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COMMON FIXED POINT RESULTS FOR WEAKLY ISOTONE INCREASING MAPPINGS IN PARTIALLY ORDERED *b*-METRIC-LIKE SPACES

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Abstract

In this paper, we establish common fixed point results for mappings satisfying new contractive conditions in partially ordered *b*-metric-like space. These results extend, generalize and improve many existing results in the literature. Also, some examples are given here to illustrate the usability of obtained results.

1. Introduction

Fixed point theory is one of the most powerful tools in nonlinear analysis. Its core is concerned with the conditions for the existence of one or more fixed points of mapping T from a topological space X into itself; that is, we can find $x \in X$ such that $Tx = x$. Recently, many researchers have focused on different contractive conditions in complete metric spaces, ordered *b*-metric spaces and obtained many fixed point results in such spaces.

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Alghamdi et al. [1] introduced the concept of b-metric-like space. Since then, several papers dealt with fixed point theory for single valued and multivalued in b-metric space [14, 15, 17-19, 23, 25, 27]. For more details on the fixed point results, their applications, comparison of different contractive conditions and related results in ordered metric spaces, we refer the reader to [3, 4, 24, 8-10, 13, 20, 24].

In this paper, we have proved some common fixed point results for g-weakly isotone increasing mappings satisfying new contractive conditions in partially ordered b-metric-like space. Our main results extend, generalize various well known results in literature. The paper is organized in four sections as follows: in Section 2, we give some required definitions and results related with b-metric-like space. Sections 3 and 4 accomplish the lemmas and main results which are the generalization of some existing results. Also, some examples are provided for the existence of our results.

2. Preliminaries

In this section, we recall some of the metric spaces and mappings as follows:

Definition 2.1 [1]**.** A b-metric-like on a nonempty set X is a function $\mathfrak{D} : X \times X \to [0, +\infty)$ such that for all $p, q, r \in X$ and a constant $K \geq 1$ the following three conditions hold true:

$(D1)$ if $\mathfrak{D}(p, q) = 0 \Rightarrow p = q$,

$(D2)$ $\mathfrak{D}(p, q) = \mathfrak{D}(q, p)$,

$(D3)$ $\mathfrak{D}(p, q) \leq K(\mathfrak{D}(p, r) + \mathfrak{D}(r, q))$.

The pair $(X, \mathfrak{D})$ is called a *b-metric-like space*.

Example 2.2 [1]**.** Let $X = [0, +\infty)$. Define the function $\mathfrak{D} : X^2 \to [0, +\infty)$ by $\mathfrak{D}(p, q) = (p + q)^2$. Then $(X, \mathfrak{D})$ is a b-metric-like space with constant $K = 2$. Clearly, $(X, \mathfrak{D})$ is not a b-metric or metric-like space. Indeed, for all $p, q, r \in X$,

$$\begin{aligned}\mathfrak{D}(p, q) = (p+q)^2 &\leq (p+r+r+q)^2 \\ &= (p+r)^2 + (r+q)^2 + 2(p+r)(r+q) \\ &\leq 2[(p+r)^2 + (r+q)^2] \\ &= 2(\mathfrak{D}(p, r) + \mathfrak{D}(r, q))\end{aligned}$$

and so $(D3)$ holds. Clearly, $(D1)$ and $(D2)$ hold.

Definition 2.3 [1]**.** Let $(X, \mathfrak{D}, K)$ be a b-metric-like space. Define $\mathfrak{D}^s$: $X^2 \to [0, \infty)$ by

$$\mathfrak{D}^s(p, q) = |\, 2\mathfrak{D}(p, q) - \mathfrak{D}(p, p) - \mathfrak{D}(q, q)\,|.$$

Clearly, $\mathfrak{D}^s(p, p) = 0$ for all $p \in X$.

Let $(x, \preccurlyeq)$ be a partially ordered set and let f, g be two self-maps on X. We will use the following terminology:

(a) elements $p, q \in X$ are called *comparable* if $p \leq q$ or $q \leq p$ holds;

(b) a subset S of X is said to be *well ordered* if every two elements of S are comparable;

(c) f is called *nondecreasing* w.r.t. $\preccurlyeq$ if $p \preccurlyeq q$ implies $fp \preccurlyeq fq$;

(d) [8] the pair (f, g) is said to be *weakly increasing* if $fp \preccurlyeq gfp$ and $gp \preccurlyeq fgp$ for all $p \in X$;

(e) [22] f is said to be g-*weakly isotone increasing* if for all $p \in X$ we have $fp \preccurlyeq gfp \preccurlyeq fgfp$.

If $f, g : X \to X$ are weakly increasing, then f is g-weakly isotone increasing. Also, in (e), if $f = g$, then we say that f is *weakly isotone increasing*. In this case, for each $p \in X$, we have $fp \preccurlyeq ffp$.

Definition 2.4 [2]. Let $(X, \preccurlyeq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preccurlyeq, d)$ is *regular* if the following conditions hold:

(1) if a non-decreasing sequence $x_n \to x$, $n \to \infty$, then $x_n \preccurlyeq x$ for all n,

(2) if a non-increasing sequence $y_n \to y$, $n \to \infty$, then $y_n \succcurlyeq y$ for all n.

3. Lemmas

The following lemmas are required in the proof of our main results.

Lemma 3.1 [1]. *Let $(X, \mathfrak{D}, K)$ be a b-metric-like space and $\{p_n\}$ be a sequence in X such that $\lim_{n\to\infty} \mathfrak{D}(p_n, p) = 0$. Moreover, $z \in X$, we have*

$$\frac{1}{K}\mathfrak{D}(p, z) \leq \liminf_{n\to\infty} \mathfrak{D}(p_n, z) \leq \limsup_{n\to\infty} \mathfrak{D}(p_n, z) \leq K\mathfrak{D}(p, z).$$

Lemma 3.2. *Let $(X, \mathfrak{D})$ be a b-metric-like space and let $\{p_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0. \tag{3.1}$$

If $\{p_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences $\mathfrak{D}(p_{m(k)}, p_{n(k)})$, $\mathfrak{D}(p_{m(k)}, p_{m(k)+1})$, $\mathfrak{D}(p_{m(k)+1}, p_{n(k)})$ and $\mathfrak{D}(p_{m(k)+1}, p_{n(k)+1})$, it holds:

$$\varepsilon \leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K^2} \le \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \le \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \le K^3\varepsilon.$$

Proof. If $\{p_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m(j)\}$ and $\{n(j)\}$ of positive integers such that

$$n(j) > m(j) > j, \quad \mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \varepsilon, \quad \mathfrak{D}(p_{m(j)}, p_{n(j)}) \ge \varepsilon \tag{3.2}$$

for all positive integers j. Now, from (3.2) and using the triangle inequality, we have

$$\begin{aligned}\varepsilon &\le \mathfrak{D}(p_{m(j)}, p_{n(j)}) \le K[\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) + \mathfrak{D}(p_{n(j)-1}, p_{n(j)})] \\ &< K\varepsilon + K\mathfrak{D}(p_{n(j)-1}, p_n(j)). \end{aligned}\tag{3.3}$$

Taking the upper and lower limits as $j \to \infty$ in (3.3), and using (3.1), we obtain that

$$\varepsilon \le \liminf_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \le \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \le K\varepsilon. \tag{3.4}$$

Using the triangle inequality again, we have

$$\begin{aligned} &\mathfrak{D}(p_{m(j)}, p_{n(j)}) \\ &\le K[\mathfrak{D}(p_{m(j)}, p_{n(j)+1}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)})] \\ &\le K^2[\mathfrak{D}(p_{m(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)})] + K\mathfrak{D}(p_{n(j)+1}, n(j)). \end{aligned}$$

Taking the upper and lower limits as $j \to \infty$, we have

$$\varepsilon \le K \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \le K^3\varepsilon$$

or

$$\frac{\varepsilon}{K} \le \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \le K^2\varepsilon.$$

The remaining two conditions of the lemma can be proved in a similar way.

