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STRONG ENDOMORPHISM KERNEL PROPERTY FOR BROUWERIAN ALGEBRAS

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Abstract

We shall show that endomorphism kernel property (EKP) and strong endomorphism kernel property (SEKP) are for some classes of universal algebras preserved by finite direct products and also by a special form of infinite direct sum construction. Full characterization of finite relative Stone algebras and **L**-algebras which have SEKP is given and a wide class of infinite relative Stone algebras which possess SEKP is described.

1. Introduction

Blyth and Silva in the paper [2] defined a strong endomorphism kernel property (SEKP) for a universal algebra (see Definition 2.2). They considered the case of Ockham algebras and in particular of MS-algebras.

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They proved, e.g., that a finite Boolean algebra has SEKP if and only if it is 2 element BA, a finite bounded distributive lattice possesses SEKP if and only if it is a chain and proved full characterization of MS-algebras having SEKP. Blyth et al. in [3] proved a full characterization of finite distributive double *p*-algebras and finite double Stone algebras having SEKP. SEKP for distributive *p*-algebras and Stone algebras has been studied and fully characterized by Fang and Fang in [4]. Fang and Sun fully characterized semilattices with SEKP in [5]. Guričan and Ploščica fully characterized unbounded distributive lattices which posses SEKP in [9]. The main approach in papers [2-4] is done by regarding algebras in question as Ockham algebras and using the duality theory of H. Priestley.

There is one important universal assumption in the original paper [2] of Blyth and Silva, namely, all algebras considered in this paper must contain two nullary operations (denoted by 0 and 1, $0 \neq 1$). This assumption is necessary to prove all important statements in their paper and therefore it seems to be impossible to directly adapt their methods to algebras which do not satisfy this assumption (e.g., $\{1\}$-lattices or unbounded lattices). Let us mention three of these results:

Theorem 1.1 [2, Theorem 1]**.** *If an algebra A has SEKP*, *then it has at most one maximal congruence*.

Corollary 1.2 [2, Corollary 1]**.** *A finite algebra that has SEKP is directly indecomposable*.

Theorem 1.3 [2, Theorem 3]**.** *A semisimple algebra has SEKP if and only if it is simple*.

As it is easy to check, $\{0, 1\}^2$ considered as 4 element distributive lattice with a top element ($\{1\}$-lattice) has SEKP and none of these statements is true for this algebra.

2. Preliminaries

Let *A* be a universal algebra. We denote $\omega_A = \{(a, a);\ a \in A\}$ and $\iota_A = A \times A$, trivial and universal congruences on *A*.

Next definition defines the notion of EKP (see [1]).

Definition 2.1. An algebra A has the endomorphism kernel property (EKP for shortening) if every congruence relation on A different from the universal congruence ι_A is the kernel of an endomorphism on A. In addition, we say that $f \in \mathrm{End}(A)$ is *associated* to $\theta \in \mathrm{Con}(A)$, if $\theta = \ker(f)$.

Next important notion is strong endomorphism property defined in [2]. Let A be a universal algebra, $f : A \to A$ be an endomorphism, $\theta \in \mathrm{Con}(A)$ be a congruence on A. f is *compatible* with θ iff $(a, b) \in \theta \Rightarrow (\varphi(a), \varphi(b)) \in \theta$. Endomorphism f is *strong* (on A), if it is compatible with every congruence $\theta \in \mathrm{Con}(A)$.

Definition 2.2. An algebra A has the strong endomorphism kernel property (SEKP for shortening) iff every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A. In addition, we say that $f \in \mathrm{End}(A)$ is *associated* to $\theta \in \mathrm{Con}(A)$, if $\theta = \ker(f)$.

Let us recall the definition of weak direct product (see Definition 22.1 in chapter 3, [6])

Definition 2.3. Let (A_i, F), $i \in I$ be algebras of the same type. Take the direct product $P = \prod(A_i;\, i \in I)$. Let $B \subseteq P$. (B, F) is called a *weak direct product* of algebras A_i if:

(1) (B, F) is a subalgebra of P,

(2) if $f, g \in B$, then the set $\{i \in I;\ f(i) \neq g(i)\}$ is finite,

(3) if $f \in B$, $g \in \prod(A_i, i \in I)$ and the set $\{i \in I;\ f(i) \neq g(i)\}$ is finite, then $g \in B$.

