Far East Journal of Mathematical Sciences (FJMS)
© 2015 Pushpa Publishing House, Allahabad, India
Published Online: June 2015
http://dx.doi.org/10.17654/FJMSJul2015_617_634
Volume 97, Number 5, 2015, Pages 617-634 **ISSN: 0972-0871**

ANALYTIC EXTENSION OF BEURLING'S ULTRADISTRIBUTIONS OF L_2-GROWTH BY CAUCHY INTEGRALS

Byung Keun Sohn

Department of Mathematics
Inje University
Kimhae, Kyungnam 621-749
Korea
e-mail: mathsohn@inje.ac.kr

Abstract

Let C be an open convex cone in $\mathbb{R}^N$ such that $\overline{C}$ does not contain any straight line and let $T^C = \mathbb{R}^N + iC \subset \mathbb{C}^N$. Let $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ be the spaces of Beurling's generalized tempered distributions and let $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ be the spaces of Beurling's ultradistributions of L_2-growth, which is a dual of $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. We show that if U is in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ such that $U * \phi \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ for $\phi \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ and is identical in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ with the Fourier-Laplace transform of the inverse Fourier transform of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$, then the Cauchy integral $C(U; z)$, $z = x + iy \in T^C$ of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ corresponding to C has U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$.

Received: February 15, 2015; Accepted: April 9, 2015

2010 Mathematics Subject Classification: 46F20, 30E20, 46F12.

Keywords and phrases: Beurling's ultradistributions, strongly elliptic, Cauchy integral, analytic extension.

Communicated by K. K. Azad

1. Introduction

Let $\mathcal{E}'(\mathbb{R}^N)$ and $\mathcal{D}'(\mathbb{R}^N)$ be the spaces of distributions of compact support and distributions, respectively, and let $\mathcal{O}'_\alpha$ be Bremermann's distribution spaces which are intermediate spaces between $\mathcal{E}'(\mathbb{R}^N)$ and $\mathcal{D}'(\mathbb{R}^N)$. Tillmann has obtained a characterization of analytic functions which represent an element of $\mathcal{E}'(\mathbb{R}^N)$ as boundary value of analytic functions in regions that are bounded by closed curve in the extended complex plane in [15] and similar analysis for an element of the Schwartz distributions $\mathcal{D}'_{L_p}(\mathbb{R}^N)$ in regions in $\mathbb{C}^n$ that are the product of unbounded domains in the plane in [16]. In [4] and [5], Bremermann has obtained the representation of elements in $\mathcal{O}'_\alpha$ as boundary values of analytic functions in half planes and tubes defined by quadrants. In both of these papers, the Cauchy integral of elements of $\mathcal{E}'(\mathbb{R}^N)$ and $\mathcal{O}'_\alpha$ plays essential roles in the analysis. In [6-11], Carmichael et al. have studied the problem discussed by Tillmann and Bremermann in the context of functions analytic in tube domains in $\mathbb{C}^n$ with base being an open convex cone in $\mathbb{R}^n$ such that $\overline{C}$ does not contain any straight line, i.e., a regular cone. We see from [11, Lemma 4], [6, Theorem 3] and [9, Lemma 5.2.3 and (5.45)] that the Cauchy integral $C(U; x+iy)$ of U in $\mathcal{E}'(\mathbb{R}^N)$ or $\mathcal{D}'(\mathbb{R}^N)$ corresponding to a regular cone C does not attain U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{D}'(\mathbb{R}^N)$. Also, in [10, Theorem 5.4 and 5.5] and [8, Theorem 4.2.5 and 4.2.6], we see that the Cauchy integral $C(U; x+iy)$ of U in $\mathcal{D}'(*, L_s(\mathbb{R}^N))$ corresponding to a regular cone C does not attain U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{D}'(*, L_s(\mathbb{R}^N))$, where $\mathcal{D}'(*, L_s(\mathbb{R}^N))$ is ultradistributions of Beurling type $\mathcal{D}'((M_p), L_s(\mathbb{R}^N))$ of L_s-growth or of Roumieu type $\mathcal{D}'(\{M_p\}, L_s(\mathbb{R}^N))$ of L_s-growth. Here M_p, $p = 0, 1, 2, \ldots$, is a certain sequence of positive numbers and $2 \le s < \infty$. In case of the Schwartz distributions $\mathcal{D}'_{L_p}(\mathbb{R}^N)$, the Cauchy integral $C(U; x+iy)$ of U in $\mathcal{D}'_{L_p}(\mathbb{R}^N)$, $1 \le p \le 2$, corresponding to a regular cone C does not attain U as boundary value as $y \to 0$, $y \in C$ in $\mathcal{S}'(\mathbb{R}^N)$ ([7, Theorem 9] and [9, Lemma 5.5.8]) but $C(U; x+iy)$ does have U as boundary value when the Cauchy integral of U is identical in tempered distributions, $\mathcal{S}'(\mathbb{R}^N)$, with the Fourier-Laplace transform of the inverse Fourier transform of U ([7, Theorem 10] and [9, Theorem 5.5.3]).