□

4. Main Results

Let $(X, \preccurlyeq, \mathfrak{D})$ be an ordered b-metric-like space with $K > 1$ and $f, g : X \to X$ be two mappings. For all $p, q \in X$, let

$$M_s(p, q) = \max\left\{\psi(\mathfrak{D}(p, q)), \psi(\mathfrak{D}(p, fp)), \psi(\mathfrak{D}(q, gq)), \psi\left(\frac{\mathfrak{D}(p, gq) + \mathfrak{D}(q, fp)}{6K}\right)\right\} \tag{4.1}$$

and

$$N_s(p, q) = \min\{\psi(\mathfrak{D}^s(p, fp)), \psi(\mathfrak{D}^s(q, gq)), \psi(\mathfrak{D}^s(p, gq)), \psi(\mathfrak{D}^s(q, fp))\}, \tag{4.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function with $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$.

Theorem 4.1. *Let $(X, \preccurlyeq, \mathfrak{D})$ be a complete partially ordered b-metric-like space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $p, q \in X$, we have*

$$K^4\mathfrak{D}(fp, gq) \leq \frac{M_s(p, q) + N_s(p, q)}{2}. \tag{4.3}$$

Then the pair (f, g) has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Let p_0 be an arbitrary point of X. Choose $p_1 \in X$ such that $fp_0 = p_1$ and $p_2 \in X$ such that $gp_1 = p_2$. Continuing in this way, construct a sequence $\{p_n\}$ defined by:

$$p_{2n+1} = fp_{2n} \quad \text{and} \quad p_{2n+2} = gp_{2n+1}$$

for all $n \geq 0$. As f is g-weakly isotone increasing, we have

$$p_1 = fp_0 \leq gfp_0 = gx_1 = x_2 \leq fgfp_0 = fp_2 = p_3.$$

Repeating this process, we obtain $p_n \leq p_{n+1}, \forall n \geq 1$.

We will prove the theorem in three steps:

Step 1. First, we prove that $\lim_{n\to\infty} \mathfrak{D}(p_n, p_{n+1}) = 0$.

Suppose $\mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$ for some j_0. Then $p_{j_0} = p_{j_0+1}$. In this case, $j_0 = 2n$, $p_{2n} = p_{2n+1}$. We need to show that $p_{2n+1} = p_{2n+2}$,

$$\begin{aligned} &K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \\ &= K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &M_s(p_{2n}, p_{2n+1}) \\ &= \max\Big\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \\ &\qquad \psi\left(\frac{\mathfrak{D}(p_{2n}, gp_{2n+1}) + \mathfrak{D}(p_{2n+1}, fp_{2n})}{6K}\right)\Big\} \\ &= \max\Big\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ &\qquad \psi\left(\frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right)\Big\} \\ &= \max\Big\{\psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ &\qquad \psi\left(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right)\Big\} \end{aligned}$$

$$= \max\left\{\psi(\mathfrak{D}(p_{2n+1},\ p_{2n+1})),\ \psi(\mathfrak{D}(p_{2n+1},\ p_{2n+2})),\right.$$

$$\left.\psi\left(\frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2}) + \mathfrak{D}(p_{2n+1},\ p_{2n+1})}{6K}\right)\right\}.$$

By $(D3)$, we have

$$\mathfrak{D}(p_{2n+1},\ p_{2n+1}) \leq 2K\mathfrak{D}(p_{2n+1},\ p_{2n+2}),$$

$$\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \leq K\mathfrak{D}(p_{2n+1},\ p_{2n+2}),$$

$$\mathfrak{D}(p_{2n+1},\ p_{2n+1}) + \mathfrak{D}(p_{2n+1},\ p_{2n+2}) \leq 3K\mathfrak{D}(p_{2n+1},\ p_{2n+2}),$$

$$\frac{\mathfrak{D}(p_{2n+1},\ p_{2n+1}) + \mathfrak{D}(p_{2n+1},\ p_{2n+2})}{6K} \leq \frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{2},$$

$$M_s(p_{2n}, p_{2n+1})$$

$$\leq \max\left\{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi\left(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}\right)\right\}.$$

Now,

$$N_s(p_{2n},\ p_{2n+1}) = \min\{\psi(\mathfrak{D}^s(p_{2n},\ fp_{2n})),\ \psi(\mathfrak{D}^s(p_{2n+1},\ gp_{2n+1})),$$

$$\psi(\mathfrak{D}^s(p_{2n},\ gp_{2n+1})),\ \psi(\mathfrak{D}^s(p_{2n+1},\ fp_{2n}))\}$$

$$= \min\{\psi(\mathfrak{D}^s(p_{2n},\ p_{2n+1})),\ \psi(\mathfrak{D}^s(p_{2n+1},\ p_{2n+2})),$$

$$\psi(\mathfrak{D}^s(p_{2n},\ p_{2n+2})),\ \psi(\mathfrak{D}^s(p_{2n+1},\ p_{2n+1}))\}.$$

If $N_s(p_{2n},\ p_{2n+1}) = \psi(\mathfrak{D}^s(p_{2n+1},\ p_{2n+1}))$,

$$\mathfrak{D}^s(p_{2n+1},\ p_{2n+1})$$

$$= |\, 2\mathfrak{D}(p_{2n+1},\ p_{2n+1}) - \mathfrak{D}(p_{2n+1},\ p_{2n+1}) - \mathfrak{D}(p_{2n+1},\ p_{2n+1})\,|$$

clearly, $N_s(p_{2n},\ p_{2n+1}) = 0,$ then from (4.3), we have

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \leq \frac{M_s(p_{2n},\ p_{2n+1}) + N_s(p_{2n},\ p_{2n+1})}{2},$$

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \le \frac{M_s(p_{2n},\ p_{2n+1}) + 0}{2},$$

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \le \frac{M_s(p_{2n},\ p_{2n+1})}{2}, \tag{4.5}$$

where

$$M_s(p_{2n},\ p_{2n+1})$$

$$\le \max\left\{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi\left(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}\right)\right\}.$$

If $M_s(p_{2n},\ p_{2n+1}) = \psi(2K\mathfrak{D}(p_{2n+1},\ p_{2n+2}))$, then from (4.5), we have

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \le \psi(2K\mathfrak{D}(p_{2n+1},\ p_{2n+2})) < \frac{2K\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{2},$$

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) - K\mathfrak{D}(p_{2n+1},\ p_{2n+2}) < 0,$$

$$K(k^3 - 1)\mathfrak{D}(p_{2n+1},\ p_{2n+2}) < 0$$

a contradiction. If $M_s(p_{2n},\ p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1},\ p_{2n+2}))$, then from (4.5), we have

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \le \psi(\mathfrak{D}(p_{2n+1},\ p_{2n+2})) < \frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{2},$$

$$\left(k^4 - \frac{1}{2}\right)\mathfrak{D}(p_{2n+1},\ p_{2n+2}) < 0$$

a contradiction. If $M_s(p_{2n},\ p_{2n+1}) = \psi\left(\frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{2}\right)$, then from (4.5), we have

$$K^4\mathfrak{D}(p_{2n+1},\ p_{2n+2}) \le \psi\left(\frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{2}\right) < \frac{\mathfrak{D}(p_{2n+1},\ p_{2n+2})}{4},$$

$$\left(k^4 - \frac{1}{4}\right)\mathfrak{D}(p_{2n+1},\ p_{2n+2}) < 0,$$

that is, $p_{2n+1} = p_{2n+2}$.

Similarly, if $j_0 = 2n+1$, then $p_{2n+1} = p_{2n+2}$ gives $p_{2n+2} = p_{2n+3}$. Consequently, the sequence $\{p_k\}$ becomes constant for $j \geq j_0$ and $\{p_{j_0}\}$ is a coincidence point of f and g. For this, let $j_0 = 2n$. Then we know that $p_{2n} = p_{2n+1} = p_{2n+2}$. Hence

$$p_{2n} = p_{2n+1} = fp_{2n} = p_{2n+2} = gp_{2n+1}.$$

This means that $fp_{2n} = gp_{2n+1}$. Now, since $p_{2n} = p_{2n+1}$, we have $fp_{2n} = gp_{2n}$.

