = 16$. Now we have

$$\begin{aligned}
\psi\left(\int_0^{d(Ax, By)} \phi(t)dt\right) &= \psi\left(\int_0^{1} e^t dt\right) \\
&= \psi(e^1 - 1) \\
&= 2(e - 1) \\
&\leq \frac{12}{7}(e^{16} - 1) \\
&= 2(e^{16} - 1) - \frac{2}{7}(e^{16} - 1) \\
&= \psi\left(\int_0^{16} e^t dt\right) - \varphi\left(\int_0^{16} e^t dt\right)
\end{aligned}$$

$$= \psi\left(\int_0^{m(x,y)} \phi(t)\,dt\right) - \varphi\left(\int_0^{m(x,y)} \phi(t)\,dt\right)$$

that is, the contractive condition (12) holds.

Hence, all the conditions of Theorem 3.2 are satisfied and 4 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence as well.

Corollary 3.1. *Let* (X, d) *be a symmetric space wherein d satisfies the conditions* (CC) *whereas Y is an arbitrary nonempty set with* $A, B, S, T : Y \to X$ *which satisfying all the hypotheses of Lemma* 3.2. *Then* (A, S) *and* (B, T) *have a coincidence point each*. *Moreover*, *if* $Y = X$, *then* A, B, S *and T have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible*.

Proof. Owing to Lemma 3.2, it follows that the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property. Hence, all the conditions of Theorem 3.2 are satisfied and A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Remark 3.1. The conclusions of Lemma 3.2, Theorem 3.2 and Corollary 3.1 remain true if we choose

$$m(x, y) = \max M^4_{A, B, S, T}(x, y) = \max M^3_{A, B, S, T}(x, y).$$

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