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QUADRUPLY INTEGRAL EQUIENERGETIC GRAPHS

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Abstract

The paper deals with (A, Q, L, D)- integral graphs and their (distance, Laplacian, signless Laplacian) energy. We construct an infinite family of pairs of non-cospectral integral, distance integral, Laplacian integral and signless Laplacian integral graphs having equal energy, distance energy, Laplacian energy and signless Laplacian energy.

1. Introduction and Preliminaries

Let A be the adjacency matrix of a simple graph $G = (V, E)$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be its eigenvalues, which are called the *eigenvalues* of G and form the spectrum of G. A graph is called *integral* if all its eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ are integers (see [9]). The *energy* $E(G)$ of G is defined as the

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sum of the absolute values of its eigenvalues. So $E(G) = \sum_{i=1}^{n} |\lambda_i|$ (see [7]). A graph G is said to be *hyperenergetic* if $E(G) > 2(n-1)$ (see [16]).

Similarly, the eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ of the distance matrix of the graph G form the distance spectrum of G, the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the Laplacian matrix of G form the Laplacian spectrum of G, and the eigenvalues $q_1, q_2, \ldots, q_n$ of the signless Laplacian matrix of the graph G form the signless Laplacian spectrum of G. A graph is called *distance integral* (or *D-integral*) if all its distance eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ are integers. The *distance energy* $E_D(G)$ of G is defined as $E_D(G) = \sum_{i=1}^{n} |\rho_i|$ (see [11]). A graph is called *Laplacian integral* (or *L-integral*) if all its Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ are integers. The *Laplacian energy* $E_L(G)$ of G is defined as $E_L(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$ (see [10]). A graph is called *signless Laplacian integral* (or *Q-integral*) if all its signless Laplacian eigenvalues $q_1, q_2, \ldots, q_n$ are integers. The *signless Laplacian energy* $E_Q(G)$ of G is defined as $E_Q(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|$ (see [1]).

The research on integral graphs started in 1974 (see [9]). There are over one hundred papers about integral graphs. The number of papers dealing with distance integral graphs, L-integral graphs and Q-integral graphs is much smaller. In recent years the research on graph energy and distance energy became a popular theme in mathematics because of their application in chemistry. The concept of graph energy originates from the Hückel molecular orbital approximation for the total π-electron energy (see [8, 10]). Distance energy is a useful molecular descriptor in QSRP modelling, as demonstrated in [2].

Two graphs G and H are called *cospectral* if their spectra coincide (see [6]). Similarly, we speak about distance cospectral graphs, Laplacian cospectral graphs and signless Laplacian cospectral graphs. Two graphs G and H are called (*distance*, *Laplacian*, *signless Laplacian*) *equienergetic* if they have the same (distance, Laplacian, signless Laplacian) energy, but that G and H are not (distance, Laplacian, signless Laplacian) cospectral (see [10]).

A graph is *regular* if each vertex has the same number of neighbors. The distance regular graphs are connected graphs having the prescribed number of vertices at any given distance from each of its vertices. The distance regular graph G has distance regularity k, if $\sum_{j=1}^{n} d(v_i, v_j) = k$ for every vertex $v_i \in V(G)$.

There are many papers containing constructions of equienergetic graphs or distance equienergetic graphs. L-equienergetic graphs are constructed, for example, in [15]. In [10], the authors constructed an infinite family of pairs of non-cospectral graphs having the same energy, distance energy and Laplacian energy. They stated that these seemed to be the first examples of 'triply equienergetic graphs'. All graphs constructed in their paper have diameter 2.

In this paper, we construct an infinite family of pairs of non-cospectral integral, distance integral, Laplacian integral and signless Laplacian integral graphs having equal energy, distance energy, Laplacian energy and signless Laplacian energy for arbitrarily large diameter. We also prove that these graphs are hyperenergetic.

2. Preliminary Results

In this section, we introduce H-join on graphs and give known results concerning to its spectrum and distance spectrum.

