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REFLECTION ON SELF-EFFICACY TRAINING AND SKILL TRAINING TO FOSTER STUDENT PERFORMANCE IN GEOMETRY: A CASE STUDY

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Abstract

This case study describes how reflection-on-self-efficacy-to-learn-mathematics training combined with skill training enhanced junior-high students' performance in geometry. Skill training was based on geometric constructions using traditional as well as computational tools. To raise students' interest, three simple and beautiful theorems presenting the preservation principle were introduced to the students: the Steiner's theorem for the trapezoid, the Napoleon's theorem for triangle and the Van Aubel's theorem for quadrilaterals. These theorems were not part of the regular curriculum. The study participants were eighth graders from a junior-high school in northern Israel $(n = 26)$. The study was designed to develop a rich perception of the factors that constructed the students' self-efficacy, self-regulation, the instruction-learning-assessment culture and progress,

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by describing and analyzing them qualitatively. Research tools were: students' reflection tasks $(n = 20)$, non-participant classroom observations $(n = 20)$, teacher reflection and field notes and a structured feedback questionnaire. The data were analyzed using constant comparative analysis and grounded theory techniques. The analysis produced a refined list of categories that were developed into conceptual abstractions. Qualitative validity of data gathering was found by triangulation. Inter-rater agreement was 86%. The means of the students' answers in the structured questionnaire were calculated and explained. Results show that these students made outstanding progress, which is rare in the current school system. The theoretical contribution of this study is the successful combination of reflection-on-self-efficacy-to-learn-mathematics training and skill training to empower geometry learning. Enhanced self-efficacy reinforced skill acquisition, which in turn contributed to higher efficacy beliefs and vice-versa.

1. Introduction

Self-efficacy (SE) is considered as a key factor of motivation, a central mechanism of personal agency [1, 2]. It refers to an individual's capacity to exercise some measure of control over his/her own functioning and over environmental events. SE beliefs are context specific evaluations of a person's capability to organize and implement actions necessary to attain designated skill performance that affect their lives. Such beliefs are claimed to affect thought patterns and performance on a wide variety of tasks. Empirical studies have lent support to the contention that SE plays a pivotal role in human functioning, and in the area of education SE has been identified as a significant predictor of achievement in various subjects [1-4].

Scholars have reported that, regardless of previous achievement of ability, high-efficacious students work harder, persist longer, persevere in the face of adversity, have greater optimism and lower anxiety, and achieve more than low-efficacious students. SE has an effect on cognitive and meta-cognitive functioning, such as problem solving, decision making, analytical strategy use, self-evaluation, time management, and self-regulation strategies, all of which affect academic achievement [1, 5, 6]. Efficacy beliefs provide students with a sense of agency to motivate their learning through use of self-regulatory processes. Greater academic self-regulation of self-efficacious students produces higher academic achievements [4].

1.1. Reflection on SE to learn mathematics

Studies on SE have consistently demonstrated that efficacy beliefs are influenced by acquisition of skills, including modeling of cognitive strategies, self-verbalization of cognitive operations and strategies, goal setting, self-monitoring, and social comparison [4]. Other studies have shown that different types of psychological influence, such as evaluative feedback and social comparative information, have an impact on efficacy beliefs [1]. The present study investigated a new type of psychological training-reflection on SE to learn mathematics. The ability to discern, to weigh, and integrate relevant sources of efficacy information improves with the development of cognitive skills for processing information. These include intentional memory, inferential, and integrative cognitive capabilities for forming self-conceptions of efficacy. The development of self-appraisal skills also relies on growth of self-reflective meta-cognitive skills to monitor one's regulative thought, to evaluate the adequacy of one's self-assessment, and to make corrective adjustments of one's appraisals if necessary [1]. Effective intellectual functioning requires meta-cognitive skills such as organizing, monitoring, evaluating, and regulating one's thinking processes [7, 8].

Studies have shown that reflection enhances meta-cognitive processes such as: self-monitoring, self-evaluation, self-reaction and attribution [9]. Since self-appraisal of efficacy is a form of meta-cognition and efficacy beliefs are structured by experience and reflective thought [1], we view reflection on SE as a forethought process so that the mental processes students will go through, while reflecting on it over a specific length of time, will have an effect on the processing of their efficacy appraisals. Their appraisals will undergo a change. Reflection involves investment of time and creative mental effort that culminates in reconstruction of knowledge and gaining new insights [10].

As students get older, their efficacy beliefs concerning specific behaviors tend to become shaped and molded. After they develop adequate ways of managing regularly recurring situations, students tend to act on their perceived efficacy without requiring continuing directive or reflective thought. As long as people continue to believe in their abilities to perform a given activity, they act habitually on that belief [1]. Routinizations can detract from the best use of personal capabilities, however, when people react in fixed ways to situations requiring discriminative adaptability. Routinization is also self-limiting when people settle for low-level pursuits on the basis of self-doubts of efficacy and no longer reappraise their capabilities or raise their self-perception. A change occurs only when the person encounters a significant experience. When routinized behavior fails to produce expected results, the cognitive control system comes back into play. New modes are considered and tested [1]. This being so, reflecting on SE forces those who tend to avoid thinking and rely on previous efficacy appraisals to rethink and repeatedly revise what is produced in order to fulfill personal standards of quality [1, 10].

1.2. Mathematics and self-regulation skill training

Successful functioning in the current era demands an adaptable, thinking, autonomous self-regulated learner [11]. Self-regulation is defined as "the degree to which students are meta-cognitively, motivationally, and behaviorally active participants in their own learning process" [9, p. 167]. The most important competencies required of such a person include: (a) cognitive competencies, meta-cognitive competencies [9], and social competencies. The need to develop these competencies expands the scope of education and thus creates a challenging enterprise for educators [12], which leads to changes in the instruction-learning-assessment culture (ILA).