Let $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ be the spaces of Beurling's generalized tempered distributions and let $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ be the spaces of Beurling's ultradistributions of L_2-growth, which is a dual of $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. Let $U \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ be a convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, i.e., $U * \phi \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ when $\phi \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. In this paper, we show that if U is a convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ and is identical in $\mathcal{S}'_{(\omega)}$ with the Fourier-Laplace transform of the inverse Fourier transform of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$, then the Cauchy integral $C(U; z)$, $z = x+iy \in T^C$ of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ corresponding to C has U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$. Since $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ is the natural generalization of the space $\mathcal{D}'_{L_2}(\mathbb{R}^N)$, our results are extensions of [7, Theorem 10] and [9, Theorem 5.5.3] in the sense of $\mathcal{D}'_{L_2}(\mathbb{R}^N)$. Also, since we can find a weight function κ such that $\mathcal{D}'((M_p), L_s(\mathbb{R}^N)) = \mathcal{D}'_{L_s,(\kappa)}(\mathbb{R}^N)$ from Remark 3.11 in [13], our results contain the conditions under which the Cauchy integral $C(U; z)$, $z = x+iy \in T^C$ of U in $\mathcal{D}'((M_p), L_2(\mathbb{R}^N))$ corresponding to C has U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{S}'^{(M_p)} = \mathcal{S}'_{(\kappa)}(\mathbb{R}^N)$.

2. Beurling's Ultradistributions of L_p-growth

We will review Beurling's ultradistributions of L_p-growth in $\mathbb{R}^N$ and establish some of their properties which will be needed in the later. Firstly, we will review Beurling's ultradistributions which introduced by Braun et al. in [3].

Definition 1. A *weight* function is an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with the following properties:

(α) There exists $L \geq 0$ with $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$,

(β) $\int_1^\infty (\omega(t)/t^2)\, dt < \infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\psi : t \to \omega(e^t)$ is convex.

For a weight function ω, we define $\tilde{\omega} : \mathbb{C}^N \to [0, \infty)$ by $\tilde{\omega}(z) = \omega(|z|)$ and call this function ω to avoid abuse of notation. Here $|z| = \sum_{j=1}^N |z_j|$.

By (δ), $\psi(0) = 0$ and $\lim_{x\to\infty} x/\psi(x) = 0$. Then we can define the Young conjugate ψ^* of ψ by

$$\psi^* : [0, \infty) \to \mathbb{R}, \quad \psi^*(y) = \sup_{x \geq 0}(xy - \psi(x)).$$

Let ω be a weight function. For a compact set $K \subset \mathbb{R}^N$, we define

$$\mathcal{D}_{(\omega)}(K) = \{f \in \mathcal{D}(K) : \| f \|_{K,\lambda} < \infty \text{ for every } \lambda > 0\},$$

where $\| f \|_{K,\lambda} = \sup_{x\in K} \sup_{n\in\mathbb{N}_0^N} | f^{(n)}(x) | \exp(-\lambda\psi^*(n/\lambda))$.

Then $\mathcal{D}_{(\omega)}(K)$ equipped with its natural topology is a Fréchet space. For a fundamental sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of $\mathbb{R}^N$, we define

$$\mathcal{D}_{(\omega)}(\mathbb{R}^N) = \operatorname{ind}_j \mathcal{D}_{(\omega)}(K_j).$$

The dual $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ of $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ is equipped with its strong topology and the elements of $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ are called *Beurling's ultradistributions*.

We denote by $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ the set of all C^∞ functions *f* in $\mathbb{R}^N$ such that $\| f \|_{K, \lambda} < \infty$ for every compact *K* and every $\lambda > 0.$ For more details about $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{E}_{(\omega)}(\mathbb{R}^N),$ we refer to [3].

A function $\phi \in C^\infty(\mathbb{R}^N)$ is in the space $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ when, for every $m \in \mathbb{N}$ and $n \in \mathbb{N}^N,$

$$p_{m, n}(\phi) = \sup_{x \in \mathbb{R}^N} e^{m\omega(x)} | \phi^{(n)}(x) | < \infty$$

and

$$\pi_{m, n}(\phi) = \sup_{x \in \mathbb{R}^N} e^{m\omega(x)} | \hat{\phi}^{(n)}(x) | < \infty.$$

$\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ is endowed with the topology generated by the family $\{p_{m, n}, \pi_{m, n}\},$ where $m \in \mathbb{N}$ and $n \in \mathbb{N}^N$ of semi-norms. Then $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ is a Fréchet space and the Fourier transform $\mathcal{F}$ defines an automorphism of $\mathcal{S}_{(\omega)}(\mathbb{R}^N).$ $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ is a dense subspace of $\mathcal{S}_{(\omega)}(\mathbb{R}^N).$ The dual space of $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ is $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ and the Fourier transformation is defined on $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ as the transposed map of $\mathcal{F},$ thus $\mathcal{F}$ defines an automorphism of the strong dual $\mathcal{S}'_{(\omega)}(\mathbb{R}^N).$ For more details about $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{S}'_{(\omega)}(\mathbb{R}^N),$ we refer to [2].