In the other case, when $k_0 = 2n+1$, similarly, it can be easily shown that p_{2n+1} is a coincidence point of the pair (f, g).

Suppose now that $\mathfrak{D}(p_{j_0}, p_{j_0+1}) > 0$ for each j_0. We claim the inequality

$$\mathfrak{D}(p_{j_0+1}, p_{j_0+2}) \leq \mathfrak{D}(p_{j_0}, p_{j_0+1}) \tag{4.6}$$

holds for each $j_0 = 1, 2, \ldots$.

Let $j_0 = 2n$ and for $n \geq 0$,

$$\mathfrak{D}(p_{2n+1}, p_{2n+2}) > \mathfrak{D}(p_{2n}, p_{2n+1}) > 0. \tag{4.7}$$

Then, as $p_{2n} \leq p_{2n+1}$, using (4.3), we obtain that

$$\begin{aligned} &K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \\ &= K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2}, \end{aligned}$$

where

$$\begin{aligned} &M_s(p_{2n}, p_{2n+1}) \\ &= \max\Big\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \\ &\qquad \psi\left(\frac{\mathfrak{D}(p_{2n}, gp_{2n+1}) + \mathfrak{D}(p_{2n+1}, fp_{2n})}{6K}\right)\Big\}, \end{aligned}$$

$$M_s(p_{2n}, p_{2n+1}) = \max\left\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi\left(\frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right)\right\}$$

and

$$N_s(p_{2n}, p_{2n+1}) = 0.$$

If $M_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (4.5), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})) < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2},$$

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) - \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2} < 0,$$

$$\left(k^4 - \frac{1}{2}\right)\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (4.5), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(\mathfrak{D}(p_{2n}, p_{2n+1})) < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2},$$

$$\left(k^4 - \frac{1}{2}\right)\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0$$

a contradiction. If $M_s(p_{2n}, p_{2n+1}) = \psi\left(\frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right)$, then from (4.5), we have

$$\begin{aligned}
&K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \\
&\leq \psi\left(\frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{6K}\right) \\
&< \frac{\mathfrak{D}(p_{2n}, p_{2n+2}) + \mathfrak{D}(p_{2n+1}, p_{2n+1})}{12K} \\
&\leq \frac{\mathfrak{D}(p_{2n}, p_{2n+1}) + \mathfrak{D}(p_{2n+1}, p_{2n+2}) + 2\mathfrak{D}(p_{2n+1}, p_{2n+2})}{12}
\end{aligned}$$

$$< \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{3},$$

$$\left(k^4 - \frac{1}{3}\right)\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0.$$

That is, $p_{2n+1} = p_{2n+2}$. Hence, (4.7) is false, that is, $\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \mathfrak{D}(p_{2n}, p_{2n+1})$ holds for all n. Therefore, (4.6) is proved for $j_0 = 2n$. Similarly, it can be shown that

$$\mathfrak{D}(p_{2n+2}, p_{2n+3}) \leq \mathfrak{D}(p_{2n+1}, p_{2n+2}).$$

Hence, $\{\mathfrak{D}(p_{j_0}, p_{j_0+1})\}$ is a nondecreasing sequence of nonnegative real numbers.

We claim that $\lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$.

Assuming that $\lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = r$, where $r > 0$, we have

$$M_s(p_{2n}, p_{2n+1}) \leq \max\left\{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi\left(\frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}\right)\right\} \tag{4.8}$$

and $N_s(p_{2n}, p_{2n+1}) = 0$.

Now, taking the upper limit as $n \to \infty$ in (4.8), we obtain

$$\limsup_{n \to \infty} M_s(p_{2n}, p_{2n+1}) \leq r.$$

Taking the upper limit, we have

$$K^4\mathfrak{D}(p_{2n}, p_{2n+1}) \leq \frac{M_s(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2},$$

$$K^4 r \leq \frac{r}{2},$$

$$\left(K^4 - \frac{1}{2}\right)r \leq 0$$

a contradiction. Hence,

$$r = \lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0.$$

Step 2. Next, we show that $\{p_n\}$ is a b-Cauchy sequence in X. That is, for every $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $m, n \geq J$, $\mathfrak{D}(p_m, p_n) < \varepsilon$.

Assume to contrary, that $\{p_n\}$ is not a b-Cauchy sequence. Then, from Lemma 3.2, there exists $\varepsilon > 0$ for which we can find subsequences $\{p_{m(j)}\}$ and $\{p_{n(j)}\}$ such that $n(j) \geq m(j) \geq j$ and:

(a) $m(j) = 2t$ and $n(j) = 2t' + 1$, where t and t' are non-negative integers,

(b) $\mathfrak{D}(p_{m(j)}, p_{n(j)}) \geq \varepsilon$ and

(c) $n(j)$ is the smallest number such that the condition (b) holds; i.e., $\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \varepsilon$. Then we have

$$\varepsilon \leq \limsup_{j \to \infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \limsup_{j \to \infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \limsup_{j \to \infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K^2} \leq \limsup_{j \to \infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\varepsilon.$$

Since $n(j) > m(j)$, we have $p_{m(j)} \preceq p_{n(j)}$,

$$\begin{aligned} K^4\mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) &= K^4\mathfrak{D}(fp_{m(j)}, gp_{n(j)}) \\ &\leq \frac{M_s(p_{m(j)}, p_{n(j)}) + N_s(p_{m(j)}, p_{n(j)})}{2}, \end{aligned}$$

where

$$M_s(p_{m(j)},\ p_{n(j)})$$

$$= \max\left\{\psi(\mathfrak{D}(p_{m(j)},\ p_{n(j)})),\ \psi(\mathfrak{D}(p_{m(j)},\ fp_{m(j)})),\ \psi(\mathfrak{D}(p_{n(j)},\ gp_{n(j)})),\right.$$

$$\left.\psi\left(\frac{\mathfrak{D}(p_{m(j)},\ gp_{n(j)}) + \mathfrak{D}(p_{n(j)},\ fp_{m(j)})}{6K}\right)\right\}$$

$$= \max\left\{\psi(\mathfrak{D}(p_{m(j)},\ p_{n(j)})),\ \psi(\mathfrak{D}(p_{m(j)},\ p_{m(j)+1})),\ \psi(\mathfrak{D}(p_{n(j)},\ p_{n(j)+1})),\right.$$

$$\left.\psi\left(\frac{\mathfrak{D}(p_{m(j)},\ p_{n(j)+1}) + \mathfrak{D}(p_{n(j)},\ p_{m(j)+1})}{6K}\right)\right\}$$

$$< \max\left\{\mathfrak{D}(p_{m(j)},\ p_{n(j)}),\ \mathfrak{D}(p_{m(j)},\ p_{m(j)+1}),\ \mathfrak{D}(p_{n(j)},\ p_{n(j)+1}),\right.$$

$$\left.\left(\frac{\mathfrak{D}(p_{m(j)},\ p_{n(j)+1}) + \mathfrak{D}(p_{n(j)},\ p_{m(j)+1})}{6K}\right)\right\}.$$

Taking the upper limit as $j \to \infty$, we have

$$\limsup_{j\to\infty} M_s(p_{m(j)},\ p_{n(j)})$$

$$\leq \max\left\{\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)},\ p_{n(j)}),\ \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)},\ p_{m(j)+1}),\right.$$

$$\limsup_{j\to\infty} \mathfrak{D}(p_{n(j)},\ p_{n(j)+1}),$$

$$\left.\left(\limsup_{j\to\infty} \frac{\mathfrak{D}(p_{m(j)},\ p_{n(j)+1}) + \mathfrak{D}(p_{n(j)},\ p_{m(j)+1})}{6K}\right)\right\}$$

$$\leq \max\left\{K\varepsilon,\ 0,\ 0,\ \frac{K^2\varepsilon + K^2\varepsilon}{6K}\right\} = K\varepsilon.$$