Definition 2.1 [17]**.** Let H be a graph with $V(H) = \{1, 2, ..., k\}$, and G_i be disjoint graphs of order n_i $(i = 1, 2, ..., k)$. Then the graph

$\vee_H (G_1, G_2, ..., G_k)$ is formed by taking the graphs $G_1, G_2, ..., G_k$ and joining every vertex of G_i to every vertex of G_j whenever *i* is adjacent to *j* in *H*.

In [13, 15], the above defined graph operation is called *generalized composition* with the notation $H[G_1, G_2, ..., G_k]$. Herein, we follow [17] and call it *H-join* of graphs $G_1, G_2, ..., G_k$, where *H* is an arbitrary graph of order *k*.

Theorem 2.1 [12]. *Let H be a graph with* $V(H) = \{1, 2, ..., k\}$, G_i *be distance t-regular graphs of order n for* $i \in \{1, 2, ..., k\}$ *and diameter at most* 2. *If* $G = \vee_H(G_1, G_2, ..., G_k)$, *then*

$$P_D(G, x) = n^k P_D\left(H, \frac{x-t}{n}\right)\prod_{i=1}^{k}\frac{P_D(G_i, x)}{x-t}.$$

Corollary 2.1 [12]. *Let H be a graph with* $V(H) = \{1, 2, ..., k\}$, G_i *be distance t-regular graphs of order* n $(i = 1, 2, ..., k)$ *and diameter at most* 2. *Let* $G = \vee_H(G_1, G_2, ..., G_k)$. *Then G is D-integral if and only if* $G_1, G_2, ..., G_k$ *and H are D-integral*.

Theorem 2.2 [13, Corollary 7a]. *Let H be a graph with* $V(H) = \{1, 2, ..., k\}$, G_i *be r-regular graphs of order n for* $i \in \{1, 2, ..., k\}$. *If* $G = \vee_H(G_1, G_2, ..., G_k)$, *then*

$$P_A(G, x) = n^k P_A\left(H, \frac{x-r}{n}\right)\prod_{i=1}^{k}\frac{P_A(G_i, x)}{x-r}.$$

Using Theorem 2.2, we have the following corollary.

Corollary 2.2 *Let H be a graph with* $V(H) = \{1, 2, ..., k\}$, G_i *be r-regular graphs of order* n $(i = 1, 2, ..., k)$ *and* $G = \vee_H(G_1, G_2, ..., G_k)$. *Then G is integral if and only if* $G_1, G_2, ..., G_k$ *and H are integral*.

Proof. If G is integral, then G_i and H are integral too. From Theorem 2.2, it follows that if H is integral with integer eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, then the roots of $n^k P_A\left(H, \frac{x-r}{n}\right)$ are also integers with the form $r + n\lambda_1$, $r + n\lambda_2, ..., r + n\lambda_k$. So $P_A(G, x)$ has integer roots. □

3. Main Result

In this section, we construct an infinite family of quadruply integral (non-cospectral) equienergetic graphs.

Notice that a circulant graph $G(n; S)$, where S is a subset of $\{1, ..., n-1\}$, is a graph on the set of n vertices $\{v_1, v_2, ..., v_n\}$ with an edge incident with v_i and v_j whenever $|i - j| \in S$ (see [14]).

Let $G_1 = G(12; \{1, 2, 5, 7, 10, 11\})$ and $G_2 = G(12; \{1, 4, 5, 7, 8, 11\})$. It is easy to verify that both the graphs are 6-regular graphs of order 12 with distance regularity 16 and diameter 2. The spectrum of G_1 is $\{-3^{(2)}, -2^{(3)}, 1^{(6)}, 6\}$ and its distance spectrum is $\{-3^{(6)}, 0^{(3)}, 1^{(2)}, 16\}$. The spectrum of G_2 is $\{-3^{(2)}, -2, -1^{(4)}, 1^{(2)}, 2^{(2)}, 6\}$ and its distance spectrum is $\{-4^{(2)}, -3^{(2)}, -1^{(4)}, 0, 1^{(2)}, 16\}$. Energy of both G_1 and G_2 is 24 and their distance energy is 36. So both graphs are integral, distance integral, non-cospectral, equienergetic and distance equienergetic.

