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REDUCED DIFFERENTIAL TRANSFORM METHOD FOR NONLINEAR SCHRÖDINGER EQUATION

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Abstract

In this paper, the reduced differential transform method (RDTM) is presented for the solutions of the nonlinear Schrödinger equations. This method, which does not require any symbolic computation, provides an iterative procedure for obtaining analytic and approximate solutions of differential equations. We demonstrate and validated the efficiency and simplicity of this method by some test examples.

0. Introduction

Consider the following nonlinear Schrödinger equation:

$$iu_t = -\frac{1}{2}\nabla^2 u + (1-\nu)u + |u|^2 u, \tag{1}$$

with initial condition

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$$u(x_1, x_2, ..., x_n, 0) = u_0(x_1, x_2, ..., x_n), \tag{2}$$

where ν is a constant, $\nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

This equation plays important roles in physics media. It can describe many nonlinear phenomena including fluid dynamics [1] and nonlinear optics [2, 3], quantum mechanics, which was first discovered by Nick Laskin as a result of extending the Feynman path integral.

The nonlinear Schrödinger equation is one of the most universal models for physical phenomena. Finding the solutions of this problem is of practical importance. Many authors have investigated its analytical and numerical solution [4, 5, 7, 8].

Reduced differential transformation method is based on Taylor series expansion. This method have many merits: it provides a straightforward means of solving linear and nonlinear differential equations without the need for linearization, perturbation, or any other transformation. It is different from the traditional high order Taylor's series method. The Taylor series method is computationally taken long time for large order differential equation and requires symbolic computation of the necessary derivatives of functions, reduced differential transformation method is without massive computations and restrictive assumptions [6]. As the RDTM is more effective than the other methods, we apply it to solve the Schrödinger equation.

1. Definition of Reduced Differential Transform

If $u(x_1, x_2, ..., x_n, t)$ is analytical and continuous function, then its reduced differential transform is expressed as follows:

$$\begin{aligned} U_k(x_1, x_2, ..., x_n) &= \frac{1}{k!}\left[\frac{\partial^k u(x_1, x_2, ..., x_n, t)}{\partial t^k}\right]\Bigg|_{t=0} \\ &= F(u(x_1, x_2, ..., x_n, t)), \end{aligned} \tag{3}$$

where

$$u(x_1, x_2, ..., x_n, t) = \sum_{k=0}^{\infty} U_k(x_1, x_2, ..., x_n) t^k$$

$$= F^{-1}(U_k(x_1, x_2, ..., x_n)), \tag{4}$$

where $U_k(x_1, x_2, ..., x_n)$ is the reduced differential transform of $u(x_1, x_2, ..., x_n, t)$.

2. Main Properties of Reduced Differential Transform

If $u(x_1, x_2, ..., x_n, t) = F^{-1}[U_k(x_1, x_2, ..., x_n)]$, $v(x_1, x_2, ..., x_n, t) = F^{-1}[V_k(x_1, x_2, ..., x_n)]$, and $\otimes$ denote the convolution, then the fundamental operations of RDTM are expressed as follows:

$$F[u(x_1, x_2, ..., x_n, t) v(x_1, x_2, ..., x_n, t)]$$

$$= \sum_{r=0}^{k} U_r(x_1, x_2, ..., x_n) V_{k-r}(x_1, x_2, ..., x_n), \tag{5}$$

$$F[\lambda_1 u(x_1, x_2, ..., x_n, t) + \lambda_2 v(x_1, x_2, ..., x_n, t)]$$

$$= \lambda_1 U_k(x_1, x_2, ..., x_n) + \lambda_2 V_k(x_1, x_2, ..., x_n), \tag{6}$$

$$F\left[\frac{\partial u(x_1, x_2, ..., x_n, t)}{\partial t}\right] = (k+1) U_{k+1}(x_1, x_2, ..., x_n). \tag{7}$$

3. Applications

In this section, we demonstrate the performance and efficiency of the proposed method by the following examples:

Example 1. We consider the one-dimensional nonlinear Schrödinger equation

$$iu_t = -\frac{1}{2} u_{xx} + u - u \sin^2 x + |u|^2 u, \quad t \geq 0, \tag{8}$$

with initial condition

$$u(x, 0) = \sin(x). \tag{9}$$

The exact solution is [7]

$$u(x, t) = \sin(x)\exp\left(-\frac{3}{2}it\right).$$

Taking the reduced differential transform of equation (8), we obtain

$$\begin{aligned} &i(k+1)U_{k+1}(x) \\ &= -\frac{1}{2}U_{kxx}(x) + U_k(x) - U_k(x)\sin^2(x) \\ &+ \sum_{s=0}^{k}\left[\sum_{r=0}^{s}U_r(x)\overline{U}_{s-r}(x)\right]U_{k-s}(x), \end{aligned} \tag{10}$$

where $\overline{U}$ is the complex conjugate of U.