Mathematics research shows that self-regulation has an effect on mathematical performance [13]. Self-regulated learners believe that opportunities to take on challenging tasks, practice their learning, develop a deep understanding of subject matter, and exert effort will enhance academic success [14]. In part, these characteristics may help to explain why self-regulated learners usually exhibit a high sense of SE [15]. These learners hold incremental beliefs about intelligence (as opposed to fixed views of intelligence) and attribute their successes or failures to factors within their control, e.g., effort expended on a task, effective use of strategies [16]. The relationship between self-regulation and SE is twofold: First, both self-regulation and SE involve meta-cognition and second, SE plays a crucial goal in every phase of self-regulation [1, 6, 7].

This challenging initiative causes changes in the ILA that may enhance mathematical achievement such as: use of technological aids in the classroom, flexible ways of teaching, teacher-student cooperation in learning, teacher considered a coach rather than a sole source of knowledge, nurturing SE and reducing anxiety, development of meta-cognitive competencies, and high order thinking performance tasks. This new culture fits the constructivists' approach to ILA in that instruction promotes the development of reflective active lifelong learners who make use of knowledge, investigate, construct meaning, and evaluate their own achievements. Task types are usually performance tasks that require integrative thinking processes, discovery of connections and relationships, elaborations, generalizations, and knowledge production [17, 11].

The combination of knowledge of one's own cognition and action control as well as its evaluation and personal effort is assumed to result in creation. Meta-cognition is a substantial ingredient of creative thinking, as it involves the knowledge of cognition and the regulation of cognition and action [18]. Creative actions might benefit from meta-cognitive skills and vice-versa, regarding the knowledge of one's own cognition and the regulation of the creative process.

In our changing technological society, innovations are recognized as the vehicle of economic and social growth and welfare. Promoting these innovations necessitates problem solving, connecting between domains and creative skills [19]. Although mathematicians and mathematics educators agree that creativity plays an essential role in doing mathematics, it is not often nurtured in schools [20]. Creativity in mathematics may be characterized in several ways, such as divergent and flexible thinking, or “unusual and insightful solutions to a given problem” [20, p. 15], connections between domains, or the ability to produce work that is both novel (e.g., original or unexpected) and appropriate (e.g., useful or adaptive to task constraints) [21]. Of all the aspects of the various existing definitions of creativity, novelty or originality is widely acknowledged as the most appropriate because creativity is generally viewed as a process related to the generation of original ideas, approaches, or actions [22]. Working on mathematical tasks may influence not only the mathematical content that is learned, but also how students experience mathematics [23]. Thus, working on appropriate creative tasks might impact students’ perceptions of mathematics as a creative domain. More recently, Sriraman [20] claimed that “mathematical creativity ensures the growth of the field of mathematics as a whole” [20, p. 13]. As such, promoting mathematical creativity is one of the aims of mathematics education. Creative students are successful self-regulated students, who control and monitor their learning environment [21, 24].

Considering that SE alone will not enhance performance if students lack specific skills needed for specific tasks, and that skill training alone might not be sufficient if a student lacks motivation to learn, both reflection on SE and mathematics and self-regulation skill training were implemented concurrently. As far as we know, this combination of types of training has not previously been used in mathematics learning.

Our aim was to follow our students’ SE, mathematics and self-regulatory skills, and progress in mathematics, while they were undergoing the combined training, and to describe and gain insights about the process. We asked the following questions:

1. What characterizes the students’ efficacy beliefs to learn mathematics?
2. What characterizes the students’ self-regulation?

3. How were the students progressing in mathematics?

4. What characterizes the climate and culture of thinking and working in the classroom?

5. How did the students feel about the experience they underwent?

2. Methods

2.1. Participants

The study participants were eighth graders from a junior-high school in northern Israel $(n = 26)$. That class was chosen because it was considered as a good class. The students' parents were invited to participate in the first meeting between the students and researchers to gain the parents' support for their children's activities in the course.

2.2. Design

This case study was designed to elicit tacit knowledge on the students' experience, and describe and analyze them qualitatively. The research design was of an inductive and emergent nature, wherein evidence of ongoing processes in the course was collected under natural conditions. The role of the researcher as a data collector was integral to the data that emerged. Guba and Lincoln [25] stressed that humans are uniquely qualified for qualitative inquiry. This is due particularly to their ability to be responsive to the cues in the natural situation, to collect information about multiple factors and across multiple levels, to take a holistic look at situations and try to reach tacit knowledge, to process data and generate and test hypotheses immediately, to ask for elaboration and clarification [25]. Throughout the data collection and analysis, an effort was made to capitalize on these qualities to develop a rich perception of the factors that constructed the students' SE, self-regulation, the ILA culture, and students' progress and achievement in this case. These methods attempted to present the data from the perspective of the observed group, so that the researchers' cultural and intellectual bias did not distort the collection, interpretation, or presentation of the data. The qualitative design consisted of systematic, yet flexible, guidelines for collecting and analyzing data to construct abstractions [26]. Listening, observing, communicating, and remaining in the field of the study for a prolonged period allowed understanding and hence creating an authentic picture of the participants' thinking in regard to their capability to perform mathematical tasks.

2.3. Research tools

2.3.1. Data collection tools

For data collection, we used four tools:

(a) Students' reflection tasks during an academic year $(n = 20)$. This tool was used for collecting data as well as for enhancing SE to learn mathematics.

(b) The skill training was followed by 20 non-participant classroom observations during the academic year.

(c) Teacher reflection written in a diary and field notes taken on the procedure during the academic year.

