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A CHARACTERIZATION OF DISTRIBUTIVE GSA

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Abstract

Throughout this paper, we concern with some concepts of R-groups, where R is a near-ring, and their related algebraic structures, that is, group semiautomata and the distributive group semiautomata. The author already studied the 1st, 2nd and 3rd isomorphic theorems of group semiautomata.

In this paper, we investigate a characterization of distributive group semiautomata.

1. Introduction

The concept of GSA is introduced by Clay and Fong [6]. The purpose of this paper is to study the algebraic theory of distributive group semiautomata, which is related with the following papers [1-5].

A *group semiautomaton* (in short, GSA) is a quadruple $(Q, +, X, \delta)$, where $(Q, +)$, as the set of states, is a group (not necessarily abelian), X is the set of inputs and $\delta : Q \times X \to Q$ is the state transition function.

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Now, we can define stronger version of GSA as following:

A GSA $(Q, +, X, \delta)$ is *distributive* if for any $a, b \in Q$ and for each $x \in X$, $(a + b)x = ax + bx$.

At first, this paper is concerned with some concepts of near-ring R, R-group and GSA, and then we introduce some properties of GSA, the fundamental isomorphic theorem and the 2nd isomorphic theorem of GSA.

Next, we will study our main theorem, that is, the 3rd isomorphic property in group semiautomata.

Let R be a (left) near-ring and G be an additive group. G is called an *R-group* if there exists a mapping $\mu : G \times R \to G$ defined by $\mu(x, a) = xa$ which satisfies (i) $x(a + b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a, b \in R$. We denote it by G_R.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N + x)a - xa \subset N$ for all $x \in G$, $a \in R$. This will be denoted by $N \lhd G$.

For the remainder basic concepts and results on near-rings, we refer to [7].

2. The Third Isomorphic Properties in GSA

From now on, the input set X will be fixed for every GSA. So a GSA on the set Q of states will be denoted by $(Q, +, \delta)$, or sometimes briefly denoted by Q if there arises no confusions. Moreover, for the brief notation $\delta(q, x)$ we shall write simply qx.

We would like to characterize some algebraic properties of the ideal of GSA which is related with the concept of ideal in R-group. Also, we will study GSA homomorphisms and the properties of ideals in GSA.

Now we give some definitions. Let $(Q, +, \delta)$ and $(Q', +, \delta')$ be two GSA with the same input set X and $f : Q \to Q'$ be a group homomorphism.

We call that f is a *GSA homomorphism* from $(Q, +, \delta)$ into $(Q', +, \delta')$ if f satisfies $(qx)f = (qf)x$ for all $q \in Q$, $x \in X$.

Let $f : (Q, +, \delta) \to (Q', +, \delta')$ be a GSA homomorphism. Then (1) f is called a *GSA monomorphism* if f is injective. (2) f is called a *GSA epimorphism* if f is surjective. (3) f is called a *GSA isomorphism* if f is bijective. In this case we denote that $Q \cong Q'$.

A normal subgroup K of $(Q, +)$ is called an *ideal* of a GSA $(Q, +, \delta)$ if $(K + q)x - qx \subseteq K$ for all $q \in Q$, $x \in X$. This will be denoted by $K \triangleleft (Q, +, \delta)$.

Clearly, 0 and Q are trivial ideals of $(Q, +, \delta)$.

Let $(Q, +, \delta)$ be a GSA with the input set X and K be an ideal of a GSA $(Q, +, \delta)$. Then the set $Q/K = \{q + K \mid q \in Q\}$ is a quotient group under addition as group theory. Define $\overline{\delta} : Q/K \times X \to Q/K$ by $(q + K, x) \mapsto qx + K$. Then $\overline{\delta}$ becomes a state transition function. This GSA $(Q/K, +\overline{\delta})$ is called a *quotient GSA*.

A subgroup S of $(Q, +)$ is called a *subgroup semiautomaton* or simply *SGSA* of a GSA $(Q, +, \delta)$ if $(S, +, \delta)$ is a GSA. This will be denoted by $S < (Q, +, \delta)$.