From initial condition equation (9), we have

$$U_0(x) = \sin(x). \tag{11}$$

From equations (10) and (11), we can obtain

$$U_1(x) = -\frac{3i}{2}\sin(x). \tag{12}$$

From equations (12) and (10), we obtain

$$U_2(x) = -\frac{9}{8}\sin(x). \tag{13}$$

Similarly we have

$$U_3(x) = -\frac{9i}{16}\sin(x), \tag{14}$$

$$U_4(x) = \left(\frac{3}{2}\right)^4\frac{1}{24}\sin(x), \tag{15}$$

$$\vdots$$

And so on, we can calculate $U_k(x)$. Substituting all values $U_k(x)$ into equation (4) we obtain

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^k = \sin(x) - \frac{3it}{2}\sin(x) - \frac{9t^2}{8}\sin(x) + \cdots = e^{-\frac{3}{2}it}\sin(x). \quad (16)$$

Example 2. Consider the two-dimensional nonlinear Schrödinger equation

$$iu_t = -\frac{1}{2}(u_{xx} + u_{yy}) + u - vu + |u|^2 u, \; (x, y) \in [0, 2\pi] \times [0, 2\pi] \quad (17)$$

with initial value

$$u(x, y, 0) = \sin(x)\sin(y), \quad (18)$$

where

$$v = \sin^2(x)\sin^2(y). \quad (19)$$

The exact solution is [8]

$$u(x, y, t) = \sin(x)\sin(y)\exp(-i2t).$$

Taking the reduced differential transform of equation (17), we can obtain the following formula:

$$i(k+1)U_{k+1}(x, y)$$

$$= -\frac{1}{2}[U_{kxx}(x, y) + U_{kyy}(x, y)] + U_k(x, y) - vU_k(x, y)$$

$$+ \sum_{s=0}^{k}\left[\sum_{r=0}^{s} U_r(x, y)\overline{U}_{s-r}(x, y)\right] U_{k-s}(x, y). \quad (20)$$

From equation (18), we have

$$U_0(x, y) = \sin(x)\sin(y). \quad (21)$$

From equations (20) and (21), we have

$$U_1(x, y) = -2i\sin(x)\sin(y). \quad (22)$$

From equations (22) and (20), we obtain

$$U_2(x, y) = -2\sin(x)\sin(y). \tag{23}$$

Similarly we have

$$U_3(x, y) = \frac{4i}{3}\sin(x)\sin(y), \tag{24}$$

$$U_4(x, y) = \frac{2}{3}\sin(x)\sin(y), \tag{25}$$

$$\vdots$$

And so on, we can calculate $U_k(x, y)$. Substituting all values $U_k(x, y)$ into equation (4) we obtain

$$\begin{aligned} u(x, y, t) &= \sum_{k=0}^{\infty} U_k(x, y)t^k \\ &= \sin(x)\sin(y)\left[1 - 2it - 2t^2 + i\frac{4t^3}{3} + \cdots\right] \\ &= e^{-2it}\sin(x)\sin(y). \end{aligned} \tag{26}$$

Example 3. Consider the three-dimensional nonlinear Schrödinger equation

$$iu_t = -\frac{1}{2}(u_{xx} + u_{yy} + u_{zz}) + u - vu + |u|^2 u, \quad 0 \le x, y, z \le 2\pi, \tag{27}$$

with initial condition

$$u(x, y, z, 0) = \sin(x)\sin(y)\sin(z), \tag{28}$$

where

$$v(x, y, z) = \sin^2(x)\sin^2(y)\sin^2(z). \tag{29}$$

The exact solution is [7]

$$u(x, y, z) = \sin(x)\sin(y)\sin(z)\exp\left(-\frac{5}{2}it\right).$$

Taking the reduced fractional differential transform of equation (27), we obtain

$$i(k+1)U_{k+1}(x,\ y,\ z)$$

$$= -\frac{1}{2}[U_{kxx}(x,\ y,\ z) + U_{kyy}(x,\ y,\ z) + U_{kzz}(x,\ y,\ z)]$$

$$+ U_k(x,\ y,\ z) - vU_k(x,\ y,\ z)$$

$$+ \sum_{s=0}^{k}\left[\sum_{r=0}^{s} U_r(x,\ y,\ z)\overline{U}_{s-r}(x,\ y,\ z)\right]U_{k-s}(x,\ y,\ z). \tag{30}$$

From equation (28), we have

$$U_0(x,\ y,\ z) = \sin(x)\sin(y)\sin(z). \tag{31}$$

From equations (30) and (31), we have

$$U_1(x,\ y) = \frac{-5i}{2}\sin(x)\sin(y)\sin(z). \tag{32}$$

From equations (32) and (30), we obtain

$$U_2(x,\ y,\ z) = \frac{-25}{4}\sin(x)\sin(y)\sin(z). \tag{33}$$

Similarly we have

$$U_3(x,\ y,\ z) = -\frac{125i}{8}\sin(x)\sin(y)\sin(z), \tag{34}$$

$$U_4(x,\ y,\ z) = \frac{625i}{16}\sin(x)\sin(y)\sin(z), \tag{35}$$

$$\vdots$$

And so on, we can calculate $U_k(x,\ y,\ z)$. Substituting all values $U_k(x,\ y,\ z)$ into equation (4), we obtain

$$u(x,\ y,\ z,\ t) = \sum_{h=0}^{\infty} U_k(x,\ y,\ z)t^k$$

$$= \sin(x)\sin(y)\sin(z)\left[1 - \frac{5}{2}it - \frac{25}{4}\frac{t^2}{2} + \frac{125}{8}i\frac{t^3}{3!} + \cdots\right]$$

$$= e^{-\frac{5it}{2}}\sin(x)\sin(y)\sin(z). \tag{36}$$

4. Conclusion

The reduced differential transform method (RDTM) is successfully applied to find the solution of the nonlinear Schrödinger equations. Since this technique does not require any discretization, linearization or small perturbations thus it reduces significantly the numerical computation. The results confirm that this method is very effective technique for obtaining the approximate analytical solutions of nonlinear differential equations.

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