(d) A nine-question structured Likert-type feedback questionnaire. At the end of the research activity, the students were asked to circle a score on a 5-point scale (5 = fully agree; 1 = disagree) for each of the nine statements.

2.3.2. Intervention tools

Intervention tools were students' reflection-on-SE-to-learn-mathematics training and skill training.

2.3.2.1. Implementing reflection on SE

The students had to write openly and in-depth, once every two weeks, about their self-beliefs to perform mathematical assignments. Each time, they were given several instructions or questions, which guided them to think in different directions, e.g., "please write openly about your thoughts, feelings, and ideas, when you think about what is currently being done in class: practicing, being tested, and try to explain to yourselves what it means to be able to learn mathematics. What does it take to succeed?" They could write whatever they wanted without answering all the questions. The teacher reinforced the students and encouraged high order thinking skills. By the end of the school year, the students had written 20 reflection tasks.

2.3.2.2. Implementing mathematics and self-regulation skill training and study procedure

The course contained six meetings of four hours each. In a preparatory meeting at the beginning of the course, the students were informed about the course content and goals and expressed their wish to participate.

The first two meetings were dedicated to familiarization with the GeoGebra software, which is a dynamic geometric software (D.G.S).

This was intended mainly to make the world of computerized technology more accessible to the students, for them to appreciate its potential and to acquire the basic skills. It should be noted that, as part of the general curriculum, 7th grade students learn how to use an electronic spreadsheet for data processing. As part of their acquaintance with the GeoGebra software, the students were presented with its geometric and computational tools, and learned the basic skills needed to use them.

The third meeting was devoted to geometric constructions, where the students learned how to use a straightedge and compass to carry out basic constructions such as: bisection of the segment, bisection of an angle, construction of a perpendicular, construction of a triangle based on the lengths of each side, etc. The students were required to prove that each construction was correct.

In the last three meetings, the students were presented with three construction tasks, based on three simple and beautiful geometric theorems, which carried the names of renowned mathematicians. Mathematicians were already using these theorems for creating multiple shapes using the preservation property, 120 years ago [27-30]. They used these theorems as problem solving tools and found many features connected to them. These theorems did not appear in the curriculum text books, therefore were new to the high school students, and even to some of the teachers. After each construction was completed and guiding questions were presented, the students moved to the computer workshop, where each pair of students shared a computer which featured an applet of the theorem. The applet allowed the students to drag vertices using the mouse and to investigate the task in a dynamic manner, while observing preservation of shapes, line lengths, obtaining different shapes for the same task, and mainly performing a generalization. The investigative task of the course involved the following procedure:

Stage 1. Acquaintance with Steiner's theorem for the trapezoid

After basic acquaintance with the fundamental constructions using a straightedge and a compass, and practice performing several constructions based on these, the students were presented with Steiner's theorem for the trapezoid: "*the straight line that connects the point of intersection of the diagonals of the trapezoid with the point of intersection of the continuations of the trapezoid's legs bisects its bases*" (Figure 1).

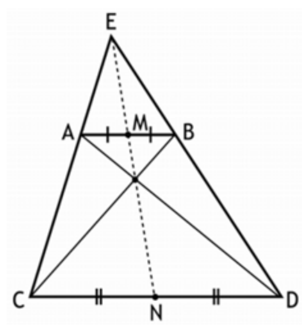

Figure 1. Steiner's theorem for the trapezoid.

To introduce the students to the world of the history of mathematics, they were briefly given details on the work of the person who discovered the theorem in the fields of mathematics and science.

The students were presented with a page on which four different trapezoids were drawn, and they were required to find the midpoint of the bases of the trapezoids using Steiner's theorem, with the help of one tool only - the straightedge (without using the scale divisions). After completing the task, they were asked to use the scale divisions of the straightedge to check that the midpoints of the bases had indeed been obtained.

To illustrate Steiner's theorem for the trapezoid to the students in a dynamic manner, a GeoGebra applet was prepared for them in which they could drag the vertices of the trapezoid on the computer screen, and thus change its dimensions. This showed that, for any trapezoid, the straight line that connects the points of intersection always bisects the bases of the trapezoid. At this stage, the teacher spoke about the phenomenon of preserving four points on a straight line (the points of intersection and the midpoints of the bases), and said that other tasks with preservation properties would be presented later on.

Link 1. Applet for demonstrating Steiner's theorem for trapezoid.

http://tube.geogebra.org/student/m761533

When it was clear that the students understood Steiner's theorem, they were required to carry out the following construction task in pairs:

Given is the triangle ABC, and lines parallel to two of its sides (as shown in Figure 2). Using only a straightedge (without its scale divisions), construct the midlines of the triangle. A hint was given to the students that a creative solution based on Steiner's theorem was needed.

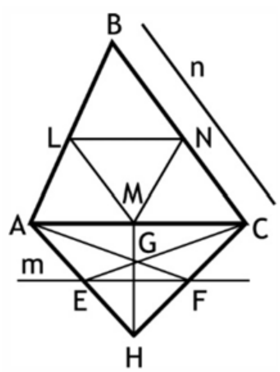

Figure 2. Triangle ABC and lines parallel to two of its sides.

After several minutes of discussion, mainly in pairs, an idea was suggested to mark any two points on each of the parallel lines, to obtain trapezoids and to bisect their bases in accordance with Steiner's theorem. The points *M* and *N* on the sides of the triangle were obtained. After these points were connected, the first midline *MN* was obtained (as shown in Figure 2). At this stage, the question of how to find the other two midlines arose, since there is no line parallel to the third side. Here, the students immediately understood that the midline *MN* is parallel to the side *AB*, and immediately found its midline.

