Advances and Applications in Discrete Mathematics
© 2015 Pushpa Publishing House, Allahabad, India Published Online: August 2015

# POWER OF GRAPH WITH (2, 2)-BIPARTITION 

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#### Abstract

We characterize the structure of graph $G$ and the integers $k$ for which the $k$ th power of the graph $G, G^{k}$ is $(2,2)$-bipartite, whenever the graph $G$ is among cycle, $n$-cube and tree.


## 1. Introduction

Graphs considered in this paper are connected, simple and without selfloops unless stated otherwise. For a graph $G, V(G)$ refers to the set of

Received: February 3, 2015; Revised: April 15, 2015; Accepted: April 29, 2015
2010 Mathematics Subject Classification: 05C75, 05C38.
Keywords and phrases: $(2,2)$-bipartite graphs, $n$-cubes, power of a graph, $d_{2}$-sequence, center of a graph.

Communicated by K. K. Azad
vertices. Adjacency between two vertices $u$ and $v$ in the graph $G$ is denoted by $u \sim_{G} v$, and nonadjacency by $u \not \psi_{G} v$. For a subset $S$ of $V(G),\langle S\rangle$ refers to the subgraph of $G$ induced by $S$. For any two vertices $u$ and $v$ in a graph $G, d_{G}(u, v)$ represents the distance between $u$ and $v$, i.e., the length of a shortest path between them. The subscripts in $\sim_{G}, \psi_{G}$ and $d_{G}$ are conveniently ignored when the graph under discussion is clear from the context. The eccentricity $e(v)$ of a vertex $v$ in a graph $G$ is the $\max _{u \in G}\{d(u, v)\}$. The radius of a graph $G$ is $\min _{v \in G} e(v)$ and the diameter of $G$ is $\max _{v \in G} e(v)$. A vertex $v$ in a graph $G$ is said to be a central vertex if its eccentricity is same as the radius of $G$, and the center of $G$ is the set of all central vertices. The notations $\bar{G}, K_{n}, P_{n}$ and $C_{n}$ refer to complement of $G$, complete graph on $n$ vertices, path on $n$ vertices and cycle on $n$ vertices, respectively. For a graph $G$ and a positive integer $k$, the $k$ th power of the graph $G$, denoted by $G^{k}$, is a simple graph with $V\left(G^{k}\right)$ same as $V(G)$ and $u \sim{ }_{G^{k}} v$ if and only if $d_{G}(u, v)$ is at most $k$. In particular, $G^{k}=G$ for $k=1$ and $G^{k}=K_{n}$ for $k \geq \operatorname{diameter}(G)$.

The study of $k$ th power of graph is found useful in several branches of applied graph theory. For example, in [1], authors have found that, a variation of frequency assignment problem considers $k$ th powers, $G^{k}(k=1,2, \ldots)$ of a given graph $G$. In [4], using chromatic properties of graphs, authors have proposed a new approach for placing the resources in symmetric networks. In [6], it has been observed that the induced subgraph of the $k$ th power of a tree, induced by leaves of the tree, may be recognized in polynomial time and this problem has applications in phylogeny.

In [5], inspired by the notion of bipartite graphs, authors have introduced the concept of $(2,2)$-bipartite graphs in which no two vertices from the same partite set are at distance two from each other. (The definitions and relevant results are given in Section 2).

In this paper, we consider the graph $G$ to be any of cycle, tree and $n$-cube, and characterize $k$ and the structure of the graph $G$ for which $G^{k}$ is (2, 2)-bipartite. Throughout this paper, $\lceil x\rceil$ represents the smallest integer not smaller than $x$ (i.e., ceiling of $x$ ), and similarly $\lfloor x\rfloor$ represents the greatest integer not greater than $x$ (i.e., floor of $x$ ). Readers are referred to [2] for all elementary notations and definitions not described but used in this paper.

