Far East J. Appl. Math. **20**(2) (2005), 169-178

MODELING THE REMOVAL OF GASEOUS POLLUTANTS AND PARTICULATE MATTERS BY HEAVY RAIN

RAM NARESH and SHYAM SUNDAR

Department of Mathematics, H. B. Technological Institute
Kanpur - 208002, India
e-mail: ramntripathi@yahoo.com

Abstract

A mathematical model for the removal of gaseous pollutants and particulate matters from the atmosphere by heavy rain is proposed and analyzed using stability theory of nonlinear differential equations. In the model, it is assumed that no absorbed phase formation takes place as there is no sufficient time left for this process and therefore, the removal phenomenon occurs mainly due to process of impaction. The atmosphere, under consideration, consists of three interacting phases, namely, raindrops phase, gaseous pollutants phase and the phase of particulate matters. It is shown that these pollutants, emitted at a constant rate from different sources, can be washed-out completely from the atmosphere under certain conditions and the remaining equilibrium level would mainly depend upon the reaction rate coefficients of these pollutants with raindrops. However, in the case of instantaneous emission, it is found that these pollutants are washed-out completely from the atmosphere by the heavy rain.

1. Introduction

The purpose of this study is to describe a nonlinear mathematical model to study the removal of gaseous pollutants and particulate matters

2000 Mathematics Subject Classification: 76R99.

Key words and phrases: heavy rain, nonlinear model, pollutants, stability analysis, raindrops density.

Received December 21, 2004

from the atmosphere by heavy rain and to keep the description as simple as possible, but to retain the main dynamical processes governing the system. Various kinds of removal mechanisms like scavenging by raindrops, dry and wet deposition occur in the atmosphere for both the gaseous pollutants and particulate matters. Scavenging of pollutants by precipitation is one of the most important cleansing mechanisms. In several experimental studies, it has been found that the atmosphere of Indian cities becomes cleaner during and after monsoon season [10, 12, 13]. Field measurements of the below cloud scavenging have been carried out by releasing particles during precipitation intervals [9]. An investigation has been carried out to study a single droplet scavenging mechanism upon atmospheric ammonia by using two-phase simulation method (TPSM) and it has been found that the gaseous ammonia is scavenged outward rapidly from the interface initially, on grounds of intrinsic solute-sink characterized by the droplet [1]. A number of investigations have been carried out to understand the scavenging of pollutants by precipitation [3, 5, 6, 11]. A model has been developed to couple Regional Atmospheric Modeling System (RAMS) with a one dimensional below cloud scavenging model in order to simulate the in-cloud and below-cloud scavenging processes [3]. Measurements have been taken on size distribution of atmospheric aerosol at Dayalbagh, Agra during July to September 1998 and resulted that aerosol exist in fine as well as coarse modes but differ in percentage contribution seasonally [11]. Mathematical models using coupled convective diffusion equations for gaseous pollutants absorbed in the raindrops have been studied by taking into account uniform distribution of raindrops [2, 4, 7].

All these models are linear but in real situations the phenomenon of interaction of pollutants with raindrops is nonlinear. The removal terms, due to precipitation, are proportional to the concentration of the pollutants as well as to the number density of raindrops. In this regard, Shukla et al. [14] proposed and analyzed a nonlinear mathematical model for the removal of air pollutants from the atmosphere. In the model they have shown that the gaseous pollutants, emitted into the atmosphere instantaneously, can be removed by rain completely. They have also shown that if the gaseous pollutants are emitted with a constant rate, they can also be washed-out from the atmosphere under appropriate conditions. A nonlinear mathematical model for the removal of air pollutants by precipitation scavenging has also been developed [8]. This model is analyzed qualitatively to see the effect of precipitation scavenging on the equilibrium level of pollutants in the atmosphere using stability theory. These models are generally applicable for ordinary rain where an absorbed phase is formed due to interaction of pollutant species with the raindrops. The removal of pollutants by the heavy rain has not been studied earlier.

Keeping in view of above, in this paper we propose and analyze a nonlinear mathematical model for scavenging of gaseous pollutants and particulate matters from the atmosphere by precipitation due to heavy rain. It is considered that the effect of these pollutants on raindrops formation is negligible and no absorbed phase formation takes place. This is because there is no sufficient time for these processes to take place. The phenomenon of removal of these pollutants takes place due to process of impaction. We assume that there exist three interacting phases in the atmosphere namely, raindrops phase, gaseous pollutants phase and the phase of particulate matters. The dynamics of these phases is assumed to be governed by the nonlinear differential equations.