Let Q_k be a hypercube. It is well known that Q_k is integral and distance integral for every k.

Theorem 3.1. *Let* $G_1 = G(12; \{1, 2, 5, 7, 10, 11\})$ *and*

$$G_2 = G(12; \{1, 4, 5, 7, 8, 11\}).$$

Let Q_k *be a hypercube. Let* $H_1 = \vee_{Q_k}(G_1, ..., G_1)$ *and* $H_2 = \vee_{Q_k}$ $(G_2, ..., G_2)$ *for every* $k \in N$. *Then both* H_1 *and* H_2 *are integral, distance* *integral*, *L-integral and Q-integral*. *Moreover*, H_1 *and* H_2 *have the same energy*, *distance energy*, *Laplacian energy and signless Laplacian energy*.

Proof. Since G_1 is a 6-regular graph and both G_1 and Q_k are integral, by Corollary 2.2, $H_1 = \vee_{Q_k}(G_1, ..., G_1)$ is integral, too. Similarly, since G_2 is 6-regular graph and both G_2 and Q_k are integral, by Corollary 2.2, $H_2 = \vee_{Q_k}(G_2, ..., G_2)$ is integral, too.

Since G_1 is distance 16-regular graph and both G_1 and Q_k are distance integral, by Corollary 2.1, H_1 is distance integral, too. Similarly, since G_2 is distance 16-regular graph and both G_2 and Q_k are distance integral, by Corollary 2.1, H_2 is distance integral, too.

Notice that both H_1 and H_2 are regular graphs. From [4, Chapter 2.2], it follows that a regular graph is integral if and only if it is Q-integral. So both H_1 and H_2 are Q-integral. From [5], it follows that a regular graph is integral if and only if it is L-integral. So both H_1 and H_2 are L-integral.

Using the formula $P_D(G, x) = n^k P_D\left(H, \frac{x-t}{n}\right)\prod_{i=1}^{k}\frac{P_D(G_i, x)}{x-t}$ from Theorem 2.1, we get that H_1 and H_2 are distance equienergetic.

Similarly, using the formula $P_A(G, x) = n^k P_A\left(H, \frac{x-r}{n}\right)\prod_{i=1}^{k}\frac{P_A(G_i, x)}{x-r}$ from Theorem 2.2, we get that H_1 and H_2 are equienergetic.

In [1], the authors state that if the graph G is regular, then $E(G) = E_L(G) = E_Q(G)$. So $E(H_1) = E_L(H_1) = E_Q(H_1)$ and $E(H_2) = E_L(H_2) = E_Q(H_2)$. Since $E(H_1) = E(H_2)$, $E_L(H_1) = E_L(H_2)$ and $E_Q(H_1) = E_Q(H_2)$. $\square$

Theorem 3.2. *Let* H_1 *and* H_2 *be those of Theorem* 3.1. *Then* H_1 *and* H_2 *are hyperenergetic*.

Proof. The spectrum of G_1 contains –3 with multiplicity 2 and –2 with multiplicity 3. So the spectrum of H_1 contains –3 with multiplicity $2 \cdot 2^k$ and –2 with multiplicity $3 \cdot 2^k$. Since the sum of all eigenvalues of H_1 is 0 (see, for example, [3]), $E(H_1) \geq 2 \cdot (3 \cdot 2 \cdot 2^k + 2 \cdot 3 \cdot 2^k) = 24 \cdot 2^k$. The number of vertices of H_1 is $12 \cdot 2^k$. So H_1 is hyperenergetic. Now, it is trivial that H_2 is also hyperenergetic. □