## 2. Definitions and Preliminaries

In the following, we recall some definitions and results related to $(2,2)$ bipartite graphs.

Definition 2.1. A graph $G$ is said to be an ( $m, n$ )-bipartite graph ( $m$ and $n$ are positive integers) if the vertex set $V(G)$ can be partitioned into a pair of nontrivial subsets $V_{1}$ and $V_{2}$ such that no two vertices are at distance $m$ from each other in $V_{1}$ and no two vertices are at distance $n$ from each other in $V_{2}$.

If $m=n=1$, the $(1,1)$-bipartite graph is the well-known bipartite graph. In [5], we have some characterizations of $(2,2)$-bipartite graphs.

Remark 2.2. Each of the following statements appeared either as a remark or as a result in the paper [5].
(i) With every bipartition of the vertex set as a choice of $(2,2)$ bipartition, a complete graph $K_{n}$ on $n(\geq 2)$ vertices is a $(2,2)$-bipartite graph. Also, every totally disconnected graph is a $(2,2)$-bipartite graph.
(ii) Every path graph $P_{n}(n \geq 2)$ is (2, 2)-bipartite.
(iii) Any graph with $C_{5}$ or $K_{1,3}$ as an induced subgraph is not $(2,2)$ bipartite.
(iv) For $n>3$, a cycle graph $C_{n}$ is a (2,2)-bipartite graph if and only if $n=4 k$ for some positive integer $k . C_{3}$ is a $(2,2)$-bipartite graph.
(v) A tree is a (2, 2)-bipartite graph if and only if it is a path.
(vi) The disjoint union of (2, 2)-bipartite graphs is a (2, 2)-bipartite graph.

Now, we shall discuss some observations useful in the further discussion.
Lemma 2.3. If a graph $G$ is $(2,2)$-bipartite, then so is any induced subgraph of $G$.

Proof. Let $G$ be a $(2,2)$-bipartite graph with a $(2,2)$-bipartition $\left\{V_{1}, V_{2}\right\}$ and let $H$ be any induced subgraph of $G$. Then a (2, 2)-bipartition of $V(H)$ is given by $\left\{U_{1}, U_{2}\right\}$, where $U_{i}=V_{i} \cap V(H)$ for $i=1,2$.

Theorem 2.4. Given a graph $G$ with diameter $D$, the diameter of $G^{k}$ is given by $\left\lceil\frac{D}{k}\right\rceil$.

Proof. If $k \geq D$, then from the definition of $G^{k}$ it follows that every pair of vertices are at distance one in $G^{k}$. In other words, $G^{k}$ is a complete graph and hence the result follows.

Now, consider the case where $k<D$. Choose any two vertices $u_{1}$ and $u_{2}$ at distance $D$ from each other in $G$, and a shortest path $P: u_{1}=v_{1}, v_{2}$, $v_{3}, \ldots, v_{D+1}=u_{2}$ of length $D$. From the definition of power graph, it follows that the distance between $u_{1}$ and $u_{2}$ in $G^{k}$ is same as the diameter of $G^{k}$. Let $D=n k+i$, where $0 \leq i<k$ and $n$ is some integer. Since $P$ is a path of shortest length in $G$, from the definition of $G^{k}$, it is clear that $d_{G^{k}}\left(u_{1}, v_{n k+1}\right)=n \quad($ where $i=0$ in $D=n k+i)$ and $d_{G^{k}}\left(u_{1}, v_{n k+j+1}\right)=$ $n+1$ for every $0<j<k$. Therefore, $d_{G^{k}}\left(u_{1}, u_{2}\right)=\left\lceil\frac{n k+i}{k}\right\rceil=\left\lceil\frac{D}{k}\right\rceil$.

The observation in the following remark is straightforward from the definition of (2, 2)-bipartite graphs.