We do remark that there is no direct connection between ideals and SGSA as we shall see in the following examples.

Example 2.1. Let us consider the integer group modulo 4 $(Q, +) = (Z_4, +) = \{0, 1, 2, 3\}$ as the set of states and $X = \{x, y\}$ the set of inputs.

Define the state transition function δ by the following rule: $(a, x) \mapsto 0$, $\forall a \in Q$, and $(a, y) \mapsto 2$, $\forall a \in Q$.

It is easy to see that $S = \{0, 2\}$ forms a SGSA of $(Q, +, \delta)$ with $SX = \{0, 2\}$. Also, S is an ideal of $(Q, +, \delta)$, for instance, $(1 + 0)x - 1x =$

$1x - 1x = 0 - 0 = 0$, $(1+0)y - 1y = 1y - 1y = 2 - 2 = 0$, $(3+2)x - 3x = 1x - 3x = 0 - 0 = 0$ and $(3+2)y - 3y = 1y - 3y = 2 - 2 = 0$. They are all elements of S.

Example 2.2. Let us consider the Klein 4-group $(Q, +) = (K_4 +) = \{0, a, b, c\}$ as the set of states and $X = \{x\}$ the set of inputs.

Define the state transition function δ by the following graph:

$$a \mapsto b \mapsto c \mapsto 0 \mapsto 0$$

using x-action. It is easy to see that $S = \{0, c\}$ forms a SGSA of $(Q, +, \delta)$ with $SX = \{0\}$. However S is not an ideal of $(Q, +, \delta)$, for $(b+c)x - bx = ax - bx = b - c = a$. This is not an element of S.

Let $f : (Q, +, \delta) \to (Q', +, \delta')$ be a GSA homomorphism. Then (1) $0f^{-1} = \{a \in Q \mid af = 0\}$ is called the *kernel* of f, which is denoted by *Kerf* and (2) *Qf* is called the *image* of f, which is denoted by *Imf*. Obviously, *Kerf* and *Imf* are GSA.

First, we introduce some basic properties of GSA which are proved usually as group and ring theory.

Theorem 2.3 [2]**.** *Let* $(Q, +, \delta)$, $(Q', +, \delta')$ *be two GSA and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA epimorphism. Then the set* $Kerf = \{q \in Q \mid qf = 0\}$ *is an ideal of* $(Q, +, \delta)$.

Conversely, if $K \lhd (Q, +, \delta)$, *then the canonical group homomorphism* $\pi : Q \to Q/K$ *by* $q\pi = q + K$ *is a GSA epimorphism from* $(Q, +, \delta)$ *onto* $(Q/K, +, \mu)$, *for all* $q \in Q$ *and* $Ker\pi = K$, *where* $\mu : Q/K \times X \to Q/K$ *via* $(q + K, x) \mapsto qx + K$.

Proposition 2.4. *Let* $(Q, +, \delta)$ *be a GSA. Then an ideal K of* $(Q, +, \delta)$ *is a SGSA if and only if* $\{0\}X \subseteq K$.

Proof. Suppose that an ideal K of $(Q, +, \delta)$ is a SGSA. Then for all

$k \in K,$ and $x \in X,$ $kx - 0x = (0 + k)x - 0x,$ which is contained in K, so $0k \in K$. Hence $\{0\}X \subseteq K$. [Since K is a SGSA, $kx \in K$, $-0x \in K - kx \subseteq K$.]

Conversely, suppose that $\{0\}X \subseteq K$. Let $k \in K,$ and $x \in X$. To show that $kx \in K,$ consider $kx - 0x = (0 + k)x - 0x \in K,$ because that $K \lhd (Q, +, \delta)$. This equation implies $kx \in K + 0x \subseteq K$. □

Obviously, we get the following important statement using the group theory.

Theorem 2.5. *Let* $(Q', +, \delta'), (Q, +, \delta),$ *be two GSA and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA homomorphism. Then the set* $Kerf\{0\}$ *if and only if* f *is a GSA monomorphism.*

We, again, introduce the fundamental theorem for a homomorphism of GSA which was proved in [2], a useful result to study the sequent statements.