For every $1 \leq p \leq \infty,$ $k \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R}^N),$ $\gamma_{k, p}(\phi)$ is defined as follows:

$$\gamma_{k,p}(\phi) = \sup_{\alpha \in \mathbb{N}^N} \| \phi^{(\alpha)} \|_p e^{-k\psi^*(\alpha/k)},$$

where $\| \cdot \|_p$ denotes the usual norm in $L_p(\mathbb{R}^N)$. ($\| f \|_\infty$ means the infimum of $\sup g(t)$ as g ranges over all functions which are equal to f almost everywhere.) If $1 \leq p < \infty$, then the space $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is the set of all C^∞-functions ϕ on $\mathbb{R}^N$ such that $\gamma_{k,p}(\phi) < \infty$ for each $k \in \mathbb{N}$. A function $\phi \in C^\infty(\mathbb{R}^N)$ is in $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ when $\gamma_{k,\infty}(\phi) < \infty$. We denote by $\mathcal{D}_{L_\infty,(\omega)}(\mathbb{R}^N)$ the subspace of $\mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ that consists of all those functions $\phi \in \mathcal{B}_{L_\infty,(\omega)}(\mathbb{R}^N)$ for which $\lim_{|x| \to \infty} \phi^{(\alpha)}(x) = 0$ for each $\alpha \in \mathbb{N}^N$. The topology of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, is generated by the family $\{\gamma_{k,p}(\phi)\}_{k \in \mathbb{N}}$ of semi-norms. Then it is obvious that $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ is continuously contained in the Schwartz's test spaces $\mathcal{D}_{L_p}$, $1 \leq p \leq \infty$. For more details about $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ and $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, we refer to [1].

From Proposition 2.1 of [1] and Proposition 2.9 of [1], we have the following:

Theorem 1. (i) $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, *are Fréchet spaces.*

(ii) $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ *is continuously contained in* $\mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N)$, *when* $1 \leq p \leq q \leq \infty$.

(iii) $\mathcal{D}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$ *with continuous and dense inclusions.*

(iv) $\mathcal{S}_\omega(\mathbb{R}^N)$ *is continuously contained in* $\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N)$, *hence in* $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$.

The dual of $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ will be denoted by $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ and it will be endowed with the strong topology. The elements of $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ are called the *Beurling's ultradistributions of* L_q*-growth* where q is the conjugate exponent of p.

From Proposition 2.2 in [1], we have the following:

Theorem 2. *Let* $T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$ *and* $\phi \in \mathcal{D}_{\omega}(\mathbb{R}^N)$ *be given. Then* $T * \phi \in L_q(\mathbb{R}^N)$, *where* $1/p + 1/q = 1$.

Definition 2. T is a *convolutor* in $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ if T is in $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ such that $T * \phi \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ for every $\phi \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$.

Assume that G is an entire function such that $\log|G(z)| = O(\omega(|z|))$ as $|z| \to \infty$. The functional T_G on $\mathcal{E}_{\omega}(\mathbb{R}^N)$ is defined by

$$\langle T_G, \phi \rangle = \sum_{\alpha=0}^{\infty} (-i)^{\alpha} \frac{G^{(\alpha)}(0)}{\alpha!} \phi^{(\alpha)}(0), \quad \phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^N).$$

The operator $G(D)$ defined on $\mathcal{D}'_{(\omega)}(\mathbb{R}^N)$ by

$$G(D) : \mathcal{D}_{(\omega)} \to \mathcal{D}_{(\omega)}, \quad \mu \to G(D)\mu = \mu * T_G,$$

is called an *ultradifferential operator of* (ω)*-class*. When $G(D)$ is restricted to $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$, $G(D)$ is a continuous operator from $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ into $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ and if for every $\phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^N)$,

$$(G(D)\phi)(x) = \sum_{\alpha=0}^{\infty} (i)^{\alpha} \frac{G^{(\alpha)}(0)}{\alpha!} \phi^{(\alpha)}(x), \quad x \in \mathbb{R}^N.$$

We see from [10, Proposition 2.4] that each ultradifferential operator $G(D)$ of (ω)-class defines a continuous linear mapping from $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ into itself for every $1 \leq p \leq \infty$.

Definition 3. An ultradifferential operator $G(D)$ of (ω)-class is said to be *strongly elliptic* if there exist $M > 0$ and $l > 0$ such that $|G(z)| \geq Me^{l\omega(|z|)}$ when $|\operatorname{Im} z| < M|Re\, z|$.