Remark 2.5. A bipartition $\left\{V_{1}, V_{2}\right\}$ of vertex set of a graph $G$ is not a $(2,2)$-bipartition if there exist vertices $v, v_{1}, v_{2}$ in $V(G)$ such that $v_{1} \in V_{i}$, $v_{2} \in V_{j}, i, j=1,2, i \neq j$ and $d\left(v, v_{1}\right)=d\left(v, v_{2}\right)=2$.

## 3. Characterization

In this main section of 'characterization', we consider three cases of $G$ being cycle, $n$-cube and tree. In the first two cases, the graphs are self centered in which every vertex is a central vertex and the diameter of the graph is same as the radius of the graph. In the third case, where the graph is a tree, the graph has at most two central vertices and the radius is the largest integer not greater than half the diameter. In each of the cases, we find close association between the center, diameter, radius and the integer $k$ for which $k$ th power of graph is $(2,2)$-bipartite.

### 3.1. Power of cycle graph

Note that every vertex $v$ in the cycle graph $C_{n}$ has the same eccentricity, and hence the graph is a self centered graph. Since the diameter of a cycle graph $C_{n}$ is $\left\lfloor\frac{n}{2}\right\rfloor, C_{n}^{k}$ is a complete graph for every $k \geq\left\lfloor\frac{n}{2}\right\rfloor$, and therefore, it is (2, 2)-bipartite.

In [5], authors have proved that $C_{n}\left(C_{n}^{k}\right.$, where $\left.k=1\right)$ is $(2,2)$-bipartite if and only if $n$ is $4 k$ for some positive integer $k$. Now, we will consider the case of $C_{n}^{k}$, where $k \geq 2$. In an attempt to characterize $k$ for which $k$ th power of $C_{n}$ is $(2,2)$-bipartite, we have the following.

Theorem 3.1. Let $G=C_{n}$ be the cycle on $n$ vertices with diameter $D$ and let $k \geq 2$. Then the following statements are equivalent.
(i) $G^{k}$ is (2, 2)-bipartite.
(ii) For any vertex $v$ in $C_{n}$, there exists at most one vertex at distance $k+1$ or more from $v$.
(iii) $k \geq\left\lceil\frac{n}{2}\right\rceil-1$.

In case of any of the above statements holds, we have the following situation. $k$ could be $D-1$ or more only when $n$ is even, in which case $G^{D-1}$ is a cocktail party graph on $n$ vertices and $G^{D}$ is the complete graph on $n$ vertices. In other case, whenever $n$ is odd, then $k$ could be only $D$ or more in which case $G^{k}$ is a complete graph.

Proof. Consider $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ such that $v_{i} \sim_{G} v_{i+1}$ for $i=1$, $2,3, \ldots,(n-1)$ and $v_{n} \sim_{G} v_{1}$.
(i) $\Rightarrow$ (ii) Let $G^{k}$ be (2, 2)-bipartite with a (2, 2)-bipartition $\left\{V_{1}, V_{2}\right\}$. Suppose that there are two distinct vertices $v_{k+2}$ and $v_{n-k}$ at distance $(k+1)$ from $v_{1}$ in $G$. If $v_{1} \in V_{1}$, then $v_{k+2} \in V_{2}$ as $d_{G^{k}}\left(v_{1}, v_{k+2}\right)=2$. Eventually, $\left\{v_{n-(k-1)}, \ldots, v_{n}, v_{1}\right\} \subseteq V_{1}$ and $\left\{v_{2}, \ldots, v_{k+1}, v_{k+2}\right\} \subseteq V_{2}$. Since $d_{G}\left(v_{n-k}, v_{1}\right)=k+1$, we have that $d_{G}\left(v_{2}, v_{n-k}\right)=k+1$ or $k+2$. For the same reason $d_{G^{k}}\left(v_{1}, v_{n-k}\right)=d_{G^{k}}\left(v_{2}, v_{n-k}\right)=2$ as $k \geq 2$ and

$$
d_{G^{k}}\left(v_{n}, v_{n-k}\right)=d_{G^{k}}\left(v_{2}, v_{n}\right)=1 .
$$

Now referring to Remark 2.5, we have a contradiction as $v_{1}$ and $v_{2}$ are in different partite sets. Hence (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (i) If there is no vertex at distance $k+1$ or more from any vertex $v$ in $G$, then $D$, the diameter of $G$, is not more than $k$. Therefore, the graph $G^{k}$ is complete and (2,2)-bipartite.