Theorem 2.6 (Fundamental theorem) [2]**.** *Let* $(Q, +, \delta), (Q', +, \delta')$ *be two GSA and* $f : (Q, +, \delta) \to (Q', +, \delta')$ *be a GSA homomorphism. Then*:

(1) *Kerf is an ideal of* $(Q, +, \delta)$.

(2) *There exists a unique GSA monomorphism* $g : Q/Kerf \to Q'$ *such that* $f = \pi g$, *where* $\pi : Q \to Q/Kerf$ *is a canonical GSA epimorphism. In particular,* $Q/Kerf \cong Imf$.

For further discussion, in a GSA $(Q, +, \delta),$ let S, T be the subsets of Q. We define their sum as the set

$$S + T = \{s + t \mid s \in S,\ t \in T\}.$$

It is not hard to see that the sum of two SGSA is not a SGSA in general, from Example 2.2. For example, $S = \{0, c\}$ and $T = \{0, b\}$ are two SGSA of $(K_4, +, \delta),$ but $S + T$ is not a SGSA of $(K_4, +, \delta)$.

However, we have the following substructures:

Theorem 2.7 [4]**.** *Let* $(Q, +, \delta)$ *be a GSA. Then*:

(1) *If S is a SGSA of* $(Q, +, \delta)$ *and I is an ideal of* $(Q, +, \delta)$, *then* $S + I$ *is a SGSA of* $(Q, +, \delta)$.

(2) *If* S *and* I *are ideals of* $(Q, +, \delta)$, *then* $S + I$ *is an ideal of* $(Q, +, \delta)$.

Theorem 2.8 (Second isomorphic theorem) [4]**.** *Let* $(Q, +, \delta)$ *be a GSA, S be a SGSA of* $(Q, +, \delta)$ *and I be an ideal of* $(Q, +, \delta)$. *Then*

(1) $S + I = I + S$, $S + I$ *is a SGSA of* $(Q, +, \delta)$, $I \lhd (S + I, +, \delta)$ *and* $I \cap S \lhd (S, +, \delta)$.

(2) $S/(I \cap S) \cong (I + S)/I$ *as quotient GSA.*

Theorem 2.9 (Third isomorphic theorem) [5]**.** *Let* $(Q, +, \delta)$ *be a GSA,* I, $K \lhd (Q, +, \delta)$ *and* $K < (Q, +, \delta)$ *with* $I \subseteq K$. *Then*

(1) $K/I \lhd Q/I$, *as GSA.*

(2) $(Q/I)/(K/I) \cong Q/K$, *as GSA-isomorphism.*

We note that the fundamental theorem, second isomorphic theorem and third isomorphic theorem in GSA are true in distributive GSA.

Now, we can introduce the following statement which is a characterization of distributive GSA.

Theorem 2.10. *Let* $(Q, +, \delta)$ *be a GSA. Then* $(Q, +, \delta)$ *is distributive if and only if for any* $x \in X$, *there exists a group homomorphism* $\phi_x : Q \to Q$.

Proof. Suppose that $(Q, +, \delta)$ is distributive. Then by definition of distributive GSA, we see that for any $a, b \in Q$ and for each $x \in X$, $(a + b)x = ax + bx$. From this equality, we can define a mapping $\phi_x : Q \to Q$ given by $a\phi_x = ax$. Then clearly, this mapping is well defined and group homomorphism, because $(a+b)\phi_x = (a+b)x = ax + ba = a\phi_x + b\phi_x$. Hence, $\phi_x : Q \to Q$ is a group homomorphism.

Conversely, suppose that there exists a group homomorphism $\phi_x : Q \to Q$ defined by $a\phi_x = ax$, for any $a \in Q$. Then for any $a, b \in Q$ and for each $x \in X$, $(a+b)x = (a+b)\phi_x = a\phi_x + b\phi_x = ax + bx$. Consequently, $(Q, +, \delta)$ is a distributive GSA. □

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