Stage 2. Acquaintance with Napoleon's theorem for triangle

The students were asked to draw any triangle ABC, and to construct equilateral triangles using a compass and the straightedge on each of its outer sides. When they had done this, the students were asked to construct two angle bisectors in each of the equilateral triangles and to mark their points of intersection by *G*, *H*, *I*, as shown in Figure 3. The students were now asked to measure the lengths of the sides of the triangle HGI using a ruler.

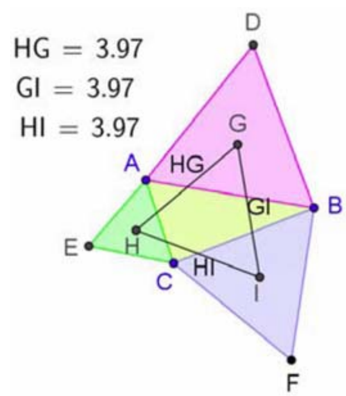

Figure 3. Napoleon triangle.

Constructing equilateral triangles on the outer sides of the original Napoleon triangle.

Surprisingly, even though each student drew a triangle ABC of different dimensions (except two students who made mistakes in the construction) everyone obtained an equilateral triangle HGI. The students encountered the preservation property: in this construction, an equilateral triangle is always obtained, irrespective of the lengths of the sides of the original triangle ABC. At this stage, the students were told that this theorem is ascribed to the Emperor Napoleon, who ruled over France and sought to expand the

French Empire by conquering and annexing other countries, including the country of the students on the course. Napoleon carried out mathematical investigations as a hobby. The students were told that various quite complicated proofs exist for this theorem, which they would encounter further up the school. One of them uses trigonometric tools with which they were not yet acquainted.

A GeoGebra applet was prepared for visualizing Napoleon's theorem in a dynamic manner, where the vertices of the triangle could be dragged on the computer screen, thus changing its dimensions. This showed that, for any triangle drawn, the centers of the equilateral triangles are the vertices of an equilateral triangle.

Link 2. Applet for demonstrating Napoleon's theorem for triangle.

http://tube.geogebra.org/student/m761545

Stage 3. Acquaintance with Van Aubel's theorem for quadrilaterals

The students were asked to draw any quadrilateral ABCD, and to construct squares on each of its outer sides, using a compass and a straightedge. After constructing the squares, the students were asked to draw their diagonals and to mark their points of intersection by the letters O_1, O_2, O_3, O_4 (the centers of the squares), as shown in Figure 4. The students were then asked to mark the segments O_1O_3 and O_2O_4 on the figure, to measure their lengths using a ruler, and to measure the angle between them.

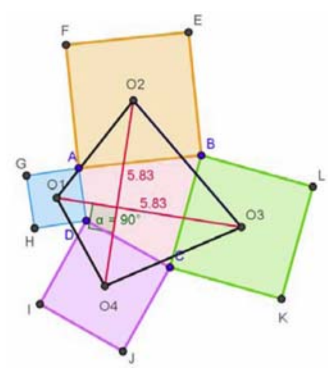

Figure 4. Van Aubel's theorem.

It was surprising that, even though each student drew a quadrilateral ABCD with different dimensions, in all the drawings, the O_1O_3 and O_2O_4 segments were the same length and the angle between them was straight. In this case too, the students encountered the preservation property.

Further activities were carried out using a computerized applet for Van Aubel's theorem.

Link 3. Applet for demonstrating Van Aubel's theorem for quadrilateral.

http://tube.geogebra.org/student/m761563

After a short time practicing use of the applet, the students were asked to make a freehand drawing of unique quadrilaterals, to draw outer squares on the sides of the quadrilateral, and write the figure that was obtained for the quadrilateral $O_1O_2O_3O_4$. They were then asked to use the applet and gridlines that appeared on the screen to draw the unique quadrilateral, while the quadrilateral $O_1O_2O_3O_4$ was obtained dynamically. For each unique quadrilateral, the students were asked to determine the shape obtained for the quadrilateral $O_1O_2O_3O_4$, and to check its correspondence to the form in the freehand drawing. The results were surprisingly creative.

Figures 5-12 demonstrate unique quadrilaterals created by attaching the centers of the squares.

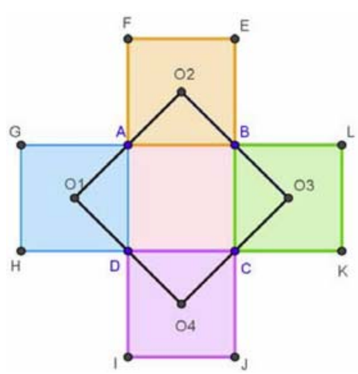

Figure 5. Quadrilateral ABCD's shape is a square. Quadrilateral $O_1O_2O_3O_4$'s shape is also a square.

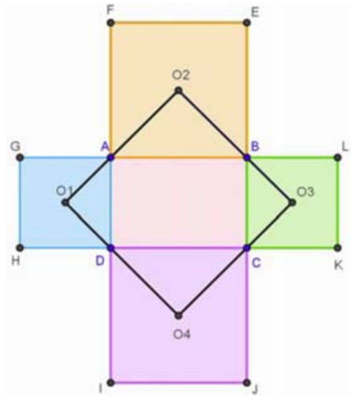

Figure 6. Quadrilateral ABCD's shape is a rectangle, and Quadrilateral $O_1O_2O_3O_4$'s shape is also a square.