Suppose that for any vertex in $G$, there exists exactly one vertex at distance $k+1$ or more in $G$. Then $d_{G}\left(v_{1}, v_{k+2}\right)=k+1$ and $d_{G}\left(v_{1}, v_{k+3}\right)$ $=k$.

Now, construct a bipartition $\left\{V_{1}, V_{2}\right\}$ of $G^{k}$ as follows. Without loss of generality, let $v_{1} \in V_{1}$. Then put $v_{k+2}$ in $V_{2}$ as $d_{G}\left(v_{1}, v_{k+2}\right)=k+1$ and
$d_{G^{k}}\left(v_{1}, v_{k+2}\right)=2$. With the observation that $d_{G}\left(v_{i}, v_{j}\right) \leq k$ and therefore, $d_{G^{k}}\left(v_{i}, v_{j}\right)=1$ for all $2 \leq i, j \leq k+2$, put $v_{i}$ in $V_{2}$ for all $2 \leq i \leq k+2$. Similarly, put all the vertices from $\left\{v_{1}, v_{k+3}, v_{k+4}, \ldots, v_{n}\right\}$ in $V_{1}$ as all those vertices are at distance $k$ or lesser from each other in $G$, and are adjacent in $G^{k}$. Clearly, $\left\{V_{1}, V_{2}\right\}$ is a (2,2)-bipartition for $G^{k}$.

The statements (ii) and (iii) are equivalent as (ii) holds with exact one vertex at distance $k+1$ from $v$ if and only if $n$ is even, in which case, $\left\lceil\frac{n}{2}\right\rceil=\left\lfloor\frac{n}{2}\right\rfloor$ and in other case $\left\lceil\frac{n}{2}\right\rceil-1=\left\lfloor\frac{n}{2}\right\rfloor$.

Further remark is easily verified from the construction of a bipartition given in the proof of (ii) $\Rightarrow$ (i).

Note 3.2. In Theorem 3.1 with $n$ is even, $G^{(D-1)}$ is clearly a graph with diameter two. Further, if $G$ is a cycle graph on $n$ vertices given by $v_{1} \sim v_{2} \sim$ $\cdots \sim v_{n} \sim v_{1}$, then $G^{(D-1)}=H$ is the graph such that $v_{i} \not_{H} v_{j}$ if and only if $i=j(\bmod k)$, where $k=\frac{n}{2}$.

Remark 3.3. The statement (ii) of Theorem 3.1 may be rewritten as 'given a central vertex $v$ there exists at most one vertex at distance $k+1$ from $v$ '. Interestingly, we find in Subsection 3.2 that the same statement provides a necessary and sufficient condition for $G^{k}$ to be (2, 2)-bipartite, whenever $G$ is $n$-cube. Further in the case of $G$ is a tree, we observe that it is a necessary condition, as we see in Subsection 3.3.

### 3.2. Power of $n$-cubes

Another important class of self centered graphs is that of $n$-cubes. In this subsection, we provide different definitions of $n$-cubes and discuss some basic properties before characterizing $k$ for which $k$ th power of $n$-cube is a (2, 2)-bipartite graph. For the definitions and several properties discussed in the following, readers are referred to Harary et al. [3].

Definition 3.4. For $n \geq 2$, an $n$-cube denoted by $Q_{n}$ is recursively defined as follows:

$$
Q_{1}=K_{2} \text { and } Q_{n}=K_{2} \times Q_{n-1}
$$

We have yet another definition of $n$-cube using the binary sequences of length $n$ with appropriate adjacency relation.