2. Mathematical Model

The atmosphere, under consideration, consists of three interacting phases as follows:

1. Raindrops phase – which occurs due to rain.

2. Gaseous pollutants phase – which occurs due to emission of gaseous pollutants into the atmosphere from different sources like industrial stacks, vehicular exhausts etc.

3. Particulate matters phase – which occurs due to emission of particulate matters.

Here we have assumed that the mechanism for the removal of the pollutants from the atmosphere is due to process of impaction caused by neighboring raindrops.

Let $C_r(t)$ be the number density of raindrops with its depletion rate coefficient r_0 and $q(t)$ is the rate of formation of raindrops, assumed to be constant (say q_0). Further, there may be some depletion of raindrops due to interaction with these pollutants (such as due to evaporation by pollutants) but in the case of heavy rain there is no time left to happen this process. Therefore, in the model we assume negligible effect of pollutants on raindrops formation. $C(t)$ and $C_p(t)$ are the concentrations of gaseous pollutants and particulate matters with their natural depletion rates δC and $\delta_p C_p$, respectively. $Q(t)$ and $Q_p(t)$ are their rates of emission, respectively, α and α_p are the removal rate coefficients of C and C_p due to interaction with C_r and this removal of pollutants is in direct proportion to the concentrations of respective pollutants as well as the number density of raindrops (i.e., $\alpha C C_r$ and $\alpha_p C_p C_r$). In this model, it has also been assumed that, due to heavy rain, there is no sufficient time for these pollutants to re-enter into the atmosphere by reversible process. Therefore, in the model recycling terms are not taken.

Under above assumptions, the system is governed by the following differential equations:

$$\frac{dC_r}{dt} = q(t) - r_0 C_r, \tag{1}$$

$$\frac{dC}{dt} = Q(t) - \delta C - \alpha C C_r, \tag{2}$$

$$\frac{dC_p}{dt} = Q_p(t) - \delta_p C_p - \alpha_p C_p C_r, \tag{3}$$

$$C_r(0) \geq 0,\ C(0) \geq 0,\ C_p(0) \geq 0.$$

3. Stability Analysis

Now we analyze the model (1)-(3) using stability theory of nonlinear differential equations under the following two conditions:

Case I. $q(t) = q_0$, $Q(t) = Q$ and $Q_p(t) = Q_p$ (constant emission),

Case II. $q(t) = q_0$, $Q(t) = 0$ and $Q_p(t) = 0$ (instantaneous emission).

Case I. When $q(t) = q_0$, $Q(t) = Q$ and $Q_p(t) = Q_p$ (constant emission). In this case, the model has only one nonnegative equilibrium point, namely, $E^*(C_r^*, C^*, C_p^*)$. The positive solution of E^* is given by the following equations:

$$C_r = \frac{q_0}{r_0}, \tag{4}$$

$$C = \frac{Q}{\delta + \alpha C_r}, \tag{5}$$

$$C_p = \frac{Q_p}{\delta_p + \alpha_p C_r}. \tag{6}$$

It is seen from equations (4)-(6) that C_r increases but C and C_p decrease with increase in q_0. Also we note that $C, C_p \to 0$ as $C_r \to \infty$.

The variational matrix corresponding to E^* is

$$M^* = \begin{bmatrix} -r_0 & 0 & 0 \\ -\alpha C^* & -(\delta + \alpha C_r^*) & 0 \\ -\alpha_p C_p^* & 0 & -(\delta_p + \alpha_p C_r^*) \end{bmatrix}.$$

From M^*, it is clear that all the eigen values are negative showing that E^* is locally asymptotically stable.

To check the global stability behavior, we need the following lemma.

Lemma 3.1. *The set* $\Omega = \left\{(C_r, C, C_p) : 0 \le C_r \le \frac{q_0}{r_0}, 0 \le C \le \frac{Q}{\delta}, 0 \le C_p \le \frac{Q_p}{\delta_p}\right\}$ *attracts all solutions initiating in the interior of positive octant.*

Proof. From equation (1), we have

$$\frac{dC_r}{dt} = q_0 - r_0 C_r.$$

From this equation, we get

$$\underset{t\to\infty}{Lim} \sup C_r(t) = \frac{q_0}{r_0}.$$

Again from equation (2),

$$\frac{dC}{dt} = Q - \delta C - \alpha C C_r,$$

$$\frac{dC}{dt} \le Q - \delta C.$$

Using comparison theorem, we get

$$\underset{t \to \infty}{Lim} \sup C(t) = \frac{Q}{\delta}.$$

Similarly, we also get

$$\underset{t \to \infty}{Lim} \sup C_p(t) = \frac{Q_p}{\delta_p}.$$

Hence the lemma.