This type of product need not exist and/or be uniquely determined. Every weak direct product is a subdirect product. Zlatoš in [14] used the name “direct sum” for a weak direct product.

Let us start with some definitions (see, e.g., paragraph 4.4 of [12]).

Definition 2.4. Let (A, F), (B, F) be algebras of the same type. The product $A \times B$ has factorable congruences, if every congruence $\theta \in \mathrm{Con}(A \times B)$ has the form $\theta_1 \times \theta_2$, where $\theta_i \in \mathrm{Con}(A_i)$ for every $i = 1, 2$.

An algebra A has factorable congruences, if whenever $A = \prod_{i=1}^{n} A_i$, then every congruence $\theta \in \mathrm{Con}(A)$ has the form $\prod_{i=1}^{n} \theta_i$, where $\theta_i \in \mathrm{Con}(A_i)$ for every $i = 1, \ldots, n$.

A variety V is said to have *factorable congruences*, if every algebra $A \in V$ has factorable congruences.

Let (B, F) be a direct sum of algebras A_i, $i \in I$ from V. (B, F) has factorable congruences, if every congruence $\theta \in \mathrm{Con}(B)$ has the form $B^2 \cap \prod(\theta_i;\ i \in I)$, where $\theta_i \in \mathrm{Con}(A_i)$ for every $i \in I$.

It is known that the factorability of congruences does not extend to infinite direct products. Zlatoš [14] considered a direct sum construction to be more fitting for this purpose, he proved the following:

Theorem 2.5 [14, Theorem 2]**.** *Let V be a variety of algebras. The following conditions are equivalent*:

(1) *V has factorable congruences*,

(2) *every direct sum of algebras from V has factorable congruences*.

3. A Construction

We shall describe a special type of a direct sum now. Let V be a variety. Let A_i, $i \in I$ be algebras from V such that they all have one element subalgebra and we have chosen (distinguished) elements $e_{A_i} \in A_i$ such that $\{e_{A_i}\}$ is one element subalgebra of A_i. (The situation is easier if the one element algebra is given by a nullary operation in V - no choice is needed in this case. If not, then we may require some other properties of distinguished elements later.)

We denote $supp(f) = \{i;\ f(i) \neq e_{A_i}\}$ for $f \in \prod(A_i,\ i \in I)$. Now let us consider the following subset B of $\prod(A_i,\ i \in I)$:

$$B = \left\{f \in \prod(A_i,\ i \in I);\ supp(f) \text{ is finite}\right\}.$$

It is easy to check that B is a subalgebra of a direct product $\prod(A_i,\ i \in I)$ and in fact that it is a direct sum of algebras A_i. We shall denote it as $\sum((A_i,\ e_{A_i});\ i \in I)$.

It is clear that for finite index set I, we do not need distinguished elements for the definition and the algebra $\sum((A_i,\ e_{A_i});\ i \in I)$ is standard direct product. But (distinguished) elements which form one element subalgebras are still necessary for our proofs also in the case of finite index set I.

Examples. (1) Let V be a variety of groups, $e_i \in G_i$ be an identity element. Then $\sum((G_i,\ e_i);\ i \in I)$ is a direct sum of groups (in a usual way).

(2) Let V be a variety of distributive lattices with top element ($\{1\}$-lattices), $1_i \in A_i$ be a top element. Then $\sum((A_i,\ 1_i);\ i \in I)$ is a direct sum of these lattices. It is easy to see that if we take $A_i = \{0, 1\}$ for $i \in I$, then the map φ defined by $\varphi(f) = \{i \in I;\ f(i) = 1\}$ is an isomorphism between this direct sum and the lattice of all cofinite subsets of the set I.

(3) Let V be a variety of unbounded distributive lattices. Let us take lattices $L_i = C_3 = \{0, a, 1\}$ (3 element chains). Then $\sum((L_i,\ a);\ i \in I)$ is a direct sum of these lattices (it means that a is a distinguished element in each L_i).

(4) Let V be a variety of Brouwerian algebras. Let us take algebras

$L_i = C_{n_i} = \{0, a_1, ..., a_{n_i - 2}, 1\}$ (n_i element chains). Take $e_i = 1$ (greatest element) for each L_i (as $\{1\}$ is the only one element subalgebra, there is no other option). Then $\sum((L_i, e_i); i \in I)$ is a direct sum of these algebras.