Definition 3.5. The hypercube (n-cube) $Q_{n}$ is a graph whose vertex set is $\{0,1\}^{n}$, and two vertices are adjacent in $Q_{n}$ if the corresponding binary sequences differ exactly at one coordinate.

Remark 3.6. Given an $n$-cube graph $Q_{n}$, many of the following properties follow directly from Definitions 3.4 and 3.5.
(i) $Q_{n}$ is an $n$-regular graph on $2^{n}$ vertices and with $n 2^{n-1}$ edges.
(ii) The neighborhood of any vertex $v$ in $Q_{n}$ is an independent set, i.e., any two vertices in the neighborhood of $v$ are not adjacent and, in fact, are at distance two from each other.
(iii) The diameter of the $n$-cube is $n$, and for every vertex $v$ of the graph there exists unique corresponding vertex at distance $n$. In fact, the mapping of each vertex to its eccentric vertex, in the set of vertices, is a bijective map on the set of vertices. Further, $Q_{n}^{n-1}$ is a $\left(2^{n}-2\right)$-regular graph.
(iv) Two vertices in $Q_{n}$ are at distance $r$ from each other if and only if the binary sequences representing the vertices differ at $r$ coordinates. Hence for any vertex $v$, there are ${ }^{n} C_{r}$ vertices at distance $r$ from it, $1 \leq r \leq n$.
(v) Let $S$ be a set of $2^{n-1}$ vertices from $Q_{n}$ which induces $Q_{n-1}$. Then there exists exactly one $i,(1 \leq i \leq n)$ such that the $i$ th coordinate of every vertex in $S$ is same. This may be easily verified by using the following facts:
(a) Given any two diametrically opposite vertices $u$ and $v$ in an $n$-cube and $w$ is any other vertex, there exists a shortest path between $u$ and $v$ passing through $w$.
(b) Consider any two diametrically opposite vertices in the graph induced by $S$ which is isomorphic to $(n-1)$-cube. These vertices can have common entry at exact one coordinate and, without loss of generality, let the vertices be $v=(0,0, \ldots, 0,0)$ and $w=(1,1, \ldots, 1,0)$. Now, for any shortest path between $u$ and $w$, a vertex before $w$ on the path is of the form $\left(k_{1}, k_{2}, \ldots, k_{n-1}, 0\right)$, where exactly one of $k_{1}, k_{2}, \ldots, k_{n-1}$ is zero. In fact, continuing this way, we find that all vertices on the path are with $n$th coordinate equal to 0 .
(vi) In the $n$-cube $Q_{n}$, if a subset $S$ of $V\left(Q_{n}\right)$ induces a $Q_{n-1}$, then its complement $V \backslash S$ also induces a $Q_{n-1}$.
(vii) The $n$-cube $Q_{n}$ has $n$ pairs of vertex disjoint $Q_{n-1}$-subgraphs.

Lemma 3.7. The graph $Q_{n}^{n-1}$ is (2, 2)-bipartite, and if $\left\{V_{1}, V_{2}\right\}$ is any (2, 2)-bipartition of $Q_{n}^{n-1}$, then each of $V_{1}$ and $V_{2}$ induces a complete graph on $2^{n-1}$ vertices. Further, $Q_{n}^{k}$ is (2, 2)-bipartite for $k \geq n-1$.

Proof. From the definition, $G=Q_{n}=K_{2} \times Q_{n-1}$. So, we have a set $S \subset V(G)$ such that $\langle S\rangle$ is $Q_{n-1}$ in $G$. Write $V_{1}=S$ and $V_{2}=V(G) \backslash S$. Clearly, both $\left\langle V_{1}\right\rangle$ and $\left\langle V_{2}\right\rangle$ are $Q_{n-1}$ in $G$ and therefore, both $V_{1}$ and $V_{2}$ induce complete graph on $2^{n-1}$ vertices in $Q_{n}^{n-1}$. So, $V_{1}$ and $V_{2}$ provide a (2,2)-bipartition for $G^{n-1}$, as the distance between any two vertices from $V_{1}$ and $V_{2}$ is at most $n$ in $G$ and at most 2 in $G^{n-1}$.