From [1, Corollary 2.8(i)], we have the following:

Theorem 3. *For a fixed* $1 \leq p < \infty$, *a function* $\phi \in L_p(\mathbb{R}^N)$ *is in* $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$ *if and only if* $G(D)\phi \in L_p(\mathbb{R}^N)$ *for every ultradifferential operator* $G(D)$ *of* (ω)*-class.*

We know from Theorem 2 that $\widetilde{T} * \phi \in L_q(\mathbb{R}^N)$ for $T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$, where $\widetilde{T}(\phi) = T(\widetilde{\phi})$ and $1/p + 1/q = 1$. For every ultradifferential operator $G(D)$, we have that $G(D)(\widetilde{T} * \phi) = \widetilde{T} * G(D)\phi \in L_q(\mathbb{R}^N)$ and we can apply Theorem 3 to conclude that $\widetilde{T} * \phi \in \mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N)$. (Remark of [1, Definition 3].)

3. Cauchy Integral of Beurling Ultradistributions of L_2-growth

In this section, we will find the condition under which the Cauchy integral $C(U;\, x + iy)$ of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ does have U as boundary value when $y \to 0$, $y \in C$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$.

Let C be a cone with vertex at 0, i.e., if $y \in C$ implies $\lambda y \in C$ for all $\lambda > 0$.

Definition 4. An open convex cone C such that $\overline{C}$ does not contain any straight line will be called a *regular cone*.

For a cone C, $\mathcal{O}(C)$ will denote the convex hull(envelop) of C and $T^C = \mathbb{R}^N + iC \subset \mathbb{C}^N$ is a tube in $\mathbb{C}^N$. The set $C^* = \{t \in \mathbb{R}^N; \langle t, y\rangle \geq 0, y \in C\}$ is the dual of the cone C.

Definition 5. Let C be a regular cone in $\mathbb{R}^N$. The *Cauchy kernel* $K(z-t)$, $z \in T^C = \mathbb{R}^N + iC$, $t \in \mathbb{R}^N$, corresponding to the tube T^C is

$$K(z-t) = \int_{C^*} e^{2\pi i\langle (z-t), \eta\rangle} d\eta, \quad z \in T^C, \quad t \in \mathbb{R}^N.$$

We see from Theorem 13 in [14] that for a regular cone C in $\mathbb{R}^N$, $K(z-t) \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \leq p \leq \infty$ as a function of $t \in \mathbb{R}^N$ for $z \in T^C$. Hence, we can define the following:

Definition 6. Let $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \leq p \leq \infty$. The *Cauchy integral* $C(U; z)$ of U in $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ corresponding to C is

$$C(U; z) = \langle U_t, K(z-t)\rangle, \quad z = x + iy \in T^C.$$

Now we will show that the Cauchy integral $C(U; z)$ of U in $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \leq p \leq \infty$, corresponding to C is analytic in T^C. If $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, then $T * \phi \in L_q$, $1/p + 1/q = 1$, by Theorem 2. Since $G(D)$ is a continuous operator from $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$ into $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$, we see from the remark of Theorem 3 and Theorem 2 that for $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \leq p \leq \infty$,

$$G(D)(T * \phi) = T * G(D)\phi \subset T * \mathcal{D}_{(\omega)}(\mathbb{R}^N) \subset L_q(\mathbb{R}^N),$$

where $1/p + 1/q = 1$. Hence, we have from Theorem 3 that for $1 \leq p \leq \infty$,

$$\text{if } T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N) \text{ and } \phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N), \text{ then } T * \phi \in \mathcal{D}'_{L_q,(\omega)}(\mathbb{R}^N),$$

where $1/p + 1/q = 1$. Then, by the exactly same lines in the proof of Lemma 8 in [14] (or Proposition 2.3 in [1]), we have the following:

Lemma 1. *Let* $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$. *Then there exist a strongly elliptic ultradifferential operator* $G(D)$ *of* (ω)*-class and* $f, g \in \mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N)$, $1/p + 1/q = 1$ *such that* $U = G(D)f + g$.

By combining Lemma 1 above and Lemma 9 in [14], we have the following:

Theorem 4. *Let* $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $1 \le p \le \infty$. *Then there exist a strongly elliptic ultradifferential operator* $G(D)$ *of* (ω)*-class and* $f \in \mathcal{D}_{L_q,(\omega)}(\mathbb{R}^N)$, $1/p + 1/q = 1$ *such that*

$$U(x) = G(D)f(x) = \sum_{\alpha=0}^{\infty} (-i)^{\alpha} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x).$$

Then, by the exactly same lines in the proof of Theorem 16 in [14], we have the following:

Theorem 5. *Let C be a regular cone in* $\mathbb{R}^N$ *and let* $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \le p \le \infty$. *Then the Cauchy integral* $C(U; z)$ *of* U *in* $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \le p \le \infty$, *corresponding to C is analytic in* T^C.