Similarly,

$$\begin{aligned}
&N_s(p_{m(j)}, p_{n(j)}) \\
&= \min\{\psi(\mathfrak{D}^s(p_{m(j)}, fp_{m(j)})), \psi(\mathfrak{D}^s(p_{n(j)}, gp_{n(j)})), \\
&\qquad \psi(\mathfrak{D}^s(p_{m(j)}, gp_{n(j)})), \psi(\mathfrak{D}^s(p_{n(j)}, fp_{m(j)}))\} \\
&= \min\{\psi(\mathfrak{D}^s(p_{m(j)}, p_{m(j)+1})), \psi(\mathfrak{D}^s(p_{n(j)}, p_{n(j)+1})), \\
&\qquad \psi(\mathfrak{D}^s(p_{m(j)}, p_{n(j)+1})), \psi(\mathfrak{D}^s(p_{n(j)}, p_{m(j)+1}))\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) \\
&\leq |2\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) - \mathfrak{D}(p_{n(j)}, p_{n(j)}) - \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1})| \\
&\leq |2\mathfrak{D}(p_{n(j)}, p_{n(j)+1}) - (\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}))| \\
&\leq |\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) - 2\mathfrak{D}(p_{n(j)}, p_{n(j)+1})|.
\end{aligned}$$

By ($D3$),

$$\mathfrak{D}(p_{n(j)}, p_{n(j)}) \leq 2K\mathfrak{D}(p_{n(j)}, p_{n(j)+1}),$$

$$\mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) \leq 2K\mathfrak{D}(p_{n(j)}, p_{n(j)+1}),$$

$$\mathfrak{D}(p_{n(j)}, p_{n(j)}) + \mathfrak{D}(p_{n(j)+1}, p_{n(j)+1}) \leq 4K\mathfrak{D}(p_{n(j)}, p_{n(j)+1}),$$

$$\mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) \leq |(4K - 2)(p_{n(j)}, p_{n(j)+1})|,$$

$$\limsup_{j\to\infty} \mathfrak{D}^s(p_{n(j)}, p_{n(j)+1}) \leq |(4K - 2) \limsup_{j\to\infty}(p_{n(j)}, p_{n(j)+1})|,$$

clearly, $N_s(p_n(j), p_m(j)) = 0$.

Hence, by taking the upper limit as $j \to \infty$, we have

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1})$$

$$\leq \frac{\limsup_{j\to\infty} M_s(p_{m(j)}, p_{n(j)}) + \limsup_{j\to\infty} N_s(p_{m(j)}, p_{n(j)})}{2},$$

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq \frac{K\varepsilon}{2} \leq K\varepsilon$$

which implies that $\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) < \frac{\varepsilon}{K^3} < \frac{\varepsilon}{K^2}$ a contradiction to (4) property proving above. Hence $\{p_n\}$ is a b-Cauchy sequence.

Step 3. In this step, we will show that f and g have a common fixed point. Since $\{p_n\}$ is a b-Cauchy sequence in the complete b-metric-like space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_{2n}, z) = \lim_{n\to\infty} \mathfrak{D}(p_{2n+1}, z) = \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z) = 0. \tag{4.9}$$

By the triangle inequality, we have

$$\mathfrak{D}(fz, z)$$

$$\leq K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(fp_{2n}, z)] = K[\mathfrak{D}(fz, fp_{2n}) + \mathfrak{D}(p_{2n+1}, z)]. \tag{4.10}$$

Suppose that f is continuous. Letting $n \to \infty$ in (4.10) and applying (4.9), we have

$$\mathfrak{D}(fz, z) \leq K[\lim_{n\to\infty} \mathfrak{D}(fz, fp_{2n}) + \lim_{n\to\infty} \mathfrak{D}(fp_{2n}, z)] = 0$$

which implies that $fz = z$.

Let $\mathfrak{D}(z, gz) > 0$. As z and gz are comparable by (4.3), we have

$$K^4\mathfrak{D}(z, gz) = K^4\mathfrak{D}(fz, gz) \leq \frac{M_s(z, z) + N_s(z, z)}{2}, \tag{4.11}$$

where

$$M_s(z, z)$$

$$= \max\left\{\psi(\mathfrak{D}(z, z)), \psi(\mathfrak{D}(z, fz)), \psi(\mathfrak{D}(z, gz)), \psi\left(\frac{\mathfrak{D}(z, gz) + \mathfrak{D}(z, fz)}{6K}\right)\right\}.$$

By the triangle inequality, we have

$$\mathfrak{D}(z, z) \leq 2K\mathfrak{D}(z, gz),$$

$$\mathfrak{D}(z, gz) \leq K\mathfrak{D}(z, gz),$$

$$\mathfrak{D}(z, z) + \mathfrak{D}(z, gz) \leq 3K\mathfrak{D}(z, gz),$$

$$\frac{\mathfrak{D}(z, z) + \mathfrak{D}(z, gz)}{6K} \leq \frac{(\mathfrak{D}(z, gz))}{2},$$

$$M_s(z, z)$$

$$\leq \max\left\{\psi(2K\mathfrak{D}(z, gz)), \psi(2K\mathfrak{D}(z, gz)), \psi(\mathfrak{D}(z, gz)), \psi\left(\frac{\mathfrak{D}(z, gz)}{2}\right)\right\}$$

$$< 2K\mathfrak{D}(z, gz).$$

Similarly,

$$N_s(z, z) = \min\{\psi(\mathfrak{D}^s(z, fz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, fz))\},$$

$$\mathfrak{D}^s(z, z) = |\, 2\mathfrak{D}(z, z) - \mathfrak{D}(z, z) - \mathfrak{D}(z, z)\,| = 0.$$

Clearly, $N_s(z, z) = 0$.

Hence, (4.11) gives $K^4\mathfrak{D}(z, gz) < K\mathfrak{D}(z, gz)$, which is a contradiction. Thus, $\mathfrak{D}(z, gz) = 0$. Similarly, if g is continuous, then the desired result is obtained. □

Theorem 4.2. *Let* $(X, \preccurlyeq, \mathfrak{D})$ *be a complete partially ordered b-metric-like space. Let* $f, g : X \to X$ *be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements* $p, q \in X$, *we have*

$$K^4\mathfrak{D}(fp,\ gq) \leq \frac{M_s(p,\ q) + N_s(p,\ q)}{2}. \tag{4.12}$$

Then the pair $(f,\ g)$ *has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.*

Proof. Following the proof of Theorem 4.1, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_n,\ z) = 0.$$

Now, we prove that z is a common fixed point of f and g. Since $p_{2n+1} \to z$ as $n \to \infty$ from regularity of X, $p_{2n+1} \leq z$. Therefore, from (4.3), we have

$$K^4\mathfrak{D}(fz,\ gp_{2n+1}) \leq \frac{M_s(z,\ p_{2n+1}) + N_s(z,\ p_{2n+1})}{2}, \tag{4.13}$$

where

$$M_s(z,\ p_{2n+1}) = \max\left\{\psi(\mathfrak{D}(z,\ p_{2n+1})),\ \psi(\mathfrak{D}(z,\ fz)),\ \psi(\mathfrak{D}(p_{2n+1},\ gp_{2n+1})),\right.$$
$$\left.\psi\left(\frac{\mathfrak{D}(z,\ gp_{2n+1}) + \mathfrak{D}(p_{2n+1},\ fz)}{6K}\right)\right\}.$$