Since the mapping of a vertex with its eccentric vertex (at distance $n$ ) in $G$ defines a one-one mapping, no set of vertices with cardinality more than $2^{n-1}$ can induce a complete graph in $G^{n-1}$. Further, the diameter of $G^{n-1}$ is two implies that every partite set in any $(2,2)$-bipartition induces a complete graph and therefore, both the partite sets have the same cardinality $2^{n-1}$.

The second part of the lemma follows as the diameter of $Q_{n}$ is $n$.

The concept of distance $r$ sequence (in short, $d_{r}$-sequence) in a given graph is useful in proving the next results.

Definition 3.8. A sequence $\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ of $k$ vertices of graph $G$ is said to be a $d_{r}$-sequence in $G$ if $d_{G}\left(v_{i}, v_{i+1}\right)=r$ for every $i, 1 \leq i \leq k-1$.

Remark 3.9. In the graph given in Figure 3.1, the sequence $\left\{v_{1}, v_{4}, v_{6}, v_{7}\right\}$ is a $d_{2}$-sequence.


Figure 3.1
Lemma 3.10. Let $G$ be a $(2,2)$-bipartite graph with (2, 2)-bipartition $\left\{V_{1}, V_{2}\right\}$ and let $\left\{v_{1}, v_{2}, \ldots, v_{r}\right\}$ be a $d_{2}$-sequence of vertices of $G$. Then the vertices indexed with odd integers are in one partite set and the rest are in other partite set.

Proof follows from the definition of $d_{2}$-sequence and definition of (2, 2)-bipartite graph.

Remark 3.11. Consider a connected graph $G$ and a $d_{2}$-sequence $\left\{v_{1}, v_{2}, \ldots, v_{r}\right\}$ of $r$ vertices of $G$. If both $i$ and $j$ are even (similarly, odd), then $d\left(v_{i}, v_{j}\right)=2$ implies that $G$ is not (2,2)-bipartite.

Lemma 3.12. The power graph $Q_{n}^{n-2}$ is not (2, 2)-bipartite for any $n \geq 3$.

Proof. To prove the result, we construct a $d_{2}$-sequence of vertices in $H=Q_{n}^{n-2}$, say $\left\{v_{1}, v_{2}, \ldots, v_{r}\right\}$, such that $d_{H}\left(v_{i}, v_{j}\right)=2$ for some $i \neq j$ with same parity.

Take $v_{1}=(0,0, \ldots, 0)$ and then construct $v_{i}$ recursively by keeping ( $i-1$ )th coordinate of $v_{i-1}$ as it is and switching the rest of the coordinates.
$n$ is even. Let $n=2 k$ for some integer $k$. From the choice of vertices, observe that $(2 k+1)$ th vertex in the sequence is the one with all coordinates equal to 1 and therefore, it is at distance two from $v_{1}$ in $H$ (they are at distance $n$ in $Q_{n}$ and $n \geq 3$ ). So, we have that 1 and $2 k+1$ are odd integers but $v_{1}$ and $v_{2 k+1}$ are at distance two from each other in $H$. Referring to Remark 3.11, we have that $H$ is not $(2,2)$-bipartite.
$n$ is odd. Let $n=2 k+1$ for some integer $k$. In this case, $v_{2}=(0,1,1, \ldots, 1)$ and $v_{2 k+2}=(0,0,0, \ldots, 0)$ which are at distance two from each other in $H$, which leads to the conclusion that $H$ is not $(2,2)$ bipartite as in the earlier case.