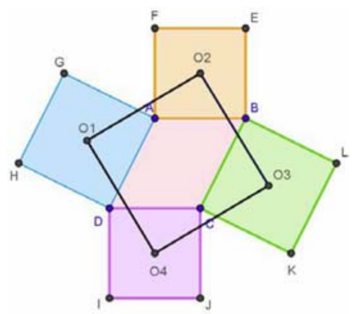

Figure 7. Quadrilateral ABCD's shape is a rhombus. Quadrilateral $O_1O_2O_3O_4$'s shape is also a square.

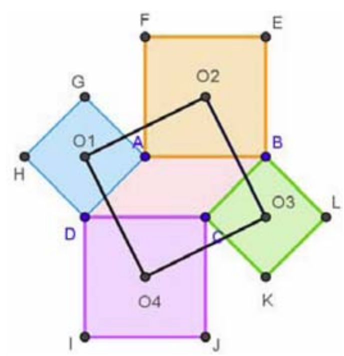

Figure 8. Quadrilateral ABCD's shape is a parallelogram, and quadrilateral $O_1O_2O_3O_4$'s shape is also a square.

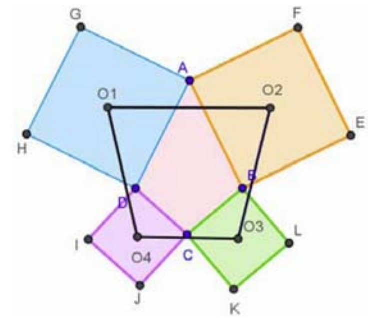

Figure 9. Quadrilateral ABCD's shape is a kite, and quadrilateral $O_1O_2O_3O_4$'s shape is an isosceles trapezoid.

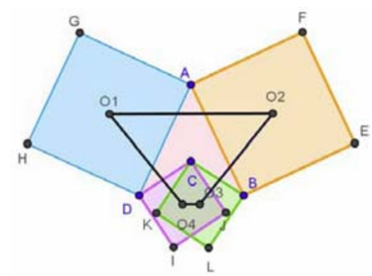

Figure 10. Quadrilateral ABCD's shape is a concave kite, and quadrilateral $O_1O_2O_3O_4$'s shape is an isosceles trapezoid.

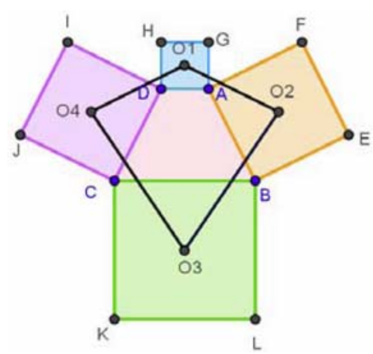

Figure 11. Quadrilateral ABCD's shape is an isosceles trapezoid, and quadrilateral $O_1O_2O_3O_4$'s shape is a kite.

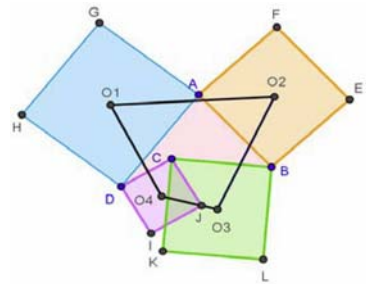

Figure 12. Quadrilateral ABCD's shape is any concave quadrilateral, and quadrilateral $O_1O_2O_3O_4$'s shape is any non concave quadrilateral.

The students were asked to formally prove that the constructions obtained (see Figures 5-11) would always be the shapes that appeared on the screen. It is important to note that any quadrilateral that appeared on the screen the O_1O_3 and O_2O_4 segments were the same length and the angle between them was straight.

During these tasks, the students were equipped, not only with mathematical skills, but also with self-regulation skills and a climate that facilitated self-regulation.

2.4. Data analysis

The data were analyzed using constant comparative analysis and grounded theory techniques [26]. The unit of analysis was an idea. The units were thematically coded into categories through three-phase coding: initial, axial and selective coding [31]. Each unit was compared with other units or with properties of a category. Analyses began during data collection and continued after its conclusion. The analysis resulted in a refined list of categories that were developed into conceptual abstractions called *constructs*. When repetition of the same constructs was obtained from multiple cases, and when new units did not point to any new aspect, then the list of constructs became theoretically saturated. The researchers stayed in the setting over time thus enabling interpretation of the meaning in individuals' lives [31].

Qualitative validity of data gathering was checked according to the qualitative rules, by triangulating evidence from the observations against the evidence from all other tools [31]. Qualitative validity of data analysis was checked by comparing researcher results with those of an external rater $(n = 10)$.

To explore the students' feedback on their experience, the means of the students' answers on the structured questionnaire were calculated and explained. The students' openly written comments were added to all data collected and were analyzed qualitatively.

3. Results and Discussion

The students' reflection tasks and comments, teacher reflection, the field notes, and the observations were analyzed and a variety of themes emerged. In this section, we will describe and discuss the specific unique course case by organizing the issues that represent it most fully. Those constructs [26] have been organized into five main aspects: the student SE aspect, the students' self-regulation skill aspect, the ILA environment aspect, the mathematics skill and achievement aspect, and students' feedback on the course.

3.1. Students' efficacy beliefs aspect

Students' SE was enhanced and the most important themes that emerged were as follows:

* The computerized technological tools opened a new window to the world of mathematics: it was fascinating to see the variety of geometric shapes and the possibility to investigate them dynamically and to reach generalizations quickly. SE was enhanced, as remarked by one student:

> "Creating those shapes was fascinating. If I could do that so fast, I would be able to do anything!."
> (Student reflection)

The use of the computer helped the students make rapid progress and they experienced success. Successful outcomes boosted SE, which in turn led to their next success. This result supports Bandura's theory that states that learners obtain information to appraise their SE from four sources of which *enactive mastery experiences* is the most important, as it serves as the best indicator of capability [1]. Research shows that mastery experience is a significant predictor of SE [32].