We can state theorem which provides the main information for direct sums over finite and also infinite index sets.

Theorem 3.1. *Let V be a variety with factorable congruences in which every algebra A has one element subalgebra, $A_i \in V$, $i \in I$ be algebras with distinguished elements e_{A_i} (it means that $\{e_{A_i}\}$ are subalgebras of A_i for $i \in I$).*

If all A_i, $i \in I$ have SEKP and for every $\iota_{A_i} \neq \theta \in \mathrm{Con}(A_i)$, there is a strong endomorphism φ such that $\varphi(e_{A_i}) = e_{A_i}$ and $\ker(\varphi) = \theta$, then $\sum((A_i, e_{A_i}); i \in I)$ has SEKP.

Conversely, if $\sum((A_i, e_{A_i}); i \in I)$ has SEKP, then each A_i, $i \in I$ has SEKP.

Proof. Let us use denote $B = \sum((A_i, e_{A_i}); i \in I)$. Let $\iota_B \neq \theta \in \mathrm{Con}(B)$. Our assumptions and Theorem 2.5 imply that there are congruences $\theta_i \in \mathrm{Con}(A_i)$ such that $\theta = B^2 \cap \prod(\theta_i; i \in I)$. For every $i \in I$ such that $\theta_i \neq \iota_{A_i}$, take $\varphi_i \in \mathrm{End}(A_i)$ which is strong on A_i, $\ker(\varphi_i) = \theta_i$ and $\varphi_i(e_{A_i}) = e_{A_i}$.

If $\theta_i = \iota_{A_i}$, then we can take $\varphi_i : A_i \to A_i$ in such a way that for $x \in A_i$, $\varphi_i(x) = e_{A_i}$. It is clear that $\varphi_i \in \mathrm{End}(A_i)$, $\ker(\varphi_i) = \iota_{A_i} = \theta_i$, φ_i is strong on A_i and also $\varphi_i(e_{A_i}) = e_{A_i}$.

Let $f \in B$. Then it means that $supp(f)$ is finite. Take a map $\varphi : B \to \prod(A_i; i \in I)$ is given by $\varphi(f)(i) = \varphi_i(f(i))$. It is easy to see that

$supp(\varphi(f)) \subseteq supp(f)$, because for all $i \in I$, we have $\varphi_i(e_{A_i}) = e_{A_i}$ so that $supp(\varphi(f))$ is finite. Moreover, we know that all projections $\pi_i : B \to A_i$ are surjective. Therefore, φ is in fact a map $\varphi : B \to B$.

Clearly, φ is an endomorphism of B. Let $f, g \in B$. Then we have

$$\begin{aligned}(f, g) \in \ker(\varphi) &\Leftrightarrow (\forall i \in I)(f(i), g(i)) \in \ker(\varphi_i)\\ &\Leftrightarrow (\forall i \in I)(f(i), g(i)) \in \theta_i\\ &\Leftrightarrow (f, g) \in \theta.\end{aligned}$$

Now, let $\iota_b \neq \Phi \in \mathrm{Con}(B)$, $\Phi_i \in \mathrm{Con}(A_i)$ be such that $\Phi = B^2 \cap \prod(\Phi_i;\, i \in I)$.

Let $(f, g) \in \Phi$. Let $i \in I$. Then $(f(i), g(i)) \in \Phi_i$ and therefore $(\varphi_i(f(i)), \varphi_i(g(i))) \in \Phi_i$, because each φ_i is strong on A_i so that $(\varphi(f), \varphi(g)) \in \Phi$.

For the converse, let $C = \sum((A_i, e_{A_i});\, i \in I)$ has SEKP. Let $i \in I$. It is clear that $C \cong A_i \times \sum((A_j, e_{A_j});\, j \in I\backslash\{i\})$.

Denote $A = A_i$ and $B = \sum((A_j, e_{A_j});\, j \in I\backslash\{i\})$. Let $A \times B$ has SEKP. We shall prove that A (B, respectively) has SEKP. Let A and B have more than one element. It is clear that B has one element subalgebra, namely, $b = f \in \prod(A_i,\, i \in I\backslash\{i\})$ such that for $i \in I\backslash\{i\}$, we have $f(i) = e_{A_i}$.