Let C be a regular cone in $\mathbb{R}^N$ and let U be a convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. Since the Cauchy kernel $K(z - t) \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $p \ge 2$,

$$K_y = K(x + iy) = \int_{C^*} e^{2\pi i \langle x+iy, \eta \rangle} d\eta \in \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N), \tag{1}$$

where $y \in C$ and $p \ge 2$. Since U is a convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, $C(U; x + iy) = (U * K_y)(x) \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, $y \in C$. Since $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$

$\subset \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ and $\mathcal{S}_{(\omega)} \subset \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$,

$$\langle C(U;\, x+iy),\, \varphi(x)\rangle = \langle (U * K_y)(x),\, \varphi(x)\rangle = \langle\langle U(t),\, K(x+iy-t)\rangle,\, \varphi(x)\rangle$$

is well-defined for $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$ and $y \in C$. From the representation of a convolutor $U \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ in Theorem 10 of [14], there exist a strongly elliptic ultradifferential operator $G(D)$ and $f \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ such that $U = G(D)f$. Since $f(t), K(z-t) \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, we see that $f^{(\alpha)}(t) \cdot K(z-t) \in L_1(\mathbb{R}^N)$ as a function of $t \in \mathbb{R}^N$ for $z \in T^C$. Hence, we see from change of order of integration that if $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$,

$$\begin{aligned}\langle C(U;\, x+iy),\, \varphi(x)\rangle &= \langle\langle U(t),\, K(x+iy-t)\rangle,\, \varphi(x)\rangle \\ &= \int\int \sum_{\alpha=0}^{\infty} i^{\alpha} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(t) K(z-t)\, dt\varphi(x)\, dx \\ &= \int \sum_{\alpha=0}^{\infty} i^{\alpha} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(t) \int K(z-t)\varphi(x)\, dxdt \\ &= \langle U(t) \langle K(z-t),\, \varphi(x)\rangle\rangle \qquad (2)\end{aligned}$$

for $z = x+iy,\ y \in C$.

We note that the reason for using $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$ in above is that the symmetry of $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ under the Fourier and inverse Fourier transforms will be needed later.

Combining (1), Lemma 19 in [14] and the continuity of convolutor U in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, we have the following:

Theorem 6. *Let C be a regular cone in $\mathbb{R}^N$. Let U be a convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. If $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$, then*

$$\lim_{y\to 0}\langle C(U;\, x+iy),\, \varphi(x)\rangle = \langle U,\, \mathcal{F}^{-1}[I_{C^*}(\eta)\hat{\varphi}(\eta) : t]\rangle,$$

where $I_{C^*}(\eta)$ *is a characteristic function of* C^*.

We see from Theorem 6 that the Cauchy integral $C(U;\, x+iy)$ of the convolutor in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ corresponding to a regular cone C in $\mathbb{R}^N$ does not attain U as boundary value when $y \to 0,\ y \in C$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$. To find the conditions under which $U \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ is represented as the boundary value of analytic function $C(U;\, z)$, the Cauchy integral of U in $\mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ corresponding to C, in tube when $y \to 0,\ y \in C$ in $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$, we need definition of the convolution $U * V$ for $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ and $V \in \mathcal{D}'_{L_q,(\omega)}(\mathbb{R}^N)$, $1/p + 1/q = 1$ and its properties.

Definition 7 [1, Definition 3]**.** Let $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ and $V \in \mathcal{D}'_{L_q,(\omega)}(\mathbb{R}^N)$, $1/p + 1/q = 1$. Then the convolution $U * V$ is the ultradistribution $U * V \in \mathcal{D}'_{(\omega)}$ given by $\langle U * V,\, \phi\rangle = \langle U,\, \tilde{V} * \phi\rangle$, where $\tilde{V}(\phi) = V(\tilde{\phi})$.

Let $\phi \in \mathcal{D}_{(\omega)}(\mathbb{R}^N)$ and $h \in L_p(\mathbb{R}^N) \subset \mathcal{S}'_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}'_{(\omega)}(\mathbb{R}^N)$. By the definition of T_G,

$$\begin{aligned}\langle G(D)h,\, \phi\rangle &= \langle h * T_G,\, \phi\rangle = \langle h(x),\, \langle T_G(y),\, \phi(x+y)\rangle\rangle \\ &= \left\langle h(x),\, \sum_{\alpha=0}^{\infty}(-i)^{\alpha}\frac{G^{(\alpha)}}{\alpha!}\phi^{(\alpha)}(x)\right\rangle \\ &= \sum_{\alpha=0}^{\infty}(-i)^{\alpha}\frac{G^{(\alpha)}}{\alpha!}\left(-\frac{\partial^{\alpha}}{\partial x^{\alpha}}\langle h(x),\, \phi(x)\rangle\right) \\ &= G(D)\langle h,\, \phi\rangle. \end{aligned} \tag{3}$$