* Introducing unknown geometric theorems made students curious and trained them to consider any new issue as a challenge, which increased SE:

> "This makes me feel I will succeed in almost any math task in the future."
> (Student reflection)

The student's use of the word "math" indicates that he has made a transfer from geometry to mathematics in general. Transfer is a thinking process which is considered rare [10], and it clearly supports Bandura's theory on the possibility of generalizing efficacy beliefs from a specific subject to the general domain [1].

* Learning new things on the basis of previous knowledge enriches one's knowledge in mathematics:

> "When I learn from the past, I understand so well that I feel I can excel next time!".
> (Student reflection)

Acquiring knowledge is a sophisticated, continuous, dynamic process of meaning transformation and construction by connecting new information to previous knowledge. Doing it well enhances SE [4].

* Efficiently learning how to use traditional drawing tools evokes positive feelings:

> "I am so happy I can draw and use a ruler and a compass in geometry, and in my calculations."
> (Student reflection)

The feelings of satisfaction and enhanced SE are the result of acquiring skills. This finding supports previous findings that show that students need both SE and skills to achieve academic success [33].

* Divergent and flexible thinking allowed the creation of multiple shapes:

> "It is because I became flexible that I became creative, I think, in many directions. I constructed a variety of shapes with the quadrilateral formed using the centers of the squares. I am comfortable, Wow! I might even have a girlfriend."
> (Student reflection)

The result of engaging the students with creative skills was twofold: first, the students' creative abilities were improved, and second, by communicating their confidence to work mathematically, many students expressed a change in attitude to their own mathematical competence and were more willing to engage with unknown or challenging mathematical tasks [3]. Working on mathematical tasks may influence not only the mathematical content that is learned, but also how students experience mathematics [23]. Thus, working on this task had an impact on students' view of mathematics as a creative domain. Students' thinking transfer from the course to social life is part of their divergent thinking, which is one aspect of creativity.

* Mathematics helps everyday life and it is beautiful.

> "I would like to investigate tasks that help solve everyday problems. Math is beautiful."
> (Student reflection)

Of note was the students' change in attitude to learning mathematics through SE and self-regulated skill training. Learning mathematics became important and pleasant.

* A calm and pleasant atmosphere on the course enhances positive efficacy beliefs:

> "I want to learn more theorems, I am doing geometry well. It is calm and nice here. I am not anxious."
> (Student reflection)

Physiological reaction is another source that affects SE. SE beliefs are based, in part, on interpretations of one's emotional and physical states during task preparation and performance. Feeling calm rather than nervous and worried when preparing for and performing a task leads to higher SE. Belief in one's ability to attain a specific goal leads to greater effort and persistence.

It is through the students' interpretation of their performance that their SE is developed, resulting in a positive correlation between SE and achievement [34]. This finding supports Bandura's theory on emotions or feelings as an important source of SE [1, 2].

∗ All students should have the opportunity to perform these types of activities in mathematics. It adds to one's self-confidence:

> "Why do not they do it with everyone? Successfully creating a few shapes from centers of squares constructed on the sides of the original quadrilateral was amazing; it showed me that I could. I am sorry for those who did not have the chance to experience this course."
> (Student reflection)

Sternberg [21] argued that all students have the capacity to be creators, and to experience the joy associated with making something new, but without efficacy beliefs, the students would not have invested time and effort [21].

3.2. Students' self-regulation skill aspect

Self-regulation was perceived when students monitored their thinking processes, thought critically about the task to find answers, asked their friends relevant questions, connected between domains, and tried new ways to solve problems. When a student failed with an incorrect shape (see Stage 2), he chose a different path. Students demonstrated meta-cognitive ability when they reflected on their capabilities. They evaluated their efforts and their experience, took control and responsibility, and knew how to defend their constructions, to develop plans for completing the assignment, to assess their own and their peer's work. The teacher helped them believe in their own ability to be creative. Results show that the students made efficient use of information, developed a deep understanding of the subject matter, exerted effort to achieve geometric success, and successfully cooperated with their friends.

3.3. The environment aspect

One of the aims of this case study was to develop a deeper understanding of the different components that made up the alternative ILA culture, which was conducive to the development of self-directed active learning. The constructs that emerged show that the students were provided with a self-regulatory environment that facilitated self-regulation skills:

* The teacher, considered a coach, encouraged the students to choose their own ways of solving problems, and to choose again when they discovered that they had erred in their selection. When a student attempted to surmount an obstacle, the teacher praised the effort. This enhanced the students' SE, regardless of whether the attempt was entirely successful.

* The variety of ways of teaching: frontal, in pairs, in groups, and individually was part of the special flexible learning environment of this classroom.

> "Working in various ways is very interesting."
> (Student reflection)

* The use of computerized technology in the classroom was innovative and attractive.

* Collaboration was one crucial element in this course.

> "I like working with friends. I learn things I did not know before."
> (Student reflection)

We gave our students the opportunity to work collaboratively to spur self-regulation. We all learn from examples, by watching other people's techniques, strategies, and approaches during the learning process. Several studies have found that collaboration may have a creativity-promoting influence. The student working with a peer, as in the present study, must evaluate the peer's idea, then his/her own idea, select appropriate ideas, and build on them. On the one hand, the peer's different background and knowledge base may contribute different perspectives for consideration; on the other hand, diversity may be very wide, causing difficulty in finding an agreed-upon solution [35]. The literature review indicates that cooperative interaction has considerable impact on the stimulation of creativity [36]. The teacher working collaboratively with the students afforded the students a genuine feeling of working as mathematicians who are looking for creativity. In addition, students absorbed the enthusiasm and joy exhibited by many creative people as they made something new [37]. The collaborative work supports Bandura's theory on *vicarious experiences*, which is also one of the sources of SE [1]. *Vicarious experience* refers to learning through observing others perform a given task. Role models are especially influential when they are perceived as similar to the observer [1], as we have seen in this study. Collaboration in this case facilitated self-regulation and SE.