Theorem 3.1. *If the following inequalities*

$$\left(\frac{\alpha Q}{\delta}\right)^2 < 2r_0(\delta + \alpha C_r^*),$$

$$\left(\frac{\alpha_p Q_p}{\delta_p}\right)^2 < 2r_0(\delta_p + \alpha_p C_r^*)$$

hold in Ω, *then* E^* *is globally asymptotically stable.*

Proof. Consider the following positive definite function about E^*,

$$V = \frac{1}{2}(C_r - C_r^*)^2 + \frac{1}{2}(C - C^*)^2 + \frac{1}{2}(C_p - C_p^*)^2. \tag{7}$$

Differentiating this with respect to 't', we get

$$\dot{V} = (C_r - C_r^*)\frac{dC_r}{dt} + (C - C^*)\frac{dC}{dt} + (C_p - C_p^*)\frac{dC_p}{dt}$$

$$= (C_r - C_r^*)(q_0 - r_0 C_r) + (C - C^*)(Q - \delta C - \alpha C C_r)$$

$$+ (C_p - C_p^*)(Q_p - \delta_p C_p - \alpha_p C_p C_r).$$

After some algebraic manipulations, it can be written as

$$\dot{V} = -r_0(C_r - C_r^*)^2 - (\delta + \alpha C_r^*)(C - C^*)^2 - (\delta_p + \alpha_p C_r^*)(C_p - C_p^*)^2$$

$$- \alpha C(C_r - C_r^*)(C - C^*) - \alpha_p C_p(C_r - C_r^*)(C_p - C_p^*).$$

Now $\dot{V}$ will be negative definite if

$$(\alpha C)^2 < 2r_0(\delta + \alpha C_r^*),$$

i.e.,

$$\left(\frac{\alpha Q}{\delta}\right)^2 < 2r_0(\delta + \alpha C_r^*) \tag{8}$$

and

$$(\alpha_p C_p)^2 < 2r_0(\delta_p + \alpha_p C_r^*),$$

i.e.,

$$\left(\frac{\alpha_p Q_p}{\delta_p}\right)^2 < 2r_0(\delta_p + \alpha_p C_r^*). \tag{9}$$

Under above conditions $\dot{V}$ is negative definite showing that V is a Liapunov's function. Hence the theorem.

This theorem shows that with increase in removal parameters α and α_p, the respective concentrations of C and C_p decrease in the atmosphere and may even tend to zero if α and α_p are very large. From the above theorem, we also note that if Q and Q_p are small, then the possibility of satisfying conditions (8) and (9) increases.

Case II. When $q(t) = q_0$, $Q(t) = 0$ and $Q_p(t) = 0$ (instantaneous emission). In this case, the model takes the form

$$\frac{dC_r}{dt} = q_0 - r_0 C_r, \tag{10}$$

$$\frac{dC}{dt} = -\delta C - \alpha C C_r, \tag{11}$$

$$\frac{dC_p}{dt} = -\delta_p C_p - \alpha_p C_p C_r. \tag{12}$$

In this case, also the model has only one nonnegative equilibrium point, namely, $E\left(\frac{q_0}{r_0}, 0, 0\right)$.

Now we compute the following variational matrix corresponding to E,

$$M = \begin{bmatrix} -r_0 & 0 & 0 \\ 0 & -\left(\delta + \alpha \frac{q_0}{r_0}\right) & 0 \\ 0 & 0 & -\left(\delta_p + \alpha_p \frac{q_0}{r_0}\right) \end{bmatrix}.$$

From M, it is clear that E is locally asymptotically stable.

Since $C_r > 0$, it can be checked that E is also globally asymptotically stable.

From equations (11) and (12), we have

$$\frac{dC}{dt} \leq -\delta C \quad \text{and} \quad \frac{dC_p}{dt} \leq -\delta_p C_p.$$

Using comparison theorem, we can show that

$$\operatorname*{Lim\,sup}_{t\to\infty} C(t) = 0 \quad \text{and} \quad \operatorname*{Lim\,sup}_{t\to\infty} C_p(t) = 0. \tag{13}$$

Thus, we conclude that in the case of instantaneous emission both of these pollutants can be washed-out completely from the atmosphere.