Hence the proof.
Example 3.13. In the present example, we demonstrate the construction of $d_{2}$-sequences which help us in concluding that $Q_{6}^{4}$ and $Q_{7}^{5}$ are not (2, 2)-bipartite.

First, consider the graph $Q_{n}^{n-2}$, where $n=6$. For $v_{1}=(0,0,0,0,0,0)$, construct $v_{2}$ by fixing the first coordinate and switching the other coordinates, we get $v_{2}=(0,1,1,1,1,1)$. Clearly, $v_{1}$ and $v_{2}$ are at distance 5 from each other in $Q_{6}$ and at distance 2 in $Q_{6}^{4}$. Similarly, continue to get a $d_{2}$ sequence with

$$
\begin{aligned}
& v_{3}=(1,1,0,0,0,0), v_{4}=(0,0,0,1,1,1), v_{5}=(1,1,1,1,0,0), \\
& v_{6}=(0,0,0,0,0,1), \text { and } v_{7}=(1,1,1,1,1,1) .
\end{aligned}
$$

Clearly, $v_{1}$ and $v_{7}$ are at distance 2 from each other in $Q_{6}^{4}$ and therefore, $Q_{6}^{4}$ is not (2, 2)-bipartite.

In the case of $n=7$, we shall start with $v_{1}=(0,0,0,0,0,0,0)$ and continue to get

$$
\begin{aligned}
& v_{2}=(0,1,1,1,1,1,1), v_{3}=(1,1,0,0,0,0,0), v_{4}=(0,0,0,1,1,1,1), \\
& v_{5}=(1,1,1,1,0,0,0), v_{6}=(0,0,0,0,0,1,1), v_{7}=(1,1,1,1,1,1,0),
\end{aligned}
$$

and $v_{8}=v_{1}=(0,0,0,0,0,0,0)$, a $d_{2}$ sequence in $Q_{7}^{5}$. Again, $v_{8}$ and $v_{2}$ are at distance 2 from each other in $Q_{7}^{5}$, which concludes that $Q_{7}^{5}$ is not a (2, 2)-bipartite graph.

Since every $Q_{n}^{m}(m \leq n-2)$ has an induced $Q_{m+2}^{m}$ subgraph, the following theorem is immediate from Lemmas 3.7 and 3.12, and the understanding that every induced subgraph of a (2,2)-bipartite graph is (2, 2)-bipartite.

Theorem 3.14. For every positive integer $n \geq 2$, the power graph $Q_{n}^{m}$ is (2, 2)-bipartite if and only if $m \geq n-1$.

Now, we have the following corollary.
Corollary 3.15. An n-cube graph $Q_{n}$ is (2, 2)-bipartite if and only if $n=2$.

### 3.3. Power of trees

In the earlier parts of the section, we have considered cycles and hypercubes, in which cases every vertex is a central vertex. Now, we consider a tree graph for our study, in which case the center of the graph is not trivial. By definition of a central vertex, it is a vertex with minimal eccentricity (radius of graph), and the number of such vertices in a tree is at most two. For our convenience, we say the vertices $x$ and $y$ in a tree graph are
diagonally opposite if the unique path between them passes through all the central vertices. Given a noncentral vertex $z$ in a tree graph $G$, define the split of graph $G$ with reference to $z$ by subgraphs $B_{z}$ and $C_{z}$, where $B_{z}$ is the maximal connected subgraph that contains $z$ but not every central vertices, and $C_{z}=G \backslash B_{z}$. Note that $C_{z}$ has exactly one central vertex irrespective of the cases whether the tree $G$ is unicentral or bicentral.

In the following, we characterize $k$ for which $G^{k}$ is $(2,2)$-bipartite whenever $G$ is a tree.

Theorem 3.16. Let $G$ be tree and $k \geq 2$ be any integer. Then the following statements are equivalent:
(i) $G^{k}$ is (2,2)-bipartite.
(ii) No three vertices are with mutual distance $k+1$ or more in $G$.