∗ Relaxation and enjoyable performance tasks facilitated learning, nurtured SE, and reduced anxiety.

In a group environment such as a classroom, the teacher has several roles in promoting successful learning. Besides choosing the task, the teacher has to foster a safe environment. Students in this study had low mathematics anxiety, were relaxed, tension free, and calm. Generating different solutions requires an open and stress-free mind, which contributes to the students' endeavor to succeed. Observations showed that the students enjoyed the activities and were highly motivated. According to Jung [24], providing a safe environment in a group situation, where individuals can try out novel ideas and question their own, may foster higher levels of thinking and creativity [24]. Levenson [38] found that when students feel free and relaxed and do not have time limits, they may come up with original ways of solving problems [38]. Part of being creative means investment of time and hard work.

∗ Developing meta-cognitive competencies and high order thinking performance tasks was part of a culture that fitted the constructivists' approach to ILA in that instruction promotes the development of reflective active lifelong learners. Learning stressed the learners' active role in making use of knowledge, investigating, constructing geometric shapes, and proving their own achievements [12].

* Task types were performance tasks that required integrative thinking processes, finding out connections and relationships, elaborations, and generalizations – the production of their knowledge [17, 11].

3.4. Students' skill and mathematics achievement aspect

It is important to note that, owing to the accelerated route in mathematics, by the end of the course, these students had completed most of the 9th grade geometry curriculum, as well as some of the curriculum for 10th grade. Course attendance was 90%, and no-one dropped out. Only three students had previous experience of geometric constructions in elementary school, but they did not remember much about it. The students' achievement on this course can be described by the following most striking issues that emerged from the data:

* Introducing innovations and issues that were not included in the current traditional curriculum enriched the students' knowledge repertoire of various theorems of creative mathematicians, enabling them to make connections between domains. Students could combine the worlds of ancient and modern mathematicians by proving their theorems, studying and working with them in various interesting ways, and reaching generalizations. They constructed forms and succeeded in proving their accuracy (see construction tasks Stages 1-3). In this course, students used previous knowledge to produce novel, original, and unexpected geometric shapes. They learned new ideas that they had not known before, such as Napoleon's Theorem, and used them for their own creations (Stages 2, 3). This gave the students the sense that mathematics is an interconnected science and not a collection of isolated topics [39]. Hence, Stupel and Ben-Chaim [39] emphasized that the NCTM standards [40] also recommend that teachers present tasks that exhibit the connection between different mathematical domains. Although this is not easy, there is certainly a need to encourage such activities in class [39]. Teachers would do well to apply knowledge from different mathematical

domains, as much as possible, to promote high order thinking, which contributes to creativity.

* Students found explanations for geometric shapes that were built on the basis of former mathematical knowledge acquired during the year and new knowledge acquired in the course, as stated by the teacher:

> "To summarize this stage, one should note the relatively quick manner in which the students managed the Stage 1 task, integrating geometrical knowledge learned during the school year with new knowledge they acquired shortly before receiving the task."
> (Teacher reflection)

Acquiring new insights on the basis of previous knowledge is typical of self-regulation. It occurred in all stages of the performance task.

* They gained the ability to write full formal proofs as well as making calculations in different triangles, the family of quadrilaterals, as well as tasks related to the circle:

> "They are able to write formal proofs. This is an amazing group of students. They show great potential for a future in science and technology."
> (Teacher reflection)

* Students experienced geometric constructions by the use of traditional tools, such as a straightedge and compass to carry bisection of the segment, bisection of an angle, construction of a perpendicular, construction of the triangle based on the lengths of each side, providing proofs for the accuracy of each construction. They constructed tasks based on renowned mathematicians' geometric theorems.

> "That was a learning experience of practicing traditional tools in geometry."
> (Teacher's reflection)

* They were capable of investigating the theorems not only using manual geometric constructions but by using computerized technology, too. Students were acquainted with the potential and basic skills in using computerized technology; they became acquainted with the GeoGebra software.

> "The computerized technology makes a great difference to students' learning."
> (Teacher's reflection)

* Students used high order thinking at every step of the activities. They tried to be creative and divergent thinkers to cope with challenging tasks in various mathematical domains, and they solved problems in interesting and surprising ways. Students investigated applets of theorems in a dynamic manner, while observing preservation of shapes, line lengths, obtaining different shapes for the same task, and mainly performing generalizations (see construction task, Stages 1-3). When students were asked to determine the shape obtained for the quadrilateral $O_1O_2O_3O_4$, and to check its correspondence to the expected form obtained from the freehand drawing, the results were creative:

"The following are the surprising results obtained by the students using the Van Aubel's theorem applet:

Quadrilateral 1. ABCD is a square – the quadrilateral $O_1O_2O_3O_4$ is also a square.

Quadrilateral 2. ABCD is a rectangle – the quadrilateral $O_1O_2O_3O_4$ is a square.

Quadrilateral 3. ABCD is a rhombus – the quadrilateral $O_1O_2O_3O_4$ is a square.

Quadrilateral 4. ABCD is a parallelogram – the quadrilateral $O_1O_2O_3O_4$ is a square.

Quadrilateral 5. ABCD is a kite – the quadrilateral $O_1O_2O_3O_4$ is an equilateral trapezoid.

Quadrilateral 6. ABCD is a concave kite – the quadrilateral $O_1O_2O_3O_4$ is an equilateral trapezoid.