Let $\iota_A \neq \theta \in \mathrm{Con}(A)$. Then $\theta \times \iota_B$ is a non-trivial congruence on $A \times B$. Therefore, we have an endomorphism $f \in \mathrm{End}(A \times B)$ which is strong on $A \times B$ and $\ker(f) = \theta \times \iota_B$.

Let $\{b\}$ be one element subalgebra of B. Then every basic n-ary operation g of B is idempotent on an element b, it means that $g(b, ..., b) = b$. Therefore, $inj_b : A \to A \times B$ given by $inj_b(x) = (x, b)$ is a homomorphism. Let $pr_A : A \times B \to A$ be a projection, i.e., $pr_A(x, y) = x$.

Let $f_1 : A \to A$ be a composition $f_1 = pr_A \circ f \circ inj_b$. Then it means that $f_1(x) = pr_A(f(x, b))$. As a composition of homomorphisms, $f_1 \in \mathrm{End}(A)$.

Let us check that $\ker(f_1) = \theta$. If $(x, y) \in \theta$, then $((x, b), (y, b)) \in \theta \times \iota_B$. Therefore, $f(x, b) = f(y, b)$ and also $f_1(x) = pr_A(f(x, b)) = pr_A(f(y, b)) = f_1(y)$. It means that $\theta \subseteq \ker(f_1)$.

For the other inclusion, we shall start with a small consideration. By the assumption, *f* is strong on $A \times B$, it means that for the congruence $\iota_A \times \omega_B$, the following holds: if $((x, b), (y, b)) \in \iota_A \times \omega_B$, then $(f(x, b), f(y, b)) \in \iota_A \times \omega_B$. This means that for any $x, y \in A$, we have $pr_B(f(x, b)) = pr_B(f(y, b)) - pr_B$ is a projection $pr_B : A \times B \to B$ here.

Now, let $(x, y) \in \ker(f_1)$ so that

$$pr_A(f(x, b)) = f_1(x) = f_1(y) = pr_A(f(y, b)).$$

By the previous arguments, we have also $pr_B(f(x, b)) = pr_B(f(y, b))$ and therefore $f(x, b) = f(y, b)$. This means that $((x, b), (y, b)) \in \theta \times \iota_B$ and by this, we have $\ker(f_1) \subseteq \theta$.

We shall prove that f_1 is strong on *A* now. Let $\Phi \in \mathrm{Con}(A)$, $(x, y) \in \Phi$. Then $((x, b), (y, b)) \in \Phi \times \omega_B$. As *f* is strong on $A \times B$, we have $(f(x, b), f(y, b)) \in \Phi \times \omega_B$. This means that

$$(f_1(x), f_1(y)) = (pr_A(f(x, b)), pr_A(f(y, b))) \in \Phi$$

(we can use any congruence on *B* instead of ω_B here). □

For varieties where one element subalgebras are defined by a nullary operation, we have

Corollary 3.2. *Let V be a variety with factorable congruences with one nullary operation e in which every algebra A has one element subalgebra given by a value* e_A *of the nullary operation e in A. Then it means that distinguished elements are* e_A.

Let $A_i \in V$, $i \in I$. *Then* $\sum((A_i, e_{A_i}); i \in I)$ *has SEKP in the variety V if and only if each* A_i, $i \in I$ *has SEKP in the variety V.*

Proof. Homomorphisms in this variety must preserve nullary operation and therefore the special condition required by Theorem 3.1 concerning strong endomorphisms and distinguished elements is automatically satisfied.

□

As an example, let us take $A_i = \{0, 1\}$, $i \in I$ in the variety of distributive lattices with top element, it is easy to see that each A_i has SEKP and therefore $\sum((A_i, 1); i \in I)$ has SEKP. As it is mentioned in the example (2), $\sum((A_i, 1); i \in I)$ is isomorphic to the lattice of all cofinite subsets of the set *I* (and dually to the lattice of all finite subsets for $\{0\}$-lattices). In fact, unbounded distributive lattices and SEKP would be good candidate to consider, but these were fully characterized in [9] by means of a Priestley duality.