Taking the limit as $n \to \infty$ in (4.13) and using Lemma 3.1, we obtain that

$$K^3\mathfrak{D}(fz,\ z)$$
$$= K^4\frac{1}{K}\mathfrak{D}(fz,\ z) \leq K^4 \limsup_{n\to\infty} \mathfrak{D}(fz,\ gp_{2n+1})$$
$$\leq \frac{\limsup_{n\to\infty} M_s(z,\ p_{2n+1}) + \limsup_{n\to\infty} N_s(z,\ p_{2n+1})}{2}$$
$$= \frac{\max\left\{\limsup_{n\to\infty} \mathfrak{D}(z,\ p_{2n+1}),\ \limsup_{n\to\infty} \mathfrak{D}(z,\ fz),\ \limsup_{n\to\infty} \mathfrak{D}(p_{2n+1},\ gp_{2n+1}),\ \limsup_{n\to\infty} \frac{\mathfrak{D}(z,\ gp_{2n+1}) + \mathfrak{D}(p_{2n+1},\ fz)}{6K}\right\} + \limsup_{n\to\infty} N_s(z,\ p_{2n+1})}{2}$$

$$\leq \frac{1}{2} \max\left\{\mathfrak{D}(z,\ z),\ \mathfrak{D}(z,\ fz),\ \mathfrak{D}(z,\ z),\ \frac{\mathfrak{D}(z,\ fz) + \mathfrak{D}(z,\ z)}{6K}\right\}$$

$$+ \limsup_{n\to\infty} N_s(z,\ p_{2n+1}).$$

By the triangle inequality, we have

$$\mathfrak{D}(z,\ z) \leq 2K\mathfrak{D}(z,\ fz),$$

$$\mathfrak{D}(z,\ fz) \leq K\mathfrak{D}(z,\ fz),$$

$$\mathfrak{D}(z,\ z) + \mathfrak{D}(z,\ fz) \leq 3K\mathfrak{D}(z,\ fz),$$

$$\frac{\mathfrak{D}(z,\ z) + \mathfrak{D}(z,\ fz)}{6K} \leq \frac{(\mathfrak{D}(z,\ fz))}{2},$$

$$M_s(z,\ z)$$

$$\leq \max\left\{\psi(2K\mathfrak{D}(z,\ fz)),\ \psi(2K\mathfrak{D}(z,\ fz)),\ \psi(\mathfrak{D}(z,\ fz)),\ \psi\left(\frac{(\mathfrak{D}(z,\ fz))}{2}\right)\right\}$$

$$< 2K\mathfrak{D}(z,\ fz).$$

Similarly,

$$N_s(z,\ z) = \min\{\psi(\mathfrak{D}^s(z,\ fz)),\ \psi(\mathfrak{D}^s(z,\ gz)),\ \psi(\mathfrak{D}^s(z,\ gz)),\ \psi(\mathfrak{D}^s(z,\ fz))\},$$

$$\mathfrak{D}^s(z,\ z) = |\, 2\mathfrak{D}(z,\ z) - \mathfrak{D}(z,\ z) - \mathfrak{D}(z,\ z)\,| = 0.$$

Clearly, $N_s(z,\ z) = 0.$

Hence, by (4.13), we have

$$K^4\mathfrak{D}(z,\ fz) < \frac{2K\mathfrak{D}(z,\ fz) + 0}{2},$$

$$K^4\mathfrak{D}(z,\ fz) - \frac{2K\mathfrak{D}(z,\ fz) + 0}{2} < 0,$$

$$K(K^3 - 1)\mathfrak{D}(z,\ fz) < 0$$

a contradiction, this implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. □

Corollary 4.3. *Let* $(X, \preccurlyeq, d)$ *be a complete partially ordered b-metric space. Let* $f, g : X \to X$ *be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements* $x, y \in X$ *and a constant* $s > 1$, *we have*

$$s^4 d(fx, gy) \leq M_s(x, y). \tag{4.14}$$

Then the pair (f, g) *has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.*

Corollary 4.4. *Let* $(X, \preccurlyeq, d)$ *be a complete partially ordered b-metric space. Let* $f : X \to X$ *be a mapping such that f is weakly isotone increasing. Suppose that for every two comparable elements* $x, y \in X$ *and a constant* $s > 1$, *we have*

$$s^4 d(fx, gy) \leq M_s(x, y),$$

where

$$M_s(x, y)$$

$$= \max\left\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy)), \psi\left(\frac{d(x, fy) + d(y, fx)}{2K}\right)\right\}.$$

Then f has a fixed point z in X if either:

(a) *f is continuous or*

(b) *X is regular.*

Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Theorem 4.5. *Let* $(X, \preccurlyeq, d)$ *be a complete partially ordered b-metric-like space with* $K > 1$. *Let* $f, g : X \to X$ *be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements* $p, q \in X$, *we have*

$$K^4\mathfrak{D}(fp,\ gq) \le \frac{N(p,\ q) + N_s(p,\ q)}{2}, \tag{4.15}$$

where

$$N(p,\ q) = \max\{\psi(\mathfrak{D}(p,\ q)),\ \psi(\mathfrak{D}(p,\ fp)),\ \psi(\mathfrak{D}(q,\ gq)),$$
$$\psi(\mathfrak{D}(p,\ gq)),\ \psi(\mathfrak{D}(q,\ fp))\} \tag{4.16}$$

and

$$N_s(p,\ q)$$
$$= \min\{\psi(\mathfrak{D}^s(p,\ fp)),\ \psi(\mathfrak{D}^s(q,\ gq)),\ \psi(\mathfrak{D}^s(p,\ gq)),\ \psi(\mathfrak{D}^s(q,\ fp))\} \tag{4.17}$$

and $\psi : [0,\ \infty) \to [0,\ \infty)$ *is a continuous function with* $\psi(t) < \frac{t}{2K}$ *for each* $t > 0$ *and* $\psi(0) = 0$. *Then the pair* $(f,\ g)$ *has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.*

Proof. Let p_0 be an arbitrary point of *X*. Choose $p_1 \in X$ such that $fp_0 = p_1$ and $p_2 \in X$ such that $gp_1 = p_2$. Continuing in this way, construct a sequence $\{p_n\}$ defined by:

$$p_{2n+1} = fp_{2n} \quad \text{and} \quad p_{2n+2} = gp_{2n+1}$$

for all $n \ge 0$. As *f* is *g*-weakly isotone increasing, we have

$$p_1 = fp_0 \le gfp_0 = gx_1 = x_2 \le fgfp_0 = fp_2 = p_3.$$

Repeating this process, we obtain $p_n \le p_{n+1}, \ \forall n \ge 1$.

We will prove the theorem in three steps:

Step 1. First, we prove that $\lim_{n\to\infty} \mathfrak{D}(p_n,\ p_{n+1}) = 0$.

Suppose $\mathfrak{D}(p_{j_0},\ p_{j_0+1}) = 0$ for some j_0. Then $p_{j_0} = p_{j_0+1}$. In this case, $p_0 = 2n$, $p_{2n} = p_{2n+1}$. We need to show that $p_{2n+1} = p_{2n+2}$. If $\mathfrak{D}(p_{2n+1}, p_{2n+2}) > 0$, then, from (4.15), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2})$$

$$= K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{N(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2}, \tag{4.18}$$

where

$$N(p_{2n}, p_{2n+1})$$

$$= \max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})),$$

$$\psi(\mathfrak{D}(p_{2n}, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, fp_{2n}))\}$$

$$= \max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})),$$

$$\psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1}))\}$$

$$= \max\{\psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})),$$

$$\psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1}))\}$$

$$= \max\{\psi(\mathfrak{D}(p_{2n+1}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))\}.$$

By $(D3)$, we have

$$\mathfrak{D}(p_{2n+1}, p_{2n+1}) \leq 2K\mathfrak{D}(p_{2n+1}, p_{2n+2}),$$

$$N(p_{2n}, p_{2n+1}) \leq \{\psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))\}$$

$$= \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})).$$

Now,

$$N_s(p_{2n}, p_{2n+1}) = \min\{\psi(\mathfrak{D}^s(p_{2n}, fp_{2n})), \psi(\mathfrak{D}^s(p_{2n+1}, gp_{2n+1})),$$

$$\psi(\mathfrak{D}^s(p_{2n}, gp_{2n+1})), \psi(\mathfrak{D}^s(p_{2n+1}, fp_{2n}))\}$$

$$= \min\{\psi(\mathfrak{D}^s(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+2})),$$

$$\psi(\mathfrak{D}^s(p_{2n}, p_{2n+2})), \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+1}))\}.$$

If $N_s(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}^s(p_{2n+1}, p_{2n+1}))$, then

$$\mathfrak{D}^s(p_{2n+1}, p_{2n+1})$$

$$= |2\mathfrak{D}(p_{2n+1}, p_{2n+1}) - \mathfrak{D}(p_{2n+1}, p_{2n+1}) - \mathfrak{D}(p_{2n+1}, p_{2n+1})|,$$

clearly, $N_s(p_{2n}, p_{2n+1}) = 0$, then from (4.15), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(2K\mathfrak{D}(p_{2n+1}, p_{2n+2})) < \frac{(2K\mathfrak{D}(p_{2n+1}, p_{2n+2}))}{4K}.$$

Hence, $[2K^4 - 1]\mathfrak{D}(p_{2n+1}, p_{2n+2}) < 0$, which is a contradiction, so $p_{2n+1} = p_{2n+2}$.