Quadrilateral 7. ABCD is an equilateral trapezoid – the quadrilateral $O_1O_2O_3O_4$ is a kite.

Quadrilateral 8. ABCD is any concave quadrilateral – the quadrilateral $O_1O_2O_3O_4$ is any non-concave quadrilateral.

It should be noted here that one pair of students dragged one of the vertices of the quadrilateral ABCD in such a manner that caused it to degenerate into a triangle, and still, the properties of the diagonals of the centers were still preserved.

Cases 1-4 allowed the students to generalize: for each parallelogram ABCD, the quadrilateral $O_1O_2O_3O_4$ shall be a square, and since the square, the rectangle and the rhombus are unique forms of the parallelogram, in these cases, the quadrilateral of the centers shall be a square.

In the cases 1-7, the students were asked to provide formal proof that the shape appearing on the screen would always be obtained.

It is important to note that for each quadrilateral presented in the applet, the lengths of the segments O_1O_3 and O_2O_4 appeared on the computer screen, and they were found to be equal and perpendicular.

Figures 4-11 describe unique quadrilaterals and the shape of the quadrilateral of the vertices obtained."
(Teacher's notes)

Several studies have indicated the importance of having students engage in tasks that might encourage some of the different aspects of mathematical creativity [22]. Divergent thinking, which facilitated production of multiple solutions, is one characteristic of creativity. This flexible thinking was typical of all the stages of the course, allowing the students to pursue many different perspectives and creating a variety of shapes while preserving the given features. The multiple possibilities demonstrated the esthetics, wealth, and elegance of geometry. Divergent thinking enhanced students' mathematical understanding when material was approached from different points of view. Mathematicians tackle a mission from different points of view. By encouraging our students to do the same as mathematicians, they, too, learned to appreciate the value of tackling a mission from different viewpoints [39].

3.5. Students' feedback aspect

The results of the 5-point scale questionnaire are shown in Table 1. Only 23 students responded.

Table 1. Feedback questions and their means $(n = 23)$

Questions	Mean
1. To what extent did you like the activities?	4.3
2. To what extent were new activities based on previous knowledge?	4.35
3. To what extent did you acquire new mathematical ideas?	4.51
4. To what extent did you acquire new mathematical knowledge?	4.56
5. To what extent did the computerized technological instrument contribute to your learning?	4.48
6. To what extent were you surprised during the activities?	4.28
7. It was a disadvantage to have the geometric theorems given without proofs.	2.37
8. To what extent was the historical knowledge on famous mathematicians important?	2.05
9. To what extent would you be interested in coping with similar challenging research tasks in the future?	4.08

The high mean scores of Questions 1-6 and 9 show that the students admitted enjoying the new challenging ideas that arose during the skill training process, and encountering the beauty of mathematics: they acquired new knowledge and ideas; they confirmed that computerized technological instruments contributed to their learning; they admitted that the new activities were based on previous knowledge, so that they could connect between old and new and make progress; they liked the mathematical activities; they were very surprised to discover creative work and innovations, and found the training interesting and challenging. They would like to be challenged by more research tasks of a similar type in the future. The mean of Question 7 shows that the students would like to use new mathematical theorems without having to prove them, and that they were less enthusiastic about the historical information (Question 8).

Data collection validity was achieved by triangulation and inter-rater agreement was 86%.

4. Conclusions and Implications for the Future

The purpose of this study was to enhance the mathematics achievements of eighth grade students using reflection-on-SE-to-learn-mathematics training and skill training. Results show that the students acquired the ability to construct shapes and write full formal proofs and calculations of various triangles, rectangles, and circles. In a relaxed, self-regulated atmosphere, they became willing to try new ways and innovations and they allowed themselves to create for pleasure. They began to like geometry. They became self-efficacious, showed responsibility and control over their process of learning, and gained remarkable attainments. The combination of reflection on SE training and skill training fostered flexible and divergent thinking, which allowed high achievements and creative work.

There is a consensus among mathematicians and mathematics educators that creativity plays an essential role in doing mathematics. Creative students are self-regulated students who take control over processes and experience high SE beliefs. Therefore, we recommend encouraging creative performance tasks in school. Leikin [22] claimed that multiple-solution tasks offer students the opportunity to come up with different solutions, in turn encouraging three hallmarks of mathematical creativity in school: fluency, flexibility, and novelty. Tasks which may produce creative divergent thinking should invite the students to search for many different solutions. Multiple ways of thinking is a basis for mathematical creativity in that it leads to unexpected novelties, which have the potential to enhance human society. An increasing number of educators, researchers, and agencies of education are asserting the need to foster mathematical creativity among students [40-42]. Furthermore, mathematical creativity ensures the growth of the field of mathematics as a whole [20]. We recommend assisting teachers to view creativity as inherent in learning and to inspire teachers to believe that all students can become creative, as creativity is not an exclusive trait of the gifted [43]. This study has revealed the potential of student engagement in creative and high order thinking mathematical work by engaging students effectively in a creative collaborative performance task, which demanded meta-awareness, SE, self-regulation and mathematical skills.

Intrinsic motivation, openness, curiosity and autonomy often play a role in children's creative efforts [44]. Silver et al. [45] found that experienced teachers have many aims when choosing tasks, including building students' self-confidence, which is negatively correlated with anxiety.

The impact of the processes explored in this study extends beyond students' academic achievements in mathematics, which were very clear. The theoretical contribution of this study is the demonstrated capability of combined reflection and skill training to enhance SE beliefs and achievements.

We recommend research efforts to continue to examine the relationships between reflection on SE and self-regulated mathematical skills on the one hand, and student performance on the other, to enhance students' attainments.

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