Let us turn to the case of EKP now. Blyth et al. have Theorems 2 and 3 in [1] which state:

Theorem 3.3 [1, Theorem 2]. *Let V be a variety that has factorable congruences and let* $A_1, ..., A_n$ *be algebras in V such that for any* $i \neq j$, *there exists a homomorphism* $f_{ij} : A_i \to A_j$. *Then if each* A_i *has EKP so also does* $\prod_{i=1}^{n} A_i$.

Theorem 3.4 [1, Theorem 3], see also [8, Theorem 4]. *Let V be a variety that has factorable congruences and in which every subalgebra of a directly indecomposable algebra is also directly indecomposable. If* $A_1, ..., A_n$ *are non-trivial directly indecomposable algebras in V, then the following statements are equivalent*:

(i) $\prod_{i=1}^{n} A_i$ *has EKP*,

(ii) *each* A_i *has EKP and there exists a homomorphism* $f_{ij} : A_i \to A_j$ *for each* $i \neq j$.

As an analog of Theorem 3.3, we can state

Theorem 3.5. *Let V be a variety with factorable congruences in which every algebra A has one element subalgebra*, $A_i \in V$, $i \in I$ *be algebras with distinguished elements* e_{A_i} *(it means that* $\{e_{A_i}\}$ *are subalgebras of* A_i *for* $i \in I$).

If all A_i, $i \in I$ *have EKP and for every* $\iota_{a_i} \neq \theta \in \mathrm{Con}(A_i)$, *there is an endomorphism* φ *such that* $\varphi(e_{A_i}) = e_{A_i}$ *and* $\ker(\varphi) = \theta$, *then* $\sum((A_i, e_{A_i});\ i \in I)$ *has EKP*.

Proof. Proof is almost identical to the proof of the first part of Theorem 3.1, we only need to omit the word "strong" where appropriate. It is also not necessary to check compatibility of constructed endomorphisms with other congruences. □

We also have a simplification for varieties where one element subalgebras are defined by a nullary operation.

Corollary 3.6. *Let V be a variety with factorable congruences with one nullary operation e in which every algebra A has one element subalgebra given by a value* e_{A_i} *of the nullary operation e in* A_i. *Then it means that distinguished elements are* e_{A_i}.

Let $A_i \in V$, $i \in I$. *If each* A_i, $i \in I$ *has EKP in the variety V, then* $\sum((A_i, e_{A_i});\ i \in I)$ *has EKP in the variety V.*

As an example, we can take finite chains C_{n_i}, $i \in I$ with top elements in the variety of distributive lattices with top element. By [1, Example 1], each C_{n_i} has EKP as $\{1\}$-lattice (even as a bounded lattice). Therefore, $\sum((C_{n_i}, 1);\ i \in I)$ has EKP as $\{1\}$-lattice.

4. Brouwerian and Relative Stone Algebras

We are going to show that all finite relative Stone algebras have SEKP.

Brouwerian algebra is an algebra $(L, \vee, \wedge, *, 1)$ of type $(2, 2, 2, 0)$ such that $(L, \vee, \wedge)$ is (necessarily distributive) lattice with greatest element 1 and $a * b$ is a greatest element of the set $\{x \in L;\ a \wedge x \leq b\}$ (a relative pseudocomplement of a in b).

Brouwerian algebras form a variety with factorable congruences (because of the lattice reduct) with one element subalgebras $\{1\}$, therefore this variety is suitable for Theorems 3.1 and 3.5.

Brouwerian algebra is called *relative Stone algebra*, if every its (closed) interval is a Stone algebra or equivalently, if it satisfies the equality

$$(x * y) \vee (y * x) = 1$$

(see [11]). By the results in [10], relative Stone algebras form a variety denoted by $\mathscr{K}_\infty$ and the lattice of subvarieties of $\mathscr{K}_\infty$ is a chain

$$\mathscr{K}_1 \subset \mathscr{K}_2 \subset \cdots \subset \mathscr{K}_n \subset \cdots \subset \mathscr{K}_\infty$$

of length $\omega + 1$. Here $L \in \mathscr{K}_n$ $(n \geq 2)$ if and only if L satisfies the equation E_n:

$$(x_1 * x_2) \vee (x_2 * x_3) \vee \cdots \vee (x_n * x_{n+1}) = 1.$$

Heyting algebra is an algebra $(L, \vee, \wedge, *, 0, 1)$ of type $(2, 2, 2, 0, 0)$ such that $(L, \vee, \wedge, *, 1)$ is a Brouwerian algebra and 0 is the smallest element of the lattice $(L, \vee, \wedge)$.