Similarly, if $j_0 = 2n+1$, then $p_{2n+1} = p_{2n+2}$ gives $p_{2n+2} = p_{2n+3}$. Consequently, the sequence $\{p_k\}$ becomes constant for $j \geq j_0$ and $\{p_{j_0}\}$ is a coincidence point of f and g. For this, let $j_0 = 2n$. Then we know that $p_{2n} = p_{2n+1} = p_{2n+2}$, hence

$$p_{2n} = p_{2n+1} = fp_{2n} = p_{2n+2} = gp_{2n+1}.$$

This means that $fp_{2n} = gp_{2n+1}$. Now, since $p_{2n} = p_{2n+1}$, we have $fp_{2n} = gp_{2n}$.

In the other case, when $j_0 = 2n+1$, similarly, it can easily be shown that p_{2n+1} is a coincidence point of the pair (f, g).

Suppose now that $\mathfrak{D}(p_{j_0}, p_{j_0+1}) > 0$ for each j_0. We claim the inequality

$$\mathfrak{D}(p_{j_0+1}, p_{j_0+2}) \leq \mathfrak{D}(p_{j_0}, p_{j_0+1}) \tag{4.19}$$

holds for each $j_0 = 1, 2, \ldots.$

Let $j = 2n$ and for an $n \geq 0$,

$$\mathfrak{D}(p_{2n+1}, p_{2n+2}) > \mathfrak{D}(p_{2n}, p_{2n+1}) > 0. \tag{4.20}$$

Then, as $p_{2n} \leq p_{2n+1}$, using (4.15), we obtain that

$$\begin{aligned} & K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \\ & = K^4\mathfrak{D}(fp_{2n}, gp_{2n+1}) \leq \frac{N(p_{2n}, p_{2n+1}) + N_s(p_{2n}, p_{2n+1})}{2}, \end{aligned} \tag{4.21}$$

where, by the definition, clearly, $N_s(p_{2n}, p_{2n+1}) = 0$ and

$$\begin{aligned} & N(p_{2n}, p_{2n+1}) \\ & = \max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n}, fp_{2n})), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})), \\ & \qquad \psi(\mathfrak{D}(p_{2n}, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, fp_{2n}))\} \\ & = \max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ & \qquad \psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+1}))\} \\ & \leq \max\{\psi(\mathfrak{D}(p_{2n}, p_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})), \\ & \qquad \psi(\mathfrak{D}(p_{2n}, p_{2n+2})), \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1}))\}. \end{aligned}$$

If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2}))$, then from (4.21), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \psi(\mathfrak{D}(p_{2n+1}, p_{2n+2})) < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{4K}$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+2}))$, then from (4.21), we have

$$\begin{aligned} & K^4\mathfrak{D}(p_{2n+1}, p_{2n+2}) \\ & \leq \psi(\mathfrak{D}(p_{2n}, p_{2n+2})) < \frac{\mathfrak{D}(p_{2n}, p_{2n+2})}{4K} \\ & \leq \frac{1}{2K} K[\mathfrak{D}(p_{2n}, p_{2n+1}) + \mathfrak{D}(p_{2n+1}, p_{2n+2})] < \mathfrak{D}(p_{2n+1}, p_{2n+2}) \end{aligned}$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (4.21), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2})$$
$$\leq \psi(\mathfrak{D}(p_{2n}, p_{2n+1})) < \frac{\mathfrak{D}(p_{2n}, p_{2n+1})}{4K} < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{4K}$$

a contradiction. If $N(p_{2n}, p_{2n+1}) = \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1}))$, then from (4.21), we have

$$K^4\mathfrak{D}(p_{2n+1}, p_{2n+2})$$
$$\leq \psi(2K\mathfrak{D}(p_{2n}, p_{2n+1})) < \frac{2K\mathfrak{D}(p_{2n}, p_{2n+1})}{4K} < \frac{\mathfrak{D}(p_{2n+1}, p_{2n+2})}{2}$$

a contradiction.

Hence, (4.20) is false, that is, $\mathfrak{D}(p_{2n+1}, p_{2n+2}) \leq \mathfrak{D}(p_{2n}, p_{2n+1})$ holds for all n. Therefore, (4.19) is proved for $j_0 = 2n$. Similarly, it can be shown that

$$\mathfrak{D}(p_{2n+2}, p_{2n+3}) \leq \mathfrak{D}(p_{2n+1}, p_{2n+2}).$$

Hence, $\{\mathfrak{D}(p_{j_0}, p_{j_0+1})\}$ is a nondecreasing sequence of nonnegative real numbers. We claim that $\lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0$.

Assume that $\lim_{j_0 \to \infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = r$, where $r > 0$, then we have

$$N(p_{2n}, p_{2n+1})$$
$$\leq \frac{1}{2K}\max\{2K\mathfrak{D}(p_{2n}, p_{2n+1}), \mathfrak{D}(p_{2n+1}, p_{2n+2}), \mathfrak{D}(p_{2n}, p_{2n+2})\} \tag{4.22}$$
$$\leq \frac{1}{2K}\max\{2K\mathfrak{D}(p_{2n}, p_{2n+1}), \mathfrak{D}(p_{2n+1}, p_{2n+2}), K\mathfrak{D}(p_{2n}, p_{2n+1})$$
$$+ K\mathfrak{D}(p_{2n+1}, p_{2n+2})\} \tag{4.23}$$

and $N_s(p_{2n}, p_{2n+1}) = 0$.

Now, taking the upper limit as $n \to \infty$ in (4.23), we obtain

$$\limsup_{n\to\infty} N(p_{2n}, p_{2n+1}) \leq \frac{1}{2K}\max\{r, 2Kr\} = r. \tag{4.24}$$

Taking the upper limit as $n \to \infty$ in (4.21) in (4.23), we have $K^4 r \leq r$. Therefore, $(K^4 - 1)r \leq 0$ a contradiction with $K > 1$. Hence,

$$r = \lim_{n\to\infty} \mathfrak{D}(p_{j_0}, p_{j_0+1}) = 0. \tag{4.25}$$

Step 2. We show that $\{p_n\}$ is a *b*-Cauchy sequence in *X*. That is, for every $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $m, n \geq j$, $\mathfrak{D}(p_m, p_n) < \varepsilon$.