If $U \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$ and $V \in \mathcal{D}'_{L_q,(\omega)}(\mathbb{R}^N)$, $1/p + 1/q = 1$, then there exist elliptic ultradifferential operators $G_1(D)$, $G_2(D)$ of (ω)-class and $f \in L_q$, $g \in L_p$ such that $U = G_1(D) f$ and $V = G_2(D) g$. Hence, we see from (3) that

$$U * V = G_1(D) G_2(D)(f * g) \text{ in } \mathcal{D}_{(\omega)}(\mathbb{R}^N). \tag{4}$$

We now consider Fourier transform of $\mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $p = 1$ or $2 \leq p < \infty$. From the proof of Proposition 2.9 in [1], we know the following:

$\mathcal{S}'_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N)$ with continuous and dense inclusion, consequently, each element of $\mathcal{D}'_{L_1,(\omega)}(\mathbb{R}^N)$ can be considered as an element of $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$. Then, if $T \in \mathcal{D}'_{L_1,(\omega)}(\mathbb{R}^N)$, then the Fourier transform $\mathcal{F}(T)$ is given by

$$\langle \mathcal{F}(T), \phi \rangle = \langle T, \hat{\phi} \rangle, \quad \phi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N).$$

By Theorem 2.5 in [1], there exist $f \in L_\infty(\mathbb{R}^N)$ and an elliptic ultradifferential operator $G(D)$ such that $T = G(D) f$. Then it follows that $\mathcal{F}(T) = G(-t)\mathcal{F}(t)$ on $\mathcal{D}_{(\omega)}(\mathbb{R}^N)$. We note that if $T \in \mathcal{D}'_{L_p,(\omega)}(\mathbb{R}^N)$, $2 \leq p < \infty$, is represented as $T = G(D) f$, $f \in L_q(\mathbb{R}^N)$, where $1/p + 1/q = 1$, then $\mathcal{F}(T) = G(-\xi)\hat{f}(\xi)$.

Now we will find the conditions under which $U \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ is represented as the boundary value of the Cauchy integral $C(U; z)$, $z = x + iy \in T^C$ of U corresponding to C in tube when $y \to 0$, $y \in C$.

Lemma 2. *Let $\mathcal{S}(\mathbb{R}^N)$ be the set of all C^∞ functions polynomially rapidly decreasing at infinity with the appropriate semi-norms in the sense of Schwarz. Then $\mathcal{S}(\mathbb{R}^N)$ is contained in $\mathcal{D}_{L_1,(\omega)}(\mathbb{R}^N)$, hence in $\mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N)$, $p \geq 1$.*

Proof. It suffices to show the results in one dimensional Euclidean space $\mathbb{R}$. Let $\phi \in \mathcal{S}(R)$. By the facts for Fourier transform, for every $n \in \mathbb{N}$,

$$-2\pi x^2 \phi^{(n)}(x) = (-n)(n-1)\int_{\mathbb{R}} (iy)^{n-2}\hat{\phi}(y)e^{ixy}dy$$
$$+ 2in\int_{\mathbb{R}} (iy)^{n-1}\hat{\phi}'(y)e^{ixy}dy$$
$$+ \int_{\mathbb{R}} (iy)^n \hat{\phi}^{(2)}(y)e^{ixy}dy. \tag{5}$$

Since $(n/k)/\psi^*(n/k)$ is decreasing and goes to 0 as n goes to ∞ for every $k \in \mathbb{N}$,

$$n < k\psi^*(n/k), \quad n \in \mathbb{N}, \tag{6}$$

for every $k \in \mathbb{N}$. From (2), (3), and the fact that $\hat{\phi} \in \mathcal{S}(R)$, we have that for every $k \in \mathbb{N}$,

$$| x^2 \phi^{(n)}(x) | \le C_\phi e^{k\psi^*(n/k)}, \quad n \in \mathbb{N},$$

where C_ϕ is a constant depending on ϕ. Thus, $\phi \in \mathcal{D}_{L_1,(\omega)}(\mathbb{R})$. □

Theorem 7. *Let C be a regular cone in* $\mathbb{R}^N$. *Let U be a convolutor in* $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ *and* $U = \mathcal{F}[V]$ *in* $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ *for* $V \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ *with supp*$(V) \subset C^*$ *and* $C(U; z)$, $z = x + iy \in T^C$, *be the Cauchy integral of U corresponding to C. Then there exist an entire function* $G(z)$ *with* $\log| G(z) | = O(\omega(| z |))$ *as* $z \to \infty$ *and* $f \in L_2$ *such that*

$$V = G(x)\hat{f}(x) \text{ in } \mathcal{S}'_{(\omega)}(\mathbb{R}^N) \tag{7}$$

and

$$C(U; z) = \langle V, e^{2\pi i\langle z, t\rangle}\rangle, \quad z = x + iy \in T^C \tag{8}$$

as elements of $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ *and*

$$C(U;\, z) \to U \ \text{in} \ \mathcal{S}'_{(\omega)}(\mathbb{R}^N) \ \text{as} \ y \to 0,\ y \in C. \tag{9}$$