L-algebra is a Heyting algebra $(L, \vee, \wedge, *, 0, 1)$, for which $(L, \vee, \wedge, *, 1)$ is a relative Stone algebra. By the results in [10], **L**-algebras form a variety denoted by $\mathscr{L}_\infty$ and the lattice of subvarieties of $\mathscr{L}_\infty$ is a chain

$$\mathscr{L}_1 \subset \mathscr{L}_2 \subset \cdots \subset \mathscr{L}_n \subset \cdots \subset \mathscr{L}_\infty$$

of length $\omega + 1$ and similarly as for relative Stone algebras, $L \in \mathscr{L}_n$ $(n \geq 2)$ if and only if L satisfies the equation E_n.

We can define a unary operation $*$ on a Heyting algebra by putting $x^* = x * 0$. An element x of a Heyting algebra L is closed, if $x = x^{**}$ and dense, if $x^* = 0$.

The set $C(L) = \{a \in L;\ a = a^{**}\}$ of all closed elements is considered as an algebra $(C(L), \sqcup, \wedge, *, 0, 1)$, where $\wedge$, $*$, 0, 1 are restrictions of corresponding operations from L and $a \sqcup b = (a^* \wedge b^*)^*$ is a Boolean algebra.

The set $D(L) = \{a \in L;\ a^* = 0\}$ of all dense elements is a filter of L which itself is a Brouwerian algebra, more precisely Brouwerian subalgebra of original Heyting algebra L.

The following theorem will be useful.

Theorem 4.1 [10, Theorem 5]**.** *Let L be a Heyting algebra. Then* $L \in \mathscr{L}_n$ $(n \geq 2)$ *if and only if the following two conditions are fulfilled*:

(1) $C(L)$ *is a subalgebra of* L,

(2) $D(L) \in \mathscr{K}_{n-1}$.

The characterization of congruences of a Brouwerian algebra was given in [13] as

Lemma 4.2. *Let L be a Brouwerian algebra. If* θ *is a congruence on* L, *then* $[1]\theta = \{x \in L;\ (x, 1) \in \theta\}$ *is a filter on L and*

$$(x, y) \in \theta \text{ if and only if } x \wedge d = y \wedge d \text{ for some } d \in [1]\theta$$

and conversely, if F is a filter on L, then the relation θ_F *given by*

$$(x, y) \in \theta_F \text{ if and only if } x \wedge d = y \wedge d \text{ for some } d \in F$$

is a congruence on L with $[1]\theta_F = F$.

Let C_n be an n element chain $0 < a_1 < \cdots < a_{n-2} < 1$ considered as a

Brouwerian algebra (it means that if $a \leq b$, then $a * b = 1$, if $a > b$, then $a * b = b$). Let $\theta \in \mathrm{Con}(L)$, denote $F = [1]\theta$. F is a filter, it means that there exists $a \in C_n$ such that $F = [a) = \{x \in C_n;\ a \leq x\}$. Clearly, blocks of the congruence θ (congruence classes) are singletons $\{b\}$ for $b < a$ and the set $[a)$. We can take a mapping $f_a : C_n \to C_n$ defined by

$$f_a(x) = \begin{cases} x, & \text{if } x < a, \\ 1, & \text{if } x \in [a). \end{cases}$$

It is easy to see that f_a is an endomorphism, $\ker(f_a) = \theta$. For every $x \in L$, we have $x \leq f(x)$, therefore it is compatible with every other congruence of L (given by another filter $[b)$) which means that it is a strong endomorphism. Therefore, C_n has SEKP as a Brouwerian algebra and also as a Heyting algebra. By Theorem 3.1, we know that any finite product and even any direct sum of finite chains with 1 as distinguished elements considered as a Brouwerian algebra have SEKP (which is not true for a finite product of at least two non-trivial finite chains considered as a Heyting algebra by Corollary 1.2). As we shall see later, any finite Brouwerian algebras can be used in place of finite chains in this example.

It is known that a non-trivial algebra L is subdirectly irreducible Brouwerian algebra if and only if it has exactly one coatom and L is subdirectly irreducible relative Stone algebra if and only if it is a chain with a coatom.