Assume to contrary, that $\{p_n\}$ is not a *b*-Cauchy sequence. Then, from Lemma 3.2, there exists $\varepsilon > 0$ for which we can find subsequences $\{p_{m(j)}\}$ and $\{p_{n(j)}\}$ such that $n(j) \geq m(j) \geq j$ and:

(a) $m(j) = 2t$ and $n(j) = 2t' + 1$, where t and t' are non-negative integers,

(b) $\mathfrak{D}(p_{m(j)}, p_{n(j)}) \geq \varepsilon$ and

(c) $n(j)$ is the smallest number such that the condition (b) holds; i.e., $\mathfrak{D}(p_{m(j)}, p_{n(j)-1}) < \varepsilon$. Then we have

$$\varepsilon \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)}) \leq K\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)}, p_{n(j)+1}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K} \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)}) \leq K^2\varepsilon,$$

$$\frac{\varepsilon}{K^2} \leq \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1}, p_{n(j)+1}) \leq K^3\varepsilon.$$

Since $n(j) > m(j)$, we have $p_{m(j)} \leq p_{n(j)}$,

$$K^4\mathfrak{D}(p_{m(j)+1}, p_{n(j)+1})$$

$$= K^4\mathfrak{D}(fp_{m(j)}, gp_{n(j)}) \leq \frac{N(p_{m(j)}, p_{n(j)}) + N_s(p_{m(j)}, p_{n(j)})}{2},$$

where

$$N(p_{m(j)}, p_{n(j)})$$

$$= \max\{\psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, fp_{m(j)})), \psi(\mathfrak{D}(p_{n(j)}, gp_{n(j)})),$$

$$\psi(\mathfrak{D}(p_{m(j)}, gp_{n(j)})), \psi(\mathfrak{D}(p_{n(j)}, fp_{m(j)}))\}$$

$$= \max\{\psi(\mathfrak{D}(p_{m(j)}, p_{n(j)})), \psi(\mathfrak{D}(p_{m(j)}, p_{m(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{n(j)+1})),$$

$$\psi(\mathfrak{D}(p_{m(j)}, p_{n(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{m(j)+1}))\}$$

$$< \frac{1}{2K}\max\{\mathfrak{D}(p_{m(j)}, p_{n(j)}), \mathfrak{D}(p_{m(j)}, p_{m(j)+1}), \mathfrak{D}(p_{n(j)}, p_{n(j)+1}),$$

$$\psi(\mathfrak{D}(p_{m(j)}, p_{n(j)+1})), \psi(\mathfrak{D}(p_{n(j)}, p_{m(j)+1}))\}.$$

Taking the upper limit as $j \to \infty$, we have

$$\limsup_{j\to\infty} N(p_{m(j)}, p_{n(j)})$$

$$\leq \frac{1}{2K}\max\{\limsup_{j\to\infty}\mathfrak{D}(p_{m(j)}, p_{n(j)}), \limsup_{j\to\infty}\mathfrak{D}(p_{m(j)}, p_{m(j)+1}),$$

$$\limsup_{j\to\infty}\mathfrak{D}(p_{n(j)}, p_{n(j)+1}), \limsup_{j\to\infty}\mathfrak{D}(p_{m(j)}, p_{n(j)+1}),$$

$$\limsup_{j\to\infty}\mathfrak{D}(p_{n(j)}, p_{m(j)+1})\}$$

$$\leq \frac{1}{2K}\max\{K\varepsilon, 0, 0, K^2\varepsilon, K^2\varepsilon\} = \frac{K\varepsilon}{2}.$$

Similarly,

$$N_s(p_{m(j)},\ p_{n(j)}) = \min\{\psi(\mathfrak{D}^s(p_{m(j)},\ fp_{m(j)})),\ \psi(\mathfrak{D}^s(p_{n(j)},\ gp_{n(j)})),$$

$$\psi(\mathfrak{D}^s(p_{m(j)},\ gp_{n(j)})),\ \psi(\mathfrak{D}^s(p_{n(j)},\ fp_{m(j)}))\}$$

$$= \min\{\psi(\mathfrak{D}^s(p_{m(j)},\ p_{m(j)+1})),\ \psi(\mathfrak{D}^s(p_{n(j)},\ p_{n(j)+1})),$$

$$\psi(\mathfrak{D}^s(p_{m(j)},\ p_{n(j)+1})),\ \psi(\mathfrak{D}^s(p_{n(j)},\ p_{m(j)+1}))\}.$$

Now,

$$\mathfrak{D}^s(p_{n(j)},\ p_{n(j)+1})$$

$$\leq |2\mathfrak{D}(p_{n(j)},\ p_{n(j)+1}) - \mathfrak{D}(p_{n(j)},\ p_{n(j)}) - \mathfrak{D}(p_{n(j)+1},\ p_{n(j)+1})|$$

$$\leq |2\mathfrak{D}(p_{n(j)},\ p_{n(j)+1}) - (\mathfrak{D}(p_{n(j)},\ p_{n(j)}) + \mathfrak{D}(p_{n(j)+1},\ p_{n(j)+1}))|$$

$$\leq |\mathfrak{D}(p_{n(j)},\ p_{n(j)}) + \mathfrak{D}(p_{n(j)+1},\ p_{n(j)+1}) - 2\mathfrak{D}(p_{n(j)},\ p_{n(j)+1})|.$$

By ($D3$),

$$\mathfrak{D}(p_{n(j)},\ p_{n(j)}) \leq 2K\mathfrak{D}(p_{n(j)},\ p_{n(j)+1}),$$

$$\mathfrak{D}(p_{n(j)+1},\ p_{n(j)+1}) \leq 2K\mathfrak{D}(p_{n(j)},\ p_{n(j)+1}),$$

$$\mathfrak{D}(p_{n(j)},\ p_{n(j)}) + \mathfrak{D}(p_{n(j)+1},\ p_{n(j)+1}) \leq 4K\mathfrak{D}(p_{n(j)},\ p_{n(j)+1}),$$

$$\mathfrak{D}^s(p_{n(j)},\ p_{n(j)+1}) \leq |(4K-2)(p_{n(j)},\ p_{n(j)+1})|,$$

$$\limsup_{j\to\infty} \mathfrak{D}^s(p_{n(j)},\ p_{n(j)+1}) \leq |(4K-2)\limsup_{j\to\infty}(p_{n(j)},\ p_{n(j)+1})|,$$

clearly, $N_s(p_n(j),\ p_m(j)) = 0.$

Hence, by taking the upper limit as $j \to \infty$, we have

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1},\ p_{n(j)+1})$$

$$\leq \frac{\limsup_{j\to\infty} N(p_{m(j)},\ p_{n(j)}) + \limsup_{j\to\infty} N_s(p_m(j),\ p_n(j))}{2},$$

$$K^4 \limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1},\ p_{n(j)+1}) \leq \frac{K\varepsilon}{2} \leq K\varepsilon$$

which implies that $\limsup_{j\to\infty} \mathfrak{D}(p_{m(j)+1},\ p_{n(j)+1}) < \frac{\varepsilon}{K^3} < \frac{\varepsilon}{K^2}$ a contradiction to (4) property proving above. Hence $\{p_n\}$ is a b-Cauchy sequence.

Step 3. In this step, we will show that f and g have a common fixed point.

Since $\{p_n\}$ is a b-Cauchy sequence in the complete b-metric-like space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_{2n},\ z) = \lim_{n\to\infty} \mathfrak{D}(p_{2n+1},\ z) = \lim_{n\to\infty} \mathfrak{D}(fp_{2n},\ z) = 0. \tag{4.26}$$

By the triangle inequality, we have

$$\begin{aligned} \mathfrak{D}(fz,\ z) &\leq K[\mathfrak{D}(fz,\ fp_{2n}) + \mathfrak{D}(fp_{2n},\ z)] \\ &= K[\mathfrak{D}(fz,\ fp_{2n}) + \mathfrak{D}(p_{2n+1},\ z)]. \end{aligned} \tag{4.27}$$

Suppose that f is continuous. Letting $n \to \infty$ in (4.27) and applying (4.26), we have

$$\mathfrak{D}(fz,\ z) \leq K[\lim_{n\to\infty} \mathfrak{D}(fz,\ fp_{2n}) + \lim_{n\to\infty} \mathfrak{D}(fp_{2n},\ z)] = 0$$

which implies that $fz = z$.