Proof. Since $U = \mathcal{F}[V]$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$ and $\mathcal{F}$ defines an automorphism of the strong dual $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$, $V = \mathcal{F}^{-1}[U]$ in $\mathcal{S}'_{(\omega)}(\mathbb{R}^N)$. From the representation theorem of U in $\mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$, we let $U = G(D)f$, where $G(D)$ is a strongly elliptic ultradifferential operator and $f \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$. Then if $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$, we see from Theorem 1(iv) that $\langle V, \varphi\rangle$ is well-defined and

$$\begin{aligned}\langle V(x), \varphi(x)\rangle &= \langle U(t), \mathcal{F}^{-1}[\varphi(x);\, t]\rangle \\ &= \langle f(t), G(D)\mathcal{F}^{-1}[\varphi(x);\, t]\rangle \\ &= G(x)\langle f(t), \mathcal{F}^{-1}[\varphi(x);\, t]\rangle \\ &= \langle G(x)\hat{f}(x), \varphi(x)\rangle,\end{aligned}$$

hence (7) is satisfied.

Now let $\alpha_\varepsilon(t) \in C^\infty(\mathbb{R}^N)$ which is 1 on an ε-neighborhood of C^* and has support in a 2ε-neighborhood of C^*. By Lemma 2,

$$\alpha_\varepsilon(t)e^{2\pi i\langle z, t\rangle} \in \mathcal{S}(\mathbb{R}^N) \subset \mathcal{D}_{L_p,(\omega)}(\mathbb{R}^N), \quad 1 \le p \le \infty \tag{10}$$

as a function of t for $z \in T^C$. Then

$$\langle V(t), e^{2\pi i\langle z, t\rangle}\rangle = \langle V(t), \alpha_\varepsilon(t)e^{2\pi i\langle z, t\rangle}\rangle$$

is well-defined from (10) and the fact that $supp(V) \subset C^*$ and analytic in T^C by Theorem 4.7.4 in [9]. If $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$, then

$$\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}\hat{\varphi}(t) \in \mathcal{S}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N), \quad y \in C,$$

hence $\langle V(t), \alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle$ is well-defined.

Since $G(t)$ satisfies $\log|G(z)| = O(\omega(|z|))$ and $\lim_{t\to\infty}\omega(t)/t = 0$ from [Remark 1.2, 11], $G(t)e^{-2\pi\langle y,t\rangle} \in L_2(\mathbb{R}^N)$, $y \in C$. Since $f \in L_2(\mathbb{R}^N)$, we see that $\hat{f} \in L_2(\mathbb{R}^N)$ by Parseval's identity. Hence, $G(t)\hat{f}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle} \in L_1(\mathbb{R}^N)$. By change of order of integration that if $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$, then

$$\begin{aligned}\langle\langle V(t), e^{2\pi i\langle z,t\rangle}\rangle, \varphi(x)\rangle &= \langle\langle V(t), \alpha_\varepsilon(t)e^{2\pi i\langle z,t\rangle}\rangle, \varphi(x)\rangle \\ &= \langle V(t), \alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle \\ &= \langle \mathcal{F}[V(t)I_{C^*}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}], \varphi(x)\rangle, \qquad (11)\end{aligned}$$

where $z = x + iy \in T^C$ and I_{C^*} is the characteristic function of C^*. On the other hand, for $\phi \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$ and $y \in C$, by change of order of integration,

$$\begin{aligned}&\langle \mathcal{F}[V] * \mathcal{F}[I_{C^*}\alpha_\varepsilon e^{-2\pi\langle y,\cdot\rangle}], \phi\rangle \\ &= \langle \mathcal{F}[V(t); x], \langle \mathcal{F}[I_{C^*}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}; x'], \phi(x+x')\rangle\rangle \\ &= \langle\langle V(t), e^{-i\langle t,x\rangle}\rangle, \langle I_{C^*}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}, \langle e^{\langle -it,x'\rangle}, \phi(x+x')\rangle\rangle\rangle \\ &= \langle V(t)I_{C^*}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y,t\rangle}, \hat{\phi}(t)\rangle \\ &= \langle \mathcal{F}[VI_{C^*}\alpha_\varepsilon e^{-2\pi\langle y,\cdot\rangle}], \phi(\cdot)\rangle. \qquad (12)\end{aligned}$$