Lemma 4.3. *Let L be a finite subdirectly irreducible Brouwerian algebra which is not relative Stone. Then L does not have EKP (SEKP).*

Proof. Algebra L is not a chain and it has one coatom. Let a be the element such that $[a)$ is a chain and there are elements $a_1 \neq a_2$ such that a covers both a_1, a_2 (it means that $a = a_1 \vee a_1$ and L can be written as a glued sum $L = (a] \dotplus [a)$ - see [6, p. 8], a is the greatest element of L which is not join irreducible). Take $F = [a)$ and the congruence θ_F. Blocks of this congruence are singletons $\{x\}$ for $x < a$ and the set $[a)$. Therefore, we have $L/\theta \cong (a]$, but clearly, $(a]$ is not (isomorphic to) a subalgebra of L. By [1, Theorem 1], L does not have EKP. □

Let L be a Brouwerian algebra with the smallest element 0. We shall call it *dense* if L has exactly one atom. If L is finite, then this is equivalent to the fact that $D(L) = L\setminus\{0\}$ or to the fact that $C(L) = \{0, 1\}$.

Next structure theorem is an analog of the structure theorem for finite Stone algebras ([6, Corollary 213], originally in [7]).

Theorem 4.4. *Let L be a non-trivial finite relative Stone algebra. Then it is a product of finitely many dense relative Stone algebras. If* $L \in \mathscr{L}_n$ $(L \in \mathscr{K}_n)$, *then all components of this direct product also belong to* $\mathscr{L}_n$ $(\mathscr{K}_n)$.

Proof. As L is finite, it has smallest element, therefore we can consider it as a Heyting algebra, as it is relative Stone, it is in fact an L-algebra. For a finite **L**-algebra L, we have n such that $L \in \mathscr{L}_n$. By Theorem 4.1, Boolean algebra $C(L)$ is a subalgebra of L. Let L be not dense. Take $a \in C(L)$, $0 \neq a \neq 1$. Then a, a^* is a pair of complemented elements, therefore $L \cong (a] \times [a)$ as a distributive lattice.

Every finite distributive lattice can be converted (in a unique way) to a Brouwerian (and also to a Heyting) algebra - the operation $*$ is uniquely determined by the lattice structure of L on L, by the lattice structure of $(a]$ on $(a]$ and by the lattice structure of $[a)$ on $[a)$. Therefore, the Heyting algebra L is a product of Heyting algebras $(a]$ and $[a)$. But this means that both Heyting algebras $(a]$ and $[a)$ are homomorphic images of a relative Stone algebra (**L**-algebra, in fact) L and therefore both are relative Stone algebras (**L**-algebras) which belong to the same variety $\mathscr{L}_n$. We can continue with this process until we get algebras with $C(-) = \{0, 1\}$. Therefore, L is a product of dense algebras which all belong to $\mathscr{L}_n$ $(\mathscr{K}_n)$. □

Now we will examine congruences on a dense Brouwerian algebra L.

Lemma 4.5. *Let* L *be a dense Brouwerian algebra*, $a \in L$ *be a unique atom of* L. *If* $\theta \in \mathrm{Con}(L)$ *is such that* $(0, a) \in \theta$, *then* $\theta = \iota_L$.

Proof. The only element $d \in L$ with the property $0 \wedge d = a \wedge d$ is $d = 0$ and therefore by Lemma 4.2, $0 \in [1]\theta$ which means that $\theta = \iota_L$. □

Now, let L be a dense Brouwerian algebra, $a \in L$ be a unique atom of L. Then it means that $D(L) = [a)$. $L_1 = D(L)$ is a Brouwerian subalgebra of L, we can write L using an ordered sum (see [6, p. 8]) as $L = \{0\} + L_1$.

Let $\theta \in \mathrm{Con}(L_1)$. Then $[1]\theta \subseteq L_1$ is also a filter on L and therefore $\overline{\theta} = \theta \cup \{(0, 0)\}$ is a congruence on L.

Let $\overline{\theta} \in \mathrm{Con}(L)$, $\overline{\theta} \neq \iota_L$. Then $[1]\overline{\theta} \subseteq L_1$ and therefore $\theta = \overline{\theta} \cap L_1^2$ is a congruence on L_1.

Using this, we can prove a lemma.