4. Numerical Example

Consider the following set of parameter values:

$$q_0 = 10{,}000.0, \;\; r_0 = 0.05, \;\; Q = 40.0, \;\; Q_p = 30.0,$$

$$\delta = 0.25, \;\; \delta_p = 0.32, \;\; \alpha = 0.60, \;\; \alpha_p = 0.80.$$

E^* can be calculated as

$$C_r^* = 2{,}00{,}000.0, \;\; C^* = 0.00033, \;\; C_p^* = 0.000187.$$

It can be easily checked that inequalities (8) and (9) are satisfied using the above set of parameters showing that E^* is globally asymptotically stable within the region of attraction Ω.

Further, in the following table, we show the variation of equilibrium values with raindrops formation q_0.

It is clear from the Table 1 that equilibrium values of raindrops density increase with increase in q_0 while the equilibrium values of gaseous pollutants and particulate matters decrease.

Table 1

q_0	C_r^*	C^*	C_p^*
10,000.0	2,00,000.0	0.00033	0.000187
15,000.0	3,00,000.0	0.0002222	0.0001249
20,000.0	4,00,000.0	0.0001667	0.00009374
25,000.0	5,00,000.0	0.0001333	0.00007499
30,000.0	6,00,000.0	0.000111	0.00006249

5. Conclusion

The washout model described in this paper is developed and analyzed to study the removal of gaseous pollutants and particulate matters due to heavy rain from the atmosphere using stability theory of nonlinear differential equations. It is shown that in the case of constant emission, the pollutants are significantly removed from the atmosphere but their removal would depend upon interaction rate coefficients with raindrops. When the pollutants are emitted into the atmosphere instantaneously, it is found that both the pollutants can be washed-out completely from the atmosphere by the process of impaction. It is also shown that if q_0 increases, the concentration of both the pollutants decreases in the atmosphere. Also if removal parameters are very large, then the possibility of removal of these pollutants is more plausible.

References

[1] W. H. Chen, Atmospheric ammonia scavenging mechanisms around a liquid droplet in convective flow, Atmos. Environ. 38(8) (2004), 1107-1116.

[2] B. E. A. Fisher, The transport and removal of sulfur dioxide in a rain system, Atmos. Environ. 16 (1982), 775-783.

[3] F. L. T. Goncalves, A. M. Ramos, S. Freitas, M. A. Silva Dias and O. Massambani, In-cloud and below-cloud numerical simulation of scavenging processes at Serra Do Mar region, SE Brazil, Atmos. Environ. 36(33) (2002), 5245-5255.

[4] S. Kumar, An Eulerian model for scavenging of pollutants by raindrops, Atmos. Environ. 19(5) (1985), 769-778.

[5] G. A. Loosmore and R. T. Cederwall, Precipitation scavenging of atmospheric aerosols for emergency response applications: testing an updated model with new real time data, Atmos. Environ. 38(7) (2004), 993-1003.

[6] K. Murano, Wet deposition of ammonium and atmospheric distribution of ammonia and particulate ammonium in Japan, Environ. Poll. 102(S1) (1998), 321-326.

[7] R. Naresh, Modeling the dispersion of air pollutant from a time dependent point source: effect of precipitation scavenging, J. MACT 32 (1999), 77-91.

[8] R. Naresh, Qualitative analysis of a nonlinear model for removal of air pollutants, Int. J. Nonlinear Sci. Numer. Simul. 4 (2003), 379-385.

[9] K. W. Nicholson, J. R. Branson and P. Giess, Field measurements of the below cloud scavenging of particulate material, Atmos. Environ. 25A(3-4) (1991), 771-777.

[10] J. Pandey, M. Agarwal, N. Khanam, D. Narayan and D. N. Rao, Air pollutant concentration in Varanasi, India, Atmos. Environ. 26B (1992), 91-98.

[11] R. S. Parmar, G. S. Satsangi, M. Kumari, A. Lakhani, S. S. Srivastava and S. Prakash, Study of size distribution of atmospheric aerosol at Agra, Atmos. Environ. 35 (2001), 693-702.

[12] P. S. Prakash Rao, L. T. Khemani, G. A. Momin, P. D. Safari and A. G. Pillai, Measurements on wet and dry deposition at an urban location in India, Atmos. Environ. 26(B) (1992), 73-78.

[13] V. P. Sharma, H. C. Arora and R. K. Gupta, Atmospheric pollution studies at Kanpur -suspended particulate matter, Atmos. Environ. 17 (1983), 1307-1314.

[14] J. B. Shukla, M. Agarwal and R. Naresh, An Ecological Type Nonlinear Model for Removal Mechanism of Air Pollutants, Precipitation Scavenging and Atmospheric-surface Exchange, Vol. 3, S. E. Schwartz and W. G. N. Slinn, eds., pp. 1245-1253, Hemisphere Pub. Corp., U.S.A., 1992.

■