Let $\mathfrak{D}(z,\ gz) > 0$. As z and gz are comparable by (4.15), we have

$$K^4\mathfrak{D}(z,\ gz) = K^4\mathfrak{D}(fz,\ gz) \leq \frac{N(z,\ z) + N_s(z,\ z)}{2}, \tag{4.28}$$

where

$$\begin{aligned} &N(z,\ z) \\ &= \max\{\psi(\mathfrak{D}(z,\ z)),\ \psi(\mathfrak{D}(z,\ fz)),\ \psi(\mathfrak{D}(z,\ gz)),\ \psi(\mathfrak{D}(z,\ gz)),\ \psi(\mathfrak{D}(z,\ fz))\}. \end{aligned}$$

By the triangle inequality, we have

$$\mathfrak{D}(z, z) \leq 2K\mathfrak{D}(z, gz),$$

$$N(z, z) \leq \max\{\psi(2K\mathfrak{D}(z, gz)), \psi(2K\mathfrak{D}(z, gz)), \psi(\mathfrak{D}(z, gz)),$$
$$\psi(\mathfrak{D}(z, gz)), \psi(2K\mathfrak{D}(z, gz))\} < \mathfrak{D}(z, gz).$$

Similarly,

$$N_s(z, z) = \min\{\psi(\mathfrak{D}^s(z, fz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, fz))\},$$

$$\mathfrak{D}^s(z, z) = |\, 2\mathfrak{D}(z, z) - \mathfrak{D}(z, z) - \mathfrak{D}(z, z)\,| = 0.$$

Clearly, $N_s(z, z) = 0.$

Hence, (4.28) gives $K^4\mathfrak{D}(z, gz) < \frac{\mathfrak{D}(z, gz)}{2}$, which is a contradiction. Thus, $\mathfrak{D}(z, gz) = 0.$ Similarly, if g is continuous, then the desired result is obtained. □

Theorem 4.6. *Let $(X, \preccurlyeq, \mathfrak{D})$ be a complete partially ordered b-metric-like space with $K > 1$. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $p, q \in X$, we have*

$$K^4\mathfrak{D}(fp, gq) \leq \frac{N(p, q) + N_s(p, q)}{2}, \tag{4.29}$$

where

$$N(p, q) = \max\{\psi(\mathfrak{D}(p, q)), \psi(\mathfrak{D}(p, fp)), \psi(\mathfrak{D}(q, gq)),$$
$$\psi(\mathfrak{D}(p, gq)), \psi(\mathfrak{D}(q, fp))\} \tag{4.30}$$

and

$$N_s(p, q)$$
$$= \min\{\psi(\mathfrak{D}^s(p, fp)), \psi(\mathfrak{D}^s(q, gq)), \psi(\mathfrak{D}^s(p, gq)), \psi(\mathfrak{D}^s(q, fp))\}. \tag{4.31}$$

Then the pair (f, g) *has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.*

Proof. Following the proof of Theorem 4.5, there exists $z \in X$ such that

$$\lim_{n\to\infty} \mathfrak{D}(p_n, z) = 0.$$

Now, we prove that z is a common fixed point of f and g. Since $p_{2n+1} \to z$ as $n \to \infty$ from regularity of X, $p_{2n+1} \leq z$. Therefore, from (4.3), we have

$$K^4\mathfrak{D}(fz, gp_{2n+1}) \leq \frac{N(z, p_{2n+1}) + N_s(z, p_{2n+1})}{2}, \tag{4.32}$$

where

$$N(z, p_{2n+1}) = \max\{\psi(\mathfrak{D}(z, p_{2n+1})), \psi(\mathfrak{D}(z, fz)), \psi(\mathfrak{D}(p_{2n+1}, gp_{2n+1})),$$

$$\psi(\mathfrak{D}(z, gp_{2n+1})), \psi(\mathfrak{D}(p_{2n+1}, fz))\}.$$

Taking the limit as $n \to \infty$ in (4.32) and using Lemma 3.1, we obtain that

$$\begin{aligned}
&K^3\mathfrak{D}(fz, z)\\
&= K^4 \frac{1}{K}\mathfrak{D}(fz, z) \leq K^4 \limsup_{n\to\infty} \mathfrak{D}(fz, gp_{2n+1})\\
&\leq \frac{\limsup_{n\to\infty} N(z, p_{2n+1}) + \limsup_{n\to\infty} N_s(z, p_{2n+1})}{2}\\
&= \frac{\begin{array}{l}\max\{\limsup_{n\to\infty}\mathfrak{D}(z, p_{2n+1}), \limsup_{n\to\infty}\mathfrak{D}(z, fz), \limsup_{n\to\infty}\mathfrak{D}(p_{2n+1}, gp_{2n+1}),\\ \quad \limsup_{n\to\infty}\mathfrak{D}(z, gp_{2n+1}), \limsup_{n\to\infty}\mathfrak{D}(p_{2n+1}, fz)\} + \limsup_{n\to\infty} N_s(z, p_{2n+1})\end{array}}{4K}\\
&\leq \frac{1}{4K}(\max\{\mathfrak{D}(z, z), \mathfrak{D}(z, fz), \mathfrak{D}(z, z), \mathfrak{D}(z, z), \mathfrak{D}(z, fz)\}\\
&\quad + \limsup_{n\to\infty} N_s(z, p_{2n+1})).
\end{aligned}$$

By the triangle inequality, we have

$$\mathfrak{D}(z, z) \le 2K\mathfrak{D}(z, fz),$$

$$\mathfrak{D}(z, fz) \le K\mathfrak{D}(z, fz),$$

$$N(z, z) \le \max\{\psi(2K\mathfrak{D}(z, fz)), \psi(K\mathfrak{D}(z, fz)), \psi(2K\mathfrak{D}(z, fz)),$$

$$\psi((\mathfrak{D}(z, fz)))\} < 2K\mathfrak{D}(z, fz).$$

Similarly,

$$N_s(z, z) = \min\{\psi(\mathfrak{D}^s(z, fz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, gz)), \psi(\mathfrak{D}^s(z, fz))\},$$

$$\mathfrak{D}^s(z, z) = |\, 2\mathfrak{D}(z, z) - \mathfrak{D}(z, z) - \mathfrak{D}(z, z)\,| = 0.$$

Clearly, $N_s(z, z) = 0$.

Hence, by (4.32), we have

$$K^4\mathfrak{D}(z, fz) < \frac{2K\mathfrak{D}(z, fz) + 0}{4K},$$

$$K^4\mathfrak{D}(z, fz) - \frac{\mathfrak{D}(z, fz) + 0}{2} < 0,$$

$$\left(K^4 - \frac{1}{2}\right)\mathfrak{D}(z, fz) < 0$$

a contradiction. This implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. □

Example 4.7. Let $X = [0, \infty)$ be equipped with the b-metric-like $\mathfrak{D}(p, q) = |\, p + q\,|^2$, $p, q \in X$, where $K = 2$ according to Example 2.2 and define a relation $\preccurlyeq$ on X by $p \preccurlyeq q$ iff $q \le p$, where $\le$ is the usual ordering on $\mathbb{R}$. Define function $f, g : X \to X$ by

$$fp = \frac{p}{9} \quad \text{and} \quad gp = \frac{p}{7}.$$

Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$. Then we have the following:

(1) $(X, \preccurlyeq, \mathfrak{D})$ is a complete partially ordered b-metric-like space.

(2) f is g-weakly isotone increasing with respect to $\preccurlyeq$.

(3) f and g are continuous.

(4) For every two comparable elements $p, q \in X$, the inequality (4.3) holds, where $M_s(p, q)$ and $N_s(p, q)$ are given by (4.1), (4.2), respectively.

Proof. Step 1. The proof of (1) is clear.

Step 2. To prove (2), for each $p \in X$, $fp = \frac{p}{9} < p$ and $gp = \frac{p}{7} < p$. Thus, for each $p \in X$, we have $gfp = g\left(\frac{p}{9}\right) = \frac{p}{63} \leq fp$ and $fgfp = f\left(\frac{p}{63}\right) = \frac{p}{567} \leq gfp$, i.e., $fp \preccurlyeq gfp \preccurlyeq fgfp$. Thus, f is g-weakly isotone increasing with respect to $\preccurlyeq$.

Step 3. To prove (3), it is easy to see that f and g are continuous.

Step 4. To prove (4), assume $p, q \in X$ with $p \preccurlyeq q$, i.e., $q