Lemma 4.6. *Let* L *be a dense Brouwerian algebra*, $L_1 = D(L)$. *If a Brouwerian algebra* L_1 *has SEKP*, *then also a Brouwerian algebra* L *has SEKP* (*even as a Heyting algebra*).

Proof. If $a \in L$ is the atom, then $L_1 = D(L) = [a)$ and $L = \{0\} + [a)$.

Let $\overline{\theta} \in \mathrm{Con}(L)$, $\theta \neq \iota_L$. Take $\theta = \overline{\theta} \cap L^2$. L_1 has SEKP, therefore for the congruence $\theta \neq \iota_{L_1}$ of L_1, there is a strong endomorphism $f : L_1 \to L_1$ such that $\ker(f) = \theta$, for $\theta = \iota_{L_1}$, we can take $f : L_1 \to L_1$ defined by $f(x) = 1$ for all $x \in L_1$. This f is also a strong endomorphism with the property $\ker(f) = \theta$. Take $\bar{f} : L \to L$ defined by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in [a), \\ 0, & \text{if } x = 0. \end{cases}$$

Routine computations show that $\bar{f}$ is an endomorphism of L and $\ker(\bar{f}) = \overline{\theta}$.

We shall show that it is compatible with every (other) congruence. Let $\overline{\Psi} \in \mathrm{Con}(L)$. If $\overline{\Psi} = \iota_L$, then clearly, $\bar{f}$ is compatible with $\overline{\Psi}$.

Let $\overline{\Psi} \neq \iota_L$, we know that $\overline{\Psi} = \Psi \cup \{(0, 0)\}$, where $\Psi = \overline{\Psi} \cap L_1^2$. Let $(b, c) \in \overline{\Psi}$. If $b = 0 = c$, then $\bar{f}(b) = 0 = \bar{f}(c)$, it means that $(\bar{f}(b), \bar{f}(c)) \in \overline{\Psi}$.

Let $b \neq 0 \neq c$. Then $f(b), f(c) \in L_1$ and as f is a strong endomorphism on L_1, we see that $\bar{f}(b) = f(b) \equiv_\Psi f(c) = \bar{f}(c)$, it means that $(\bar{f}(b), \bar{f}(c)) \in \overline{\Psi}$. □

We are ready to prove the main result of this section.

Theorem 4.7. *Let L be a non-trivial finite relative Stone algebra. Then it has SEKP.*

Proof. We shall combine previous statements and proceed by an induction.

If a relative Stone algebra L is finite, then there is n such that $L \in \mathscr{K}_n$. The induction will go through n.

Let $n = 2$, $L \in \mathscr{K}_2$. As L has the smallest element, in fact, we know that $L \in \mathscr{L}_2$, and by Theorem 4.1, $D(L) \in \mathscr{K}_1$ which means that $D(L)$ is trivial and $L = C(L)$ which is a Boolean algebra. It means that a Brouwerian algebra L is a finite product of some copies of C_2 considered as a Brouwerian algebra. We see that L has SEKP by Theorem 3.1.

For the induction. Let every finite algebra in $\mathscr{K}_n$ has SEKP (it means as a Brouwerian algebra). Let $L \in \mathscr{K}_{n+1}$ be finite. We can consider it as an L-algebra, $L \in \mathscr{L}_{n+1}$.

By Theorem 4.4, we can write $L \cong L_1 \times \cdots \times L_k$, where each L_i is dense and $L_i \in \mathscr{L}_{n+1}$ (for $i = 1, ..., k$).

By Theorem 4.1, we know that $D(L_i) \in \mathscr{K}_n$ for $i = 1, ..., k,$ therefore each $D(L_i)$ has SEKP by an induction assumption. By Lemma 4.6, each L_i has SEKP as well. Therefore, L has SEKP by Theorem 3.1. □

Corollary 4.8. *Let* L *be a non-trivial finite* **L**-*algebra*. *Then* L *has SEKP if and only if it is dense*.

Proof. A non-trivial finite **L**-algebra has two nullary operations $0 \neq 1$ and by Corollary 1.2, if L has SEKP, then L is directly indecomposable. Therefore, L is dense by Theorem 4.4.

Let L be a dense **L**-algebra. Then $D(L)$ is a finite relative Stone algebra which has SEKP by Theorem 4.7 and therefore L has SEKP by Lemma 4.6.

□

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