4. Conclusion

In the paper, we constructed an infinite family of pairs of non-cospectral integral, distance integral, Laplacian integral and signless Laplacian integral graphs having equal energy, distance energy, Laplacian energy and signless Laplacian energy for arbitrarily large diameter. Moreover, we give a method how to construct such classes of graphs. In Theorem 3.1, it is possible to use any pair of graphs G_1 and G_2 that

(i) are regular with the same regularity,

(ii) are distance regular with the same distance regularity,

(iii) have diameter 2,

(iv) are integral,

(v) are distance integral,

(vi) have the same energy,

(vii) have the same distance energy.

Other examples of such pairs of graphs are, for example,

(a) $G_1 = G(12; \{1, 2, 3, 5, 7, 9, 10, 11\})$ and

$G_2 = G(12; \{1, 3, 4, 5, 7, 8, 9, 11\})$,

(b) $G_1 = G(18; \{1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17\})$ and

$G_2 = G(18; \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 16, 17\})$,

(c) $G_1 = \vee_{P_2}(\overline{K_4}, 2K_3)$ and $G_2 = \vee_{P_2}(\overline{K_4}, C_6)$.

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References

[1] N. Abreu, D. M. Cardoso, I. Gutman, E. A. Martins and M. Robiano, Bounds for the signless Laplacian energy, Linear Algebra Appl. 435 (2011), 2365-2374.

[2] V. Consonni and R. Todeschini, New spectral indices for molecule description, MATCH Commun. Math. Comput. Chem. 60 (2008), 3-14.

[3] D. M. Cvetković, M. Doob and H. Sachs, Spectra of Graphs - Theory and Application, Academic Press, New York, 1980.

[4] D. Cvetković and S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, I, Publications de L'Institute Mathématique, Nouvelle Série, Tome 85(99) (2009), 19-33.

[5] D. Cvetković, Spectral theory of graphs based on the signless Laplacian (a quick outline), Research Report, http://www.sgt.pep.ufrj.br/pdf/signless_theory.pdf

[6] D. Cvetković, P. Rowlinson and S. K. Simić, Introduction to the Theory of Graph Spectra, Cambridge Univ. Press, Cambridge, 2009.

[7] I. Gutman, The energy of a graph, Ber. Math.-Statist. Sekt. Forschungsz. Graz 103 (1978), 1-22.

[8] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, J. Serb. Chem. Soc. 70 (2005), 441-456.

[9] F. Harary and A. J. Schwenk, Which graphs have integral spectra? Graphs Combinatorics, Lecture Notes in Mathematics 406, Springer-Verlag, Berlin, 1974, pp. 45-51.

[10] A. Ilić, M. Bašić and I. Gutman, Triply equienergetic graphs, MATCH Commun. Math. Comput. Chem. 64 (2010), 189-200.

[11] G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008), 461-472.

[12] M. Pokorný and P. Híc, Distance spectra of *H*-join graphs, Filomat, to appear.

[13] A. J. Schwenk, Computing the characteristic polynomial of a graph, R. Bary and F. Harary, eds., Graphs Combinatorics, Lecture Notes in Mathematics 406, Springer-Verlag, Berlin, 1974, pp. 153-172.

[14] N. Saxena, S. Severini and I. E. Shparlinski, Parameters of integral circulant graphs and periodic quantum dynamics, Int. J. Quantum Inf. 5 (2007), 417-430.

[15] D. Stevanović, Large sets of long distance equienergetic graphs, Ars Math. Contemp. 2 (2009), 35-40.

[16] D. Stevanović and I. Stanković, Remarks on hyperenergetic circulant graphs, Linear Algebra Appl. 400 (2005), 345-348.

[17] B. F. Wu, Y. Y. Lou and Ch. X. He, Signless Laplacian and normalized Laplacian on the *H*-join operation of graphs, Discrete Math. Algorithm Appl. 06 (2014), 14750046, doi: 10.1142/S1793830914500463.