Since $U * \int_{C^*} e^{2\pi i\langle(x+iy),\eta\rangle}d\eta \in \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N)$ by (1) and hypothesis on U and $\mathcal{S}_{(\omega)}(\mathbb{R}^N) \subset \mathcal{D}_{L_2,(\omega)}(\mathbb{R}^N) \subset \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$, we see that

$$\left\langle U * \int_{C^*} e^{2\pi i \langle (x+iy), \eta \rangle} d\eta, \varphi(x) \right\rangle$$

is well-defined for $\varphi(x) \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$. From (11) and (12), we have that for $\varphi(x) \in \mathcal{S}_{(\omega)}(\mathbb{R}^N)$,

$$\begin{aligned} &\langle\langle V(t), e^{2\pi i \langle z, t \rangle}\rangle, \varphi(x)\rangle \\ &= \langle \mathcal{F}[V(t); x] * \mathcal{F}[I_{C^*}(t)\alpha_\varepsilon(t)e^{-2\pi\langle y, t\rangle}; x], \phi(x)\rangle \\ &= \left\langle U * \int_{C^*} e^{2\pi i \langle (x+iy), \eta\rangle} d\eta, \varphi(x)\right\rangle \\ &= \langle C(U; x+iy), \varphi(x)\rangle, \quad y \in C, \end{aligned} \tag{13}$$

hence (8) is satisfied.

Since $\alpha_\varepsilon(t)e^{\langle -2\pi y, t\rangle}\hat{\varphi}(t) \to \alpha_\varepsilon(t)\hat{\varphi}(t)$ in $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$ as $y \to 0$, $y \in C$, we see from (11), (13), and the continuity of $V \in \mathcal{D}'_{L_2,(\omega)}(\mathbb{R}^N)$ that for $\varphi(x) \in \mathcal{S}_{(\omega)}$,

$$\begin{aligned} \langle C(U; x+iy), \varphi(x)\rangle &= \langle\langle V(t), e^{2\pi i\langle z, t\rangle}\rangle, \varphi(x)\rangle \\ &= \langle V(t), \alpha_\varepsilon(t)e^{-2\pi\langle y, t\rangle}\hat{\varphi}(t)\rangle \\ &\to \langle V(t), \alpha_\varepsilon(t)\hat{\varphi}(t)\rangle \\ &= \langle \hat{V}, \varphi\rangle = \langle U, \varphi\rangle \end{aligned}$$

as $y \to 0$, $y \in C$. Thus, (9) is satisfied. □

References

[1] J. J. Betancor, C. Fernádez and A. Gabis, Beurling ultradistributions of L_p-growth, J. Math. Anal. Appl. 279 (2003), 246-265.

[2] G. Björck, Linear partial differential operators and generalized distributions, Ark. Mat. 6 (1965), 351-407.

[3] R. W. Braun, R. Meise and B. A. Tayor, Ultradifferentiable functions and Fourier analysis, Results Math. 17 (1990), 206-237.

[4] H. J. Bremermann, Schwartz distributions as boundary values of *n*-harmonic functions, J. Anal. Math. 14 (1965), 5-13.

[5] H. J. Bremermann, Distributions, Complex Variables, and Fourier Transforms, Addison-Wesley, New York, 1965.

[6] R. D. Carmichael, Representation of distributions in $\mathcal{O}'_\alpha$ as boundary values of functions in tube domain, Glas. Mat. Ser. III 7 (1972), 213-228.

[7] R. D. Carmichael, Generalized Cauchy and Poisson integrals and distributional boundary values, SIAM J. Math. Anal. 4(1) (1973), 198-219.

[8] R. D. Carmichael, A. Kamiński and S. Pilipović, Boundary Values and Convolution in Ultradistribution Spaces Series on Analysis, Applications and Computation, Vol. 1, World Scientific, 2007.

[9] R. D. Carmichael and D. Mitrović, Distributions and Analytic Functions, Longman Scientific and Technical, 1989.

[10] R. D. Carmichael, R. S. Pathak and S. Pilipović, Cauchy and Poisson integrals of ultradistributions, Complex Var. Theory Appl. 21 (1993), 49-72.

[11] R. D. Carmichael and W. W. Walker, Representation of distributions with compact support, Manuscripta Math. 11 (1974), 305-338.

[12] M. G. Gómez-Collado, Almost periodic ultradistributions of Beurling and of Roumieu type, Proc. Amer. Math. Soc. 129(8) (2000), 2319-2329.

[13] R. Meise and B. A. Taylor, Whitney's extension theorem for ultradifferentiable functions of Beurling type, Ark. Mat. 262 (1988), 265-287.

[14] B. K. Sohn, Cauchy and Poisson integral of the convolutor in Beurling ultradistributions of L_2-growth, Int. J. Math. Math. Sci. 2014 (2014), Art. ID 926790, 8 pp.

[15] H. G. Tillmann, Randverteilungen analytischer funktionen und distributionen, Math. Z. 59 (1953), 61-83.

[16] H. G. Tillmann, Distributionen als randverteilungen analytischer funktionen, II, Math. Z. 76 (1961), 5-21.