Proof. Let $G^{k}$ be a (2, 2)-bipartite graph. (i) $\Rightarrow$ (ii) holds trivially as we cannot have three vertices with mutual distance 2 or more in any (2, 2)bipartite graph.
(ii) $\Rightarrow$ (i) Let no three vertices be with mutual distance $k+1$ or more. Consider a path of maximal length which passes through all the central vertices and with end vertices $x$ and $y$. Note that $x$ and $y$ must be among pendant vertices in the tree $G$.

Without loss of generality, let $d_{G}(x, u) \leq d_{G}(y, u)$ for a central vertex $u$. If the strict inequality holds, then there exists another central vertex $v$ such that $d_{G}(y, v) \leq d_{G}(x, v)$. Now, consider the partition $V_{1}$ and $V_{2}$ of $V(G)$ given by the set of vertices of $B_{y}$ and $C_{y}$, respectively. Since $d_{G}(x, u)$ $\leq d_{G}(y, u)$ and, $x$ and $y$ are pendant vertices, irrespective of the case whether $G$ is unicentral or bicentral, central vertex $u$ and $x$ are in $V_{2}=V\left(C_{y}\right)$. Since $x$ and $y$ are eccentric vertices of each other, (ii) implies that one of the following holds:
(a) $d_{G}(x, y) \geq k+1$, in which case no vertex from $V_{1}$ is at distance $k+1$ from $y$ and no vertex from $V_{2}$ is at distance $k+1$ from $x$. This is because $d(w, x) \leq d(w, u)+d(u, x) \leq d(w, u)+d(u, y)=d(w, y)$ for all $w \in V_{2}$, and similarly, $d(z, y) \leq d(z, x)$ for all $z \in V_{1}$. So, $\left\langle V_{1}\right\rangle$ and $\left\langle V_{2}\right\rangle$ induce complete graphs in $G^{k}$, and hence $G^{k}$ is $(2,2)$-bipartite.
(b) $d_{G}(x, y) \leq k$, in which case $G^{k}$ is a complete graph and hence $G^{k}$ is $(2,2)$-bipartite.

The proof of following corollary is immediate from the above Theorem 3.16 and the fact that there exists exactly one pair of pendant vertices with distance $n-1$ from each other in a path graph on $n$ vertices.

Corollary 3.17. Let $G=P_{n}$ be a path graph on $n$ vertices. Then $G^{k}$ is (2, 2)-bipartite if and only if $k \geq\left\lceil\frac{n}{2}\right\rceil-1$.

Remark 3.18. From Theorem 3.16, we could relate $r$, the radius of tree graph $G$ on $n$ vertices, to some extent with the $k$ for which $k$ th power of $G$ is (not!) $(2,2)$-bipartite, as given below:
(i) If $k \leq r-2$, then $G^{k}$ is not a $(2,2)$-bipartite graph.
(ii) If $k=r-1$ and the diameter is even, in which case the graph is unicentral, we find three vertices with mutual distance $r=k+1$ in the graph. Examples for such vertices are the central vertex and diagonally opposite pendant vertices on a path of maximal length. So in this case, $G^{k}$ is not a (2, 2)-bipartite graph.
(iii) If $k=r-1$ and the diameter is odd, in which case the graph is bicentral, we are not in a position to make any affirmative statement different from the one in Theorem 3.16. For example, in the case of $G$ is path graph on even number of vertices we have that $G^{k}$ is a $(2,2)$-bipartite graph. At the
same time, one may easily construct an example of a tree graph $G$ with diameter equals to an odd integer, but one of the central vertices having degree three or more, in which case $G^{k}$ is not a $(2,2)$-bipartite graph.
(iv) If $r \leq k<d$, where $d$ is the diameter of tree, we refer only to Theorem 3.16. For each such $k$, we have examples of tree graphs satisfying (also, not satisfying) the conditions given in Theorem 3.16.

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