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ON SOLITON SOLUTIONS FOR A CLASS OF DISCRETE SCHRÖDINGER SYSTEMS

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Abstract

In this paper, we consider a class of discrete Schrödinger systems. We divide the discussion into two cases. In the first case, we consider the system with unbounded potentials, the existence of a nontrivial solution is obtained, both of its two components are nonzero. In the second case, we consider the system with radially symmetric coefficients, by proving a compactness result, we get the existence of a nontrivial radially symmetric solution.

1. Introduction

Soliton solutions for discrete Schrödinger equations and systems on

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infinite lattices have been studied by many authors from the aspects of mathematics in the last decades. In [17], the following equation was considered

$$a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n - \omega u_n = \sigma \chi_n |u_n|^2 u_n, \quad n \in \mathbf{Z},$$

where a_n, b_n and χ_n are periodic, the author proved the existence of gap solutions (standing waves with frequency ω in a spectral gap). This result was improved in [18] by proving the existence of ground state solutions. In [22], problems of the form

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \sigma \chi_n g_n(u_n) u_n, \quad n \in \mathbf{Z},$$

were studied, where $\Delta u_n = u_{n+1} + u_{n-1} - 2u_n$, ε_n and χ_n were assumed to be periodic and the existence of soliton solutions when ω is a lower edge of a finite spectral gap was proved via the generalized linking method. In [25], the following equation was considered:

$$a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n - \omega u_n = \sigma f_n(u_n), \quad n \in \mathbf{Z},$$

where a_n and b_n are periodic, the authors obtained the existence of soliton solutions when ω is below some constant. In [24], discrete Schrödinger equation in infinite m dimensional lattices of the form

$$-(\mathcal{A}u)_{\mathbf{n}} + v_{\mathbf{n}} u_{\mathbf{n}} - \omega u_{\mathbf{n}} - \sigma \gamma_{\mathbf{n}} f(u_{\mathbf{n}}) = 0, \quad \mathbf{n} \in \mathbf{Z}^m$$

was considered, where $\mathcal{A}$ is the discrete Laplacian operator on $\mathbf{Z}^m$ (see (1.2) below) and $\lim_{|\mathbf{n}| \to \infty} v_{\mathbf{n}} = \infty$. Under some growth conditions on f the author got the existence of the so-called nontrivial breather solutions. In [14], the authors considered the following system:

$$\begin{cases} \varepsilon(\mathcal{A}\varphi)_k + \varepsilon\omega_1 \varphi_k - V_k^1 \varphi_k + V_k^2 \psi_k - a_1 \varphi_k^3 + a_2 \psi_k^2 \varphi_k = 0, \\ \varepsilon(\mathcal{A}\psi)_k + \varepsilon\omega_2 \psi_k - V_k^3 \psi_k + V_k^2 \varphi_k - a_3 \psi_k^3 + a_2 \varphi_k^2 \psi_k = 0, \end{cases} \quad k \in \mathbf{Z},$$

where $\lim_{k \to \infty} V_k^j = \infty$ for $j = 1, 2, 3,$ they obtained the existence of standing waves via the Nehari manifold approach. There are many other works or survey on the discrete Schrödinger equations and systems, for example, [7, 8, 10, 11, 20, 23] and so on.

In this paper, we consider the following discrete Schrödinger system:

$$\begin{cases} -(\mathcal{A}u)_{\mathbf{n}} + \varepsilon_{\mathbf{n}} u_{\mathbf{n}} = u_{\mathbf{n}}^3 - \beta u_{\mathbf{n}} v_{\mathbf{n}}, \\ -(\mathcal{A}v)_{\mathbf{n}} + \tau_{\mathbf{n}} v_{\mathbf{n}} = \frac{1}{2} | v_{\mathbf{n}} | v_{\mathbf{n}} - \frac{\beta}{2} u_{\mathbf{n}}^2, \end{cases} \quad \mathbf{n} \in \mathbf{Z}^m. \tag{1.1}$$

Here the operator $\mathcal{A}$ is defined as

$$(\mathcal{A}\psi)_{\mathbf{n}} = \psi_{\mathbf{n}-e_1} + \psi_{\mathbf{n}-e_2} + \cdots + \psi_{\mathbf{n}-e_m} - 2m\psi_{\mathbf{n}} + \psi_{\mathbf{n}+e_1}$$

$$+ \psi_{\mathbf{n}+e_2} + \dots + \psi_{\mathbf{n}+e_m}, \tag{1.2}$$

where $e_i = (0, ..., 1, ..., 0)$ the unit vector in $\mathbf{R}^m$ with the ith component 1 for $i = 1, ..., m$. For an element $\mathbf{n} = (n_1, n_2, ..., n_m) \in \mathbf{Z}^m$, $| \mathbf{n} | = | n_1 | + | n_2 | + \cdots + | n_m |$.

The discrete Schrödinger system (1.1) can be viewed as the discretization of some Schrödinger-KdV systems (see [13]). As for the continuous Schrödinger-KdV system, there are many references, see, for example, [2, 3, 5, 6, 9, 12, 13, 15, 16, 19].

We assume the following conditions:

(C1) $\lim_{|\mathbf{n}| \to \infty} \varepsilon_{\mathbf{n}} = \infty$, $\lim_{|\mathbf{n}| \to \infty} \tau_{\mathbf{n}} = \infty$ and $\varepsilon_{\mathbf{n}} > 0$, $\tau_{\mathbf{n}} > 0$ for all $\mathbf{n} \in \mathbf{Z}^m$.

(C2) $\varepsilon_{\mathbf{n}} = \varepsilon_{\mathbf{m}}$, $\tau_{\mathbf{n}} = \tau_{\mathbf{m}}$ when $| \mathbf{n} | = | \mathbf{m} |$, $\mathbf{n}, \mathbf{m} \in \mathbf{Z}^m$.

(C3) $\varepsilon_{\mathbf{n}} \geq a_1$, $\tau_{\mathbf{n}} \geq a_2$, $\forall \mathbf{n} \in \mathbf{Z}^m$, $a_i > 0$, $i = 1, 2$ are constants.

Using the Nehari manifold technique, we first prove the following result in the unbounded potential case.

Theorem 1.1. *If* $|\beta| > \gamma_1$ *with the constant* γ_1 *defined in* (2.12), *problem* (1.1) *possesses a nontrivial solution* $\{(u_{\mathbf{n}}, v_{\mathbf{n}})\}_{\mathbf{n}\in\mathbf{Z}^m}$ *under the condition* (C1).

Then we prove the following result in the radially symmetric case.

Theorem 1.2. *Suppose the conditions* (C2), (C3) *hold. If* $|\beta| > \gamma_2$ *with the constant* γ_2 *defined in* (3.15), *problem* (1.1) *possesses a nontrivial radially symmetric solution* $\{(u_{\mathbf{n}}, v_{\mathbf{n}})\}_{\mathbf{n}\in\mathbf{Z}^m}$ *provided* $m \geq 2$.

Remark. In Theorems 1.1 and 1.2, a nontrivial solution means a solution $(\mathbf{u}, \mathbf{v}) = \{(u_{\mathbf{n}}, v_{\mathbf{n}})\}_{\mathbf{n}\in\mathbf{Z}^m}$ with $\mathbf{u} \neq 0$ and $\mathbf{v} \neq 0$. We note that system (1.1) possesses two nonzero solutions of the form $(0, \pm\mathbf{v})$ by solving a single equation (see Theorem 2.6 and Theorem 3.7 below).

2. The Unbounded Potential Case

2.1. Functional framework

In this subsection, we explain the framework of our problem. We adopt the notation defined in [7, 11] and [24]. For a positive integer *m*, we consider the real sequence spaces

$$l^p \equiv l^p(\mathbf{Z}^m)$$

$$= \left\{ u = \{u_n\}_{n\in\mathbf{Z}^m} \mid \forall n \in \mathbf{Z}^m,\ u_n \in \mathbf{R},\ \|u\|_{l^p} = \left(\sum_{n\in\mathbf{Z}^m} |u_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Between l^p spaces, the following elementary embedding relation holds:

$$l^q \subset l^p, \quad \|u\|_{l^p} \leq C\|u\|_{l^q}, \quad 1 \leq q \leq p \leq \infty.$$

Define $L_\nu^\pm : l^2 \to l^2$ for each $\nu = 1, \ldots, m$ as

$$(L_\nu^\pm u)_n = u_{n\pm e_\nu} - u_n.$$

Note that $(L_\nu^{\pm} u, v)_{l^2} = (u, L_\nu^{\pm} v)_{l^2}$, $\forall u, v \in l^2$, we can rewrite the operator $\mathcal{A}$ as

$$(-\mathcal{A}u, v)_{l^2} = \sum_{\nu=1}^{m} (L_\nu^{+} u, L_\nu^{+} v)_{l^2}, \quad \forall u, v \in l^2.$$

Hence

$$0 \le (-\mathcal{A}u, u)_{l^2} \le 4m \| u \|_{l^2}^2, \quad \forall u \in l^2,$$

thus $-\mathcal{A} : l^2 \to l^2$ is a bounded operator and $\sigma(-\mathcal{A}) \subset [0, 4m]$.

Let us denote $V_1 = \{\varepsilon_n\}_{n \in \mathbf{Z}^m}$, $V_2 = \{\tau_n\}_{n \in \mathbf{Z}^m}$ and define $H_1 = -\mathcal{A} + V_1$ and $H_2 = -\mathcal{A} + V_2$ which are self-disjoint operators defined on l^2. Let

$$E_1 = \{u \in l^2 \mid H_1^{\frac{1}{2}} u \in l^2\}, \quad \| u \|_{E_1} = \| H_1^{\frac{1}{2}} u \|_{l^2},$$

$$E_2 = \{u \in l^2 \mid H_2^{\frac{1}{2}} u \in l^2\}, \quad \| u \|_{E_2} = \| H_2^{\frac{1}{2}} u \|_{l^2}.$$

Then, by Theorem 2.2 in [24], E_1 and E_2 are compactly embedded into l^p for $p \in [2, +\infty]$.

Define $\Phi_1 : E_1 \times E_2 \to \mathbf{R}$ as

$$\Phi_1(u, v)$$

$$= \frac{1}{2}(H_1 u, u)_{l^2} + \frac{1}{2}(H_2 v, v)_{l^2} - \frac{1}{4} \sum_{n \in \mathbf{Z}^m} u_n^4 - \frac{1}{6} \sum_{n \in \mathbf{Z}^m} | v_n |^3 + \frac{\beta}{2} \sum_{n \in \mathbf{Z}^m} u_n^2 v_n.$$

Then, by the comments above, the sums in the definition of Φ_1 are finite. Moreover, we have $\Phi_1 \in C^1(E_2 \times E_2, \mathbf{R})$.

Define $J_1 : E_1 \times E_2 \to \mathbf{R}$ as

$$J_1(u, v)$$

$$= (H_1 u, u)_{l^2} + (H_2 v, v)_{l^2} - \sum_{n\in\mathbf{Z}^m} u_n^4 - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n.$$

Then we can define the Nehari manifold as follows:

$$\mathcal{M}_1 = \{(u, v) \in E_1 \times E_2 \backslash\{(0, 0)\} \,|\, J_1(u, v) = 0\}.$$

We have the following result on $\mathcal{M}_1$.

Lemma 2.1. *$\mathcal{M}_1$ is a C^1 complete manifold. For every $(u, v) \in E_1 \times E_2\backslash\{(0, 0)\}$, there exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}_1$. Moreover, the maximum of $g(t) = \Phi_1(tu, tv)$ for $t \geq 0$ is achieved at $\bar{t}$.*

Proof. For $(u, v) \in E_1 \times E_2\backslash\{(0, 0)\}$, we consider the equation on $t > 0$:

$$J_1(tu, tv) = t^2(H_1 u, u)_{l^2} + t^2(H_2 v, v)_{l^2} - t^4\sum_{n\in\mathbf{Z}^m} u_n^4$$

$$-\frac{t^3}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{3t^3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n = 0.$$

Hence

$$\varrho(t) := (H_1 u, u)_{l^2} + (H_2 v, v)_{l^2}$$

$$- t^2\sum_{n\in\mathbf{Z}^m} u_n^4 - \frac{t}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{3t\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n$$

$$= \left(-\sum_{n\in\mathbf{Z}^m} u_n^4\right)t^2 + \left(\frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 u_n - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3\right)t$$

$$+ (H_1 u, u)_{l^2} + (H_2 v, v)_{l^2} = 0. \tag{2.3}$$

If $-\sum_{n\in\mathbf{Z}^m} u_n^4 \neq 0$, since $\varrho(0) = (H_1 u, u)_{l^2} + (H_2 v, v)_{l^2} > 0$, there is a unique

$\bar{t} > 0$ such that (2.3) holds, that is to say, $(\bar{t}u, \bar{t}v) \in \mathcal{M}_1$. If $-\sum_{n\in\mathbf{Z}^m} u_n^4 = 0$, then $v \neq 0$, hence, there also exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}_1$.

For any element $(\tilde{u}, \tilde{v}) \in \mathcal{M}_1$, it holds that

$$J_1(\tilde{u}, \tilde{v}) = (H_1\tilde{u}, \tilde{u})_{l^2} + (H_2\tilde{v}, \tilde{v})_{l^2}$$
$$- \sum_{n\in\mathbf{Z}^m} \tilde{u}_n^4 - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |\tilde{v}_n|^3 + \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} \tilde{u}_n^2\tilde{v}_n = 0$$

and

$$\Phi_1(\tilde{u}_1, \tilde{u}_2) = \frac{1}{6}(H_1\tilde{u}, \tilde{u})_{l^2} + \frac{1}{6}(H_2\tilde{v}, \tilde{v})_{l^2} + \frac{1}{12}\sum_{n\in\mathbf{Z}^m} \tilde{u}_n^4. \tag{2.4}$$

Then

$$(H_1\tilde{u}, \tilde{u})_{l^2} + (H_2\tilde{v}, \tilde{v})_{l^2}$$
$$= \sum_{n\in\mathbf{Z}^m} \tilde{u}_n^4 + \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |\tilde{v}_n|^3 - \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} \tilde{u}_n^2\tilde{v}_n$$
$$\leq C_1(H_1\tilde{u}, \tilde{u})_{l^2}^2 + C_2(H_2\tilde{v}, \tilde{v})_{l^2}^{\frac{3}{2}} + C_3(H_1\tilde{u}, \tilde{u})_{l^2}^{\frac{3}{2}} + C_4(H_2\tilde{v}, \tilde{v})_{l^2}^{\frac{3}{2}}. \tag{2.5}$$

So, by (2.4),

$$(H_1\tilde{u}, \tilde{u})_{l^2} + (H_2\tilde{v}, \tilde{v})_{l^2} \geq \rho \text{ for some } \rho > 0 \text{ and } \Phi_1(\tilde{u}_1, \tilde{u}_2) \geq \frac{\rho}{6}. \tag{2.6}$$

Moreover, there holds

$$\langle J_1'(\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v})\rangle$$
$$= 2(H_1\tilde{u}, \tilde{u})_{l^2} + 2(H_2\tilde{v}, \tilde{v})_{l^2} - 4\sum_{n\in\mathbf{Z}^m} \tilde{u}_n^4 - \frac{3}{2}\sum_{n\in\mathbf{Z}^m} |\tilde{v}_n|^3 + \frac{9\beta}{2}\sum_{n\in\mathbf{Z}^m} \tilde{u}_n^2\tilde{v}_n$$
$$= -(H_1\tilde{u}, \tilde{u})_{l^2} - (H_2\tilde{v}, \tilde{v})_{l^2} - \sum_{n\in\mathbf{Z}^m} \tilde{u}_n^4 < -\rho. \tag{2.7}$$

Hence, by the implicit function theorem, $\mathcal{M}_1 = J_1^{-1}(0)$ is a C^1 complete manifold.

Since

$$g(t) = \frac{t^2}{2}(H_1 u, u)_{l^2} + \frac{t^2}{2}(H_2 v, v)_{l^2} - \frac{t^4}{4}\sum_{n\in\mathbf{Z}^m} u_n^4$$

$$- \frac{t^3}{6}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{t^3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n,$$

$g(t) \to 0$ as $t \to 0$, $g(t) \to -\infty$ as $t \to +\infty$ and $g(\bar{t}) \geq \frac{1}{6}\rho$, so the maximum of $g(t)$ is achieved in the interior of $[0, +\infty]$. Assume that $g(\tilde{t}) = \max g(t)$, then $g'(\tilde{t}) = 0$, hence

$$J_1(\tilde{t}u, \tilde{t}v) = \langle \Phi_1'(\tilde{t}u, \tilde{t}v), (\tilde{t}u, \tilde{t}v)\rangle = 0,$$

so $\tilde{t} = \bar{t}$. □

Lemma 2.2. (u, v) *is a nontrivial critical point of* Φ_1 *if and only if* (u, v) *is a constrained critical point of* Φ_1 *on* $\mathcal{M}_1$.

Proof. Suppose $\Phi_1'|_{\mathcal{M}_1}(u, v) = 0$. By the Lagrangian multiplier method, one can write $\Phi_1'|_{\mathcal{M}_1}(u, v) = \Phi_1'(u, v) - \omega J_1'(u, v)$. So we have

$$\langle \Phi_1'|_{\mathcal{M}_1}(u, v), (u, v)\rangle = \langle \Phi_1'(u, v), (u, v)\rangle - \omega\langle J_1'(u, v), (u, v)\rangle = 0.$$

By the definition of $\mathcal{M}_1$, it holds that

$$\langle \Phi_1'|_{\mathcal{M}_1}(u, v), (u, v)\rangle = -\omega\langle J_1'(u, v), (u, v)\rangle = 0.$$

From (2.7) in the proof of Lemma 2.1, $\omega = 0$. Thus, $\Phi_1'(u, v) = 0$. If (u, v) is a nontrivial critical point of Φ_1, then it is, of course, a critical point of Φ_1 constrained on $\mathcal{M}_1$. □

Lemma 2.3. Φ_1 *satisfies the Palais-Smale condition on* $\mathcal{M}_1$.

Proof. Suppose that $\{(u^k, v^k)\}$ is a Palais-Smale sequence. Then $\Phi_1(u^k, v^k) \to c$ and $\Phi_1'(u^k, v^k)|_{\mathcal{M}_1} \to c$ for some $c \in \mathbf{R}$. Then, from (2.4), we can infer that (u^k, v^k) is bounded in $E_1 \times E_2$. So we can suppose that $(u^k, v^k) \rightharpoonup (u, v)$ for some $(u, v) \in E_1 \times E_2$. Since E_1 and E_2 can be compactly embedded into l^p for $2 \le p \le \infty$, we can infer that

$$u^k \to u,\ v^k \to v \text{ in } l^p \text{ for } 2 \le p \le \infty,\ u_n^k \to u_n,\ v_n^k \to v_n$$

$$\text{for any } n \in \mathbf{Z}^m, \tag{2.8}$$

$$\sum_{n\in\mathbf{Z}^m} (u_n^k)^4 + \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n^k|^3 - \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} (u_n^k)^2(v_n^k)$$

$$\to \sum_{n\in\mathbf{Z}^m} u_n^4 + \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 - \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n$$

and by (2.6) together with the definition of $\mathcal{M}_1$, there holds

$$\sum_{n\in\mathbf{Z}^m} u_n^4 + \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 - \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n \ge \rho.$$

By the Lagrangian multiplier method, we have

$$\Phi_1'(u^k, v^k)|_{\mathcal{M}_1} = \Phi_1'(u^k, v^k) - \omega_k J_1'(u^k, v^k) \to 0$$

for a sequence of real numbers $\{\omega_k\}$. Then it holds that

$$\langle \Phi_1'(u^k, v^k)|_{\mathcal{M}_1}, (u^k, v^k)\rangle$$

$$= \langle \Phi_1'(u^k, v^k), (u^k, v^k)\rangle - \omega_k \langle J_1'(u^k, v^k), (u^k, v^k)\rangle \to 0.$$

By the definition of $\mathcal{M}_1$, we have $\omega_k \langle J_1'(u^k, v^k), (u^k, v^k)\rangle \to 0$. From (2.7), we can infer that $\omega_k \to 0$.

Denote

$$\Phi_1(u, v) = \frac{1}{2}(H_1 u, u)_{l^2} + \frac{1}{2}(H_2 v, v)_{l^2} - F(u, v) - G(u, v) + \beta T(u, v),$$

where $F(u, v) = \frac{1}{4}\sum_{n\in\mathbf{Z}^m} u_n^4$, $G(u, v) = \frac{1}{6}\sum_{n\in\mathbf{Z}^m} |v_n|^3$ and

$$T(u, v) = \frac{1}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n,$$

then

$$\Phi_1'(u^k, v^k) = (u^k, v^k) - F'(u^k, v^k) - G'(u^k, v^k) + \beta T'(u^k, v^k)$$

and

$$J_1'(u^k, v^k) = 2(u^k, v^k) - 4F'(u^k, v^k) - 3G'(u^k, v^k) + 3\beta T'(u^k, v^k).$$

Hence

$$(1 - 2\omega_k)(u^k, v^k) = (1 - 4\omega_k)F'(u^k, v^k) + (1 - 3\omega_k)G'(u^k, v^k)$$
$$+ (3\omega_k - 1)\beta T'(u^k, v^k) + o(1).$$

Now we prove that F', G' and T' are compact operators. As $k \to +\infty$, by (2.8), Sobolev embedding and Hölder inequalities, we have

$$\begin{aligned}
&\langle T_u'(u^k, v^k) - T_u'(u, v), h\rangle \\
&= \sum_{n\in\mathbf{Z}^m} (u_n^k v_n^k - u_n v_n)h_n \le \sum_{n\in\mathbf{Z}^m} |u_n^k v_n^k - u_n v_n| \cdot |h_n| \\
&\le \sum_{n\in\mathbf{Z}^m} |u_n^k v_n^k - u_n^k v_n| \cdot |h_n| + \sum_{n\in\mathbf{Z}^m} |u_n^k v_n - u_n v_n| \cdot |h_n| \\
&= \sum_{n\in\mathbf{Z}^m} |v_n^k - v_n| \cdot |u_n^k| \cdot |h_n| + \sum_{n\in\mathbf{Z}^m} |u_n^k - u_n| \cdot |v_n| \cdot |h_n|
\end{aligned}$$

$$\le \left(\sum_{n\in\mathbf{Z}^m} |v_n^k - v_n|^3\right)^{\frac{1}{3}} \left(\sum_{n\in\mathbf{Z}^m} |u_n^k|^3\right)^{\frac{1}{3}} \left(\sum_{n\in\mathbf{Z}^m} |h_n|^3\right)^{\frac{1}{3}}$$

$$+ \left(\sum_{n\in\mathbf{Z}^m} |u_n^k - u_n|^3\right)^{\frac{1}{3}} \left(\sum_{n\in\mathbf{Z}^m} |v_n|^3\right)^{\frac{1}{3}} \left(\sum_{n\in\mathbf{Z}^m} |h_n|^3\right)^{\frac{1}{3}} = o(1)\|h\|_{E_1},$$

$\forall h \in E_1$.

Similarly, we have $\forall h \in E_2$, it holds that

$$\langle T_v'(u^k, v^k) - T_v'(u, v), h\rangle$$

$$= \frac{1}{2}\sum_{n\in\mathbf{Z}^m} ((u_n^k)^2 - (u_n)^2)h_n \le \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |(u_n^k)^2 - (u_n)^2| \cdot |h_n|$$

$$\le \frac{1}{2}\left(\sum_{n\in\mathbf{Z}^m} |(u_n^k)^2 - (u_n)^2|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbf{Z}^m} |h_n|^2\right)^{\frac{1}{2}}$$

$$\le \frac{1}{2}\left(\sum_{n\in\mathbf{Z}^m} |(u_n^k)^4 - (u_n)^4|\right)^{\frac{1}{2}} \|h\|_{E_2}.$$

Since $(u_n^k)^4 + u_n^4 - |(u_n^k)^4 - u_n^4| \ge 0$, we have

$$\sum_{n\in\mathbf{Z}^m} \liminf_{k\to+\infty}((u_n^k)^4 + u_n^4 - |(u_n^k)^4 - u_n^4|)$$

$$\le \liminf_{k\to+\infty} \sum_{n\in\mathbf{Z}^m} ((u_n^k)^4 + u_n^4 - |(u_n^k)^4 - u_n^4|)$$

and so $\lim_{k\to+\infty} \sum_{n\in\mathbf{Z}^m} |(u_n^k)^4 - (u_n)^4| \to 0$ by (2.8). Thus,

$$\langle T_v'(u^k, v^k) - T_v'(u, v), h\rangle = o(1)\|h\|_{E_2}.$$

So we have proved the compactness of T'. Similarly, we have $\forall h \in E_1$,

$$\begin{aligned}
&\langle F_u'(u^k, v^k) - F_u'(u, v), h\rangle \\
&= \sum_{n\in\mathbf{Z}^m} ((u_n^k)^3 - u_n^3)h_n \le \sum_{n\in\mathbf{Z}^m} |(u_n^k)^3 - u_n^3||h_n| \\
&\le \left(\sum_{n\in\mathbf{Z}^m} |(u_n^k)^3 - u_n^3|^{\frac{4}{3}}\right)^{\frac{3}{4}} \left(\sum_{n\in\mathbf{Z}^m} |h_n|^4\right)^{\frac{1}{4}} \\
&\le C\left(\sum_{n\in\mathbf{Z}^m} |(u_n^k)^4 - (u_n)^4|\right)^{\frac{3}{4}} \|h\|_{E_1} = o(1)\|h\|_{E_1}.
\end{aligned}$$

And, for any $h \in E_2$,

$$\begin{aligned}
&\langle G_v'(u^k, v^k) - G_v'(u, v), h\rangle \\
&= \frac{1}{2}\sum_{n\in\mathbf{Z}^m} (|v_n^k|v_n^k - |v_n|v_n)h_n \le \frac{1}{2}\sum_{n\in\mathbf{Z}^m} ||v_n^k|v_n^k - |v_n|v_n||h_n| \\
&\le \frac{1}{2}\left(\sum_{n\in\mathbf{Z}^m} ||v_n^k|v_n^k - |v_n|v_n|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbf{Z}^m} |h_n|^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\left(\sum_{n\in\mathbf{Z}^m} |(v_n^k)^4 - (v_n)^4|\right)^{\frac{1}{2}} \|h\|_{E_2}.
\end{aligned}$$

By the same reason as above, we can infer that $\langle G_v'(u^k, v^k) - G_v'(u, v), h\rangle = o(1)\|h\|_{E_2}$. Here, we have used the fundamental inequality $a^{\frac{q}{p}} - b^{\frac{q}{p}} \ge (a^q - b^q)^{\frac{1}{p}}$ for $a > b > 0$ and $p > 0$, $q > 0$. Thus, F' and G' are also compact operators.

Hence $(u^k, v^k) \to F'(u, v) + G'(u, v) - \beta T'(u, v)$, so

$$(u, v) = F'(u, v) + G'(u, v) - \beta T'(u, v).$$

Thus, $(H_1 u, u)_{l^2} + (H_2 v, v)_{l^2} = 4F(u, v) + 3G(u, v) - 3\beta T(u, v)$ which means $(u, v) \in \mathcal{M}_1$. □

2.2. On the equation $-(\mathcal{A}v)_n + \tau_n v_n = \frac{1}{2}|v_n|v_n$

In this subsection, we study the equation

$$-(\mathcal{A}v)_n + \tau_n v_n = \frac{1}{2}|v_n|v_n. \tag{2.9}$$

Similar to Subsection 2.1, we define a functional $\Psi_1 : E_2 \to \mathbf{R}$ as follows:

$$\Psi_1(v) = \frac{1}{2}(H_2 v, v)_{l^2} - \frac{1}{6}\sum_{n \in \mathbf{Z}^m} |v_n|^3.$$

Then we have $\Psi_1 \in C^1(E_2, \mathbf{R})$. Define the Nehari manifold as follows:

$$\mathcal{N}_1 = \left\{ v \in E_2 \backslash \{0\} \,|\, P_1(v) := (H_2 v, v)_{l^2} - \frac{1}{2}\sum_{n \in \mathbf{Z}^m} |v_n|^3 = 0 \right\}.$$

On the Nehari manifold, we have the following lemma.

Lemma 2.4. *$\mathcal{N}_1$ is a C^1 nonempty manifold. For every $v \in E_2 \backslash \{0\}$, there exists a unique $\bar{t} = \bar{t}(v) > 0$ such that $\bar{t}v \in \mathcal{N}_1$. The maximum of $g(t) = \Psi_1(tv)$ for $t \geq 0$ is achieved at $\bar{t}$.*

Proof. For $v \in E_2 \backslash \{0\}$, consider the equation on $t > 0$,

$$P_1(tv) = t^2 (H_2 v, v)_{l^2} - \frac{t^3}{2}\sum_{n \in \mathbf{Z}^m} |v_n|^3 = 0.$$

Since $\sum_{n \in \mathbf{Z}^m} |v_n|^3 \neq 0$, there exists a unique $\bar{t} = \bar{t}(v) > 0$ such that $\bar{t}v \in \mathcal{N}_1$.

For an element $\widetilde{v} \in \mathcal{N}_1$, it holds that

$$P_1(\widetilde{v}) = (H_2\widetilde{v}, \widetilde{v})_{l^2} - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |\widetilde{v}_n|^3 = 0.$$

Then

$$(H_2\widetilde{v}, \widetilde{v})_{l^2} = \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |\widetilde{v}_n|^3 \le C_5(H_2\widetilde{v}, \widetilde{v})_{l^2}^{\frac{3}{2}}. \tag{2.10}$$

So $(H_2\widetilde{v}, \widetilde{v})_{l^2} \ge \rho$ for some $\rho > 0$ and $\Psi_1(\widetilde{v}) = \frac{1}{6}(H_2\widetilde{v}, \widetilde{v})_{l^2} \ge \frac{1}{6}\rho$. Moreover, it holds that

$$\langle P_1'(\widetilde{v}), \widetilde{v}\rangle = 2(H_2\widetilde{v}, \widetilde{v})_{l^2} - \frac{3}{2}\sum_{n\in\mathbf{Z}^m} |\widetilde{v}_n|^3 = -(H_2\widetilde{v}, \widetilde{v})_{l^2} < -\rho. \tag{2.11}$$

Hence $\mathcal{N}_1$ is a C^1 complete manifold.

Since

$$g(t) = \frac{t^2}{2}(H_2v, v)_{l^2} - \frac{t^3}{6}\sum_{n\in\mathbf{Z}^m} |v_n|^3,$$

$g(t) \to 0$ as $t \to 0$, $g(t) \to -\infty$ as $t \to +\infty$ and $g(\bar{t}) \ge \frac{1}{6}\rho$, so the maximum of $g(t)$ is achieved in the interior of $[0, +\infty]$. Assume that $g(\widetilde{t}) = \max g(t)$. Then $g'(\widetilde{t}) = 0$, hence $P_1(\widetilde{t}v) = \langle \Psi_1'(\widetilde{t}v), \widetilde{t}v\rangle = 0$, so $\widetilde{t} = \bar{t}$. □

Lemma 2.5. *Let* $v \in E_2$ *be a minimizer of the functional* Ψ_1 *constrained on the Nehari manifold* $\mathcal{N}_1$, *that is*, $\Psi_1(v) = \inf_{\mathcal{N}_1} \Psi_1(w)$. *Then v is a weak solution to equation* (2.9).

Proof. By the Lagrangian multiplier method, we can infer that v is a critical point of the functional $\Psi_1(w) + \lambda P_1(w)$. Thus, $\forall w \in E_2$, it holds that

$$\langle \Psi_1'(v), w\rangle + \lambda\langle P_1'(v), w\rangle = 0.$$

Taking $w = v$, we have

$$\langle \Psi_1'(v), v\rangle + \lambda\langle P_1'(v), v\rangle = 0.$$

So $\lambda\langle P_1'(v), v\rangle = 0$. From (2.11), it holds that $\langle P_1'(v), v\rangle < 0$, hence $\lambda = 0$. Therefore, $\forall w \in E_2$, $\langle \Psi_1'(v), w\rangle = 0$. Thus, v is a weak solution of (2.9). □

Theorem 2.6. *There exists a minimizer* v' *of the functional* Ψ_1 *constrained on the Nehari manifold* $\mathcal{N}_1$, *that is,* $\Psi_1(v') = \inf_{\mathcal{N}_1} \Psi_1(v)$, *then* v' *is a weak solution to equation* (2.9) *by Lemma* 2.5.

Proof. By Ekeland variational principle (see [4]), we can infer that there is a sequence $\{v_n\} \subset E_2$ such that $\Psi_1(v_n) = \inf_{\mathcal{N}_1} \Psi_1(v)$ and $\Psi_1'(v_n)|_{\mathcal{N}_1} \to 0$. Similar to Lemma 2.3, the (PS) condition is satisfied in this case, so up to a subsequence, $\{v_n\}$ converges to some $v' \in \mathcal{N}_1$. □

It is clear that $-v'$ is also a minimizer of the functional Ψ_1 constrained on the Nehari manifold $\mathcal{N}_1$.

2.3. Proof of Theorem 1.1

In this subsection, we show that there exists a nontrivial solution of (1.1) different from $\mathbf{u}_0$ and $\mathbf{u}_1$. Here $\mathbf{u}_0 = (0, v')$, $\mathbf{u}_1 = (0, -v')$ and v' is the solution obtained in Theorem 2.6. Our method is inspired by [1]. Set

$$\gamma_1 = \inf_{\varphi \in E_1 \setminus \{0\}} \frac{\|\varphi\|_{E_1}^2}{\sum_{n \in \mathbf{Z}^m} |v_n'||\varphi_n|^2}. \tag{2.12}$$

By the Sobolev embedding theorem, $\gamma_1 > 0$.

Theorem 2.7. *The following statements are true*:

(i) *If* $\beta < -\gamma_1$, *then* $\mathbf{u}_0$ *is a saddle point of* Φ_1 *on* $\mathcal{M}_1$. *In particular*, $\inf_{\mathcal{M}_1} \Phi_1 < \Phi_1(\mathbf{u}_0)$.

(ii) *If* $\beta > \gamma_1$, *then* $\mathbf{u}_1$ *is a saddle point of* Φ_1 *on* $\mathcal{M}_1$. *In particular*, $\inf\limits_{\mathcal{M}_1} \Phi_1 < \Phi_1(\mathbf{u}_1)$.

Let $D^2\Phi_1|_{\mathcal{M}_1}(\mathbf{u}_0)$ and $D^2\Phi_1|_{\mathcal{M}_1}(\mathbf{u}_1)$ denote the second derivatives of Φ_1 constrained on $\mathcal{M}_1$. Since $\Phi_1'(\mathbf{u}_0) = 0$ and $\Phi_1'(\mathbf{u}_1) = 0$, one has that

$$D^2\Phi_1|_{\mathcal{M}_1}(\mathbf{u}_0)(\mathbf{h}, \mathbf{h}) = \Phi_1''(\mathbf{u}_0)(\mathbf{h}, \mathbf{h}), \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M}_1,$$

$$D^2\Phi_1|_{\mathcal{M}_1}(\mathbf{u}_1)(\mathbf{h}, \mathbf{h}) = \Phi_1''(\mathbf{u}_1)(\mathbf{h}, \mathbf{h}), \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}_1.$$

Similarly, we have

$$D^2\Psi_1|_{\mathcal{N}_1}(v')(h, h) = \Psi_1''(v')(h, h), \quad \forall \mathbf{h} = (h_1, h_2) \in T_{v'}\mathcal{N}_1,$$

$$D^2\Psi_1|_{\mathcal{N}_1}(-v')(h, h) = \Psi_1''(-v')(h, h), \quad \forall \mathbf{h} = (h_1, h_2) \in T_{-v'}\mathcal{N}_1.$$

Lemma 2.8. *There hold*

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M}_1 \Leftrightarrow h_2 \in T_{v'}\mathcal{N}_1,$$

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}_1 \Leftrightarrow h_2 \in T_{-v'}\mathcal{N}_1.$$

Proof. $h_2 \in T_{v'}\mathcal{N}_1$ if and only if $(H_2v', h_2)_{l^2} = \frac{3}{4}\sum\limits_{n \in \mathbf{Z}^m} |v_n'|v_n'(h_2)_n$, $h_2 \in T_{-v'}\mathcal{N}_1$ if and only if $(H_2(-v'), h_2)_{l^2} = -\frac{3}{4}\sum\limits_{n \in \mathbf{Z}^m} |v_n'|v_n'(h_2)_n$, while $\mathbf{h} \in T_{(u,v)}\mathcal{M}_1$ if and only if

$$(H_1u, h_1)_{l^2} + (H_2v, h_2)_{l^2} = 2\sum_{n \in \mathbf{Z}^m} u_n^3(h_1)_n + \frac{3}{4}\sum_{n \in \mathbf{Z}^m} |v_n|v_n(h_2)_n$$
$$- \frac{3\beta}{2}\sum_{n \in \mathbf{Z}^m} u_nv_n(h_1)_n - \frac{3\beta}{4}\sum_{n \in \mathbf{Z}^m} u_n^2(h_2)_n.$$

Hence we have $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M}_1$ if and only if $(H_2 v', h_2)_{l^2} = \frac{3}{4}\sum_{n\in\mathbf{Z}^m} |v'_n| v'_n (h_2)_n$, and $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}_1$ if and only if $(H_2(-v'), h_2)_{l^2} = -\frac{3}{4}\sum_{n\in\mathbf{Z}^m} |v'_n| v'_n (h_2)_n$. These complete the proof. □

Proof of Theorem 2.7. If $(u, v) \in E_1 \times E_2$ and $\mathbf{h} = (h_1, h_2) \in E_1 \times E_2$, then one has

$$\begin{aligned}
&\Phi_1''(u, v)(\mathbf{h}, \mathbf{h}) \\
&= (H_1 h_1, h_1)_{l^2} + (H_2 h_2, h_2)_{l^2} - 3\sum_{n\in\mathbf{Z}^m} u_n^2 (h_1)_n^2 - \sum_{n\in\mathbf{Z}^m} |v_n| (h_2)_n^2 \\
&\quad + \beta\sum_{n\in\mathbf{Z}^m} v_n (h_1)_n^2 + \beta\sum_{n\in\mathbf{Z}^m} u_n (h_1)_n (h_2)_n. \qquad (2.13)
\end{aligned}$$

In particular, if $(u, v) = (0, v') = \mathbf{u}_0$, then we get

$$\begin{aligned}
&\Phi_1''(\mathbf{u}_0)(\mathbf{h}, \mathbf{h}) \\
&= (H_1 h_1, h_1)_{l^2} + (H_2 h_2, h_2)_{l^2} - \sum_{n\in\mathbf{Z}^m} |v'_n| (h_2)_n^2 + \beta\sum_{n\in\mathbf{Z}^m} v'_n (h_1)_n^2.
\end{aligned}$$

Taking $\mathbf{h}_0 = (h_1, 0) \in T_{\mathbf{u}_0}\mathcal{M}_1$, we have

$$\Phi_1''(\mathbf{u}_0)(\mathbf{h}_0, \mathbf{h}_0) = (H_1 h_1, h_1)_{l^2} + \beta\sum_{n\in\mathbf{Z}^m} v'_n (h_1)_n^2.$$

By the definition of γ_1, we can infer that when $\beta < -\gamma_1$, there exists $\widetilde{h}_1 \in E_1\setminus\{0\}$ such that

$$\gamma_1 \le \frac{\|\widetilde{h}_1\|_{E_1}^2}{\sum_{n\in\mathbf{Z}^m} |v'_n| (\widetilde{h}_1)_n^2} < -\beta.$$

So, for $\widetilde{\mathbf{h}}_0 = (\widetilde{h}_1, 0)$, there holds

$$\Phi_1''(\mathbf{u}_0)(\widetilde{\mathbf{h}}_0, \widetilde{\mathbf{h}}_0) \le (H_1\widetilde{h}_1, \widetilde{h}_1)_{l^2} + \beta \sum_{n\in\mathbf{Z}^m} |v_n'|(\widetilde{h}_1)_n^2 < 0.$$

If $(u, v) = \mathbf{u}_1 = (0, -v')$, then we get

$$\begin{aligned}
&\Phi_1''(\mathbf{u}_1)(\mathbf{h}, \mathbf{h})\\
&= (H_1h_1, h_1)_{l^2} + (H_2h_2, h_2)_{l^2} - \sum_{n\in\mathbf{Z}^m} |v_n'|(h_2)_n^2 - \beta \sum_{n\in\mathbf{Z}^m} v_n'(h_1)_n^2.
\end{aligned}$$

Taking $\mathbf{h}_1 = (h_1, 0) \in T_{\mathbf{u}_0}\mathcal{M}_1$, we have

$$\Phi_1''(\mathbf{u}_1)(\mathbf{h}_1, \mathbf{h}_1) = (H_1h_1, h_1)_{l^2} - \beta \sum_{n\in\mathbf{Z}^m} v_n'(h_1)_n^2.$$

By the definition of γ_1, we can infer that when $\beta > \gamma_1$, there exists $\widetilde{h}_2 \in E_1\backslash\{0\}$ such that

$$\gamma_1 \le \frac{\|\widetilde{h}_2\|_{E_1}^2}{\sum_{n\in\mathbf{Z}^m} |v_n'|(\widetilde{h}_2)_n^2} < \beta.$$

So, for $\widetilde{\mathbf{h}}_1 = (\widetilde{h}_2, 0)$, there holds

$$\Phi_1''(\mathbf{u}_1)(\widetilde{\mathbf{h}}_1, \widetilde{\mathbf{h}}_1) = (H_1\widetilde{h}_2, \widetilde{h}_2)_{l^2} - \beta \sum_{n\in\mathbf{Z}^m} v_n'(\widetilde{h}_2)_n^2 < 0.$$

Combining with Theorem 2.6, the proof of Theorem 2.7 is complete. □

The following result is a direct consequence of Theorem 2.7.

Lemma 2.9. *The following statements are true*:

(i) *If* $\beta < -\gamma_1$, *then* Φ_1 *has a global minimum* $\widetilde{\mathbf{u}}$ *on* $\mathcal{M}_1$ *and* $\Phi_1(\widetilde{\mathbf{u}}) < \Phi_1(\mathbf{u}_0) = \inf_{\mathcal{N}_1} \Psi_1$.

(ii) *If* $\beta > \gamma_1$, *then* Φ_1 *has a global minimum* $\tilde{\mathbf{u}}$ *on* $\mathcal{M}_1$ *and* $\Phi_1(\tilde{\mathbf{u}}) < \Phi_1(\mathbf{u}_1) = \inf_{\mathcal{N}_1} \Psi_1$.

Proof. The functional Φ_1 is bounded from below on $\mathcal{M}_1$ by (2.6). By Lemma 2.3, the Palais-Smale condition holds, hence $\inf_{\mathcal{M}_1} \Phi_1$ is achieved at some $\tilde{\mathbf{u}} \neq 0$ by Ekeland variational principle (see [4]). Moreover, if $\beta < -\gamma_1$, by Theorem 2.7(i), then $\Phi_1(\tilde{\mathbf{u}}) < \Phi_1(\mathbf{u}_0)$. If $\beta > \gamma_1$, by Theorem 2.7(ii), then $\Phi_1(\tilde{\mathbf{u}}) < \Phi_1(\mathbf{u}_1)$. □

Proof of Theorem 1.1. If $\tilde{\mathbf{u}} = (u, v) = (0, v)$, then $\forall w \in \mathcal{N}_1$, $(0, w) \in \mathcal{M}_1$ and $v \in \mathcal{N}_1$ hence $\Psi_1(v) = \Phi_1(0, v) \leq \Phi_1(0, w) = \Psi_1(w)$. So v achieves the minimum of Ψ_1 on $\mathcal{N}_1$. Hence $\inf_{\mathcal{M}_1} \Phi_1 = \inf_{\mathcal{N}_1} \Psi_1$. This contradicts to Lemma 2.9. If $\tilde{\mathbf{u}} = (u, v) = (u, 0)$, then from the system (1.1), $u = 0$. So $\tilde{\mathbf{u}} = 0 \notin \mathcal{M}_1$. This is also a contradiction. □

3. The Radially Symmetric Case

3.1. Functional framework and a compactness result

Under the conditions (C2) and (C3), we take the radially symmetric spaces as

$$E_{1,r} = \{u \in E_1 \mid \forall n_1, n_2 \in \mathbf{Z}^m \text{ with } |n_1| = |n_2|, u_{n_1} = u_{n_2}\},$$

$$E_{2,r} = \{u \in E_2 \mid \forall n_1, n_2 \in \mathbf{Z}^m \text{ with } |n_1| = |n_2|, u_{n_1} = u_{n_2}\}.$$

Theorem 3.1. *For* $m \geq 2$, *the embedding* $E_{i,r}$ *into* $l^p(\mathbf{Z}^m)$ *is compact*, $2 < p \leq \infty$, $i = 1, 2$.

Proof. We follow the idea of [21]. Assume that $u^k \rightharpoonup 0$ in $E_{i,r}$, u^k is bounded in $E_{i,r}$. It is clear that

$$|u_n^k|^2 \le \sup_k |u^k|_{l^2}^2 / m(n),$$

where $m(n) = \sharp\{r \in \mathbf{Z}^m \,\|\, r| = |n|\}$. Clearly, we have $m(n) \to \infty$ as $|n| \to \infty$. So, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{|n| \ge R} |u_n^k|^2 \le \varepsilon.$$

Since $u^k \rightharpoonup 0$, we have

$$\sup_{|n| \le R} |u_n^k|^2 \to 0 \text{ as } k \to \infty.$$

Hence

$$\sup_{n \in \mathbf{Z}^m} |u_n^k|^2 \to 0 \text{ as } k \to \infty.$$

For $p \in (2, \infty)$, we have

$$\sum_{n \in \mathbf{Z}^m} |u_n^k|^p \le \left(\sup_{n \in \mathbf{Z}^m} |u_n^k|^2 \right)^{\frac{p-2}{2}} \sum_{n \in \mathbf{Z}^m} |u_n^k|^2 = \left(\sup_{n \in \mathbf{Z}^m} |u_n^k|^2 \right)^{\frac{p-2}{2}} \|u^k\|_{l^2}^2$$

$$\le C \left(\sup_{n \in \mathbf{Z}^m} |u_n^k|^2 \right)^{\frac{p-2}{2}} \|u^k\|_{E_{i,r}}^2,$$

here $C > 0$ is a constant depending on a_i. So $\|u^k\|_{l^p}^p = \sum_{n \in \mathbf{Z}^m} |u_n^k|^p \to 0$ as $k \to \infty$. Since $l^p \hookrightarrow l^\infty$ is continuous for $1 \le p < \infty$, $u^k \to 0$ in l^∞. This completes the proof of Theorem 3.1. □

In the rest of this paper, we assume that $m \ge 2$. Define $\Phi_2 : E_{1,r} \times E_{2,r} \to \mathbf{R}$ as

$$\Phi_2(u, v) = \frac{1}{2}(H_1 u, u)_{l^2} + \frac{1}{2}(H_2 v, v)_{l^2} - \frac{1}{4}\sum_{n\in\mathbf{Z}^m} u_n^4$$

$$-\frac{1}{6}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n.$$

Then, by Theorem 3.1, the sums in the definition of Φ_2 are finite. Moreover, we have $\Phi_2 \in C^1(E_{1,r} \times E_{2,r}, \mathbf{R})$ and critical points of Φ_2 are solutions of (1.1).

Define $J_2 : E_{1,r} \times E_{2,r} \to \mathbf{R}$ by

$$J_2(u, v)$$

$$= (H_1 u, u)_{l^2} + (H_2 v, v)_{l^2} - \sum_{n\in\mathbf{Z}^m} u_n^4 - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |v_n|^3 + \frac{3\beta}{2}\sum_{n\in\mathbf{Z}^m} u_n^2 v_n.$$

Then the Nehari manifold is defined as follows:

$$\mathcal{M}_2 = \{(u, v) \in E_{1,r} \times E_{2,r} \setminus \{(0, 0)\} \mid J_2(u, v) = 0\}.$$

To prove Theorem 1.2, we will use Theorem 3.1. The process and the proofs are almost the same as that in Section 2, in the following we only state some lemmas and the proofs are omitted.

Lemma 3.2. *$\mathcal{M}_2$ is a C^1 complete manifold. For every $(u, v) \in E_{1,r} \times E_{2,r} \setminus \{(0, 0)\}$, there exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}_2$. The maximum of $g(t) = \Phi_2(tu, tv)$ for $t \geq 0$ is achieved at $\bar{t}$.*

Lemma 3.3. *(u, v) is a nontrivial critical point of Φ_2 if and only if (u, v) is a constrained critical point of Φ_2 on $\mathcal{M}_2$.*

Lemma 3.4. *Φ_2 satisfies the Palais-Smale condition on $\mathcal{M}_2$.*

3.2. On the equation $-(\mathcal{A}u)_n + \tau_n u_n = \frac{1}{2}|u_n|u_n$

In this subsection, under conditions (C2) and (C3), we study the equation in $E_{2,r}$,

$$-(\mathcal{A}u)_n + \tau_n u_n = \frac{1}{2}|u_n|u_n. \tag{3.14}$$

Similar to Subsection 3.1, we define a functional $\Psi_2 : E_{2,r} \to \mathbf{R}$ as follows:

$$\Psi_2(u) = \frac{1}{2}(H_2 u, u)_{l^2} - \frac{1}{6}\sum_{n\in\mathbf{Z}^m} |u_n|^3.$$

Then, by Theorem 3.1, we have $\Psi_2 \in C^1(E_{2,r}, \mathbf{R})$. Define the Nehari manifold as follows:

$$\mathcal{N}_2 = \left\{ u \in E_{2,r}\backslash\{0\} \,|\, P_2(u) := (H_2 u, u)_{l^2} - \frac{1}{2}\sum_{n\in\mathbf{Z}^m} |u_n|^3 = 0 \right\}.$$

Similar to Lemmas 2.4 and 2.5, the following results are true.

Lemma 3.5. *$\mathcal{N}_2$ is a C^1 nonempty manifold. For every $u \in E_{2,r}\backslash\{0\}$, there exists a unique $\bar{t} = \bar{t}(u) > 0$ such that $\bar{t}u \in \mathcal{N}_2$. The maximum of $g(t) = \Psi_2(tu)$ for $t \geq 0$ is achieved at $\bar{t}$.*

Lemma 3.6. *Let $u \in E_{2,r}$ be a minimizer of the functional Ψ_2 constrained on the Nehari manifold $\mathcal{N}_2$, that is, $\Psi_2(u) = \inf_{\mathcal{N}_2} \Psi_2(v)$. Then u is a weak solution to equation* (3.14).

As Theorem 2.6, the following result is also true.

Theorem 3.7. *There exists a minimizer v'' of the functional Ψ_2 constrained on the Nehari manifold $\mathcal{N}_2$, that is, $\Psi_2(v'') = \inf_{\mathcal{N}_2} \Psi_2(v)$, then v'' is a weak solution to equation* (3.14) *by Lemma* 3.6.

3.3. Proof of Theorem 1.2

In this subsection, we show that there exists a nontrivial solution of (1.1) different from $\overline{\mathbf{u}}_0$ and $\overline{\mathbf{u}}_1$. Here $\overline{\mathbf{u}}_0 = (0, v'')$, $\overline{\mathbf{u}}_1 = (0, -v'')$ and v'' is the solution obtained in Theorem 3.7. Set

$$\gamma_2 = \inf_{\varphi \in E_{1,r} \setminus \{0\}} \frac{\| \varphi \|_{E_{1,r}}^2}{\sum_{n \in \mathbf{Z}^m} | v_n'' || \varphi_n |^2}. \tag{3.15}$$

By Theorem 3.1, $\gamma_2 > 0$. Then, similar to Theorem 2.7, we have the following result.

Theorem 3.8. *We have the following statements*:

(i) *If* $\beta < -\gamma_2$, *then* $\widetilde{\mathbf{u}}_0$ *is a saddle point of* Φ_2 *on* $\mathcal{M}_2$. *In particular*, $\inf_{\mathcal{M}_2} \Phi_2 < \Phi_2(\overline{\mathbf{u}}_0)$.

(ii) *If* $\beta > \gamma_2$, *then* $\overline{\mathbf{u}}_1$ *is a saddle point of* Φ_2 *on* $\mathcal{M}_2$. *In particular*, $\inf_{\mathcal{M}_2} \Phi_2 < \Phi_2(\overline{\mathbf{u}}_1)$.

The following lemma is a direct consequence of Theorem 3.8.

Lemma 3.9. *We have the following statements*:

(i) *If* $\beta < -\gamma_2$, *then* Φ_2 *has a global minimum* $\overline{\mathbf{u}}$ *on* $\mathcal{M}_2$ *and* $\Phi_2(\overline{\mathbf{u}}) < \Phi_2(\overline{\mathbf{u}}_0) = \inf_{\mathcal{N}_2} \Psi_2$.

(ii) *If* $\beta > \gamma_2$, *then* Φ_2 *has a global minimum* $\overline{\mathbf{u}}$ *on* $\mathcal{M}_2$ *and* $\Phi_2(\overline{\mathbf{u}}) < \Phi_2(\overline{\mathbf{u}}_1) = \inf_{\mathcal{N}_2} \Psi_2$.

Then we can prove Theorem 1.2.

Proof of Theorem 1.2. If $\overline{\mathbf{u}} = (\overline{u}, \overline{v}) = (0, v)$, then $\forall u \in \mathcal{N}_2$, $(0, u) \in \mathcal{M}_2$ and $v \in \mathcal{N}_2$. Hence $\Psi_2(v) = \Phi_2(\overline{u}, \overline{v}) \leq \Phi_2(0, u) = \Psi_2(u)$, so v achieves the minimum of Ψ_2 on $\mathcal{N}_2$. Hence $\inf_{\mathcal{M}_2} \Phi_2 = \inf_{\mathcal{N}_2} \Psi_2$. This contradicts to Lemma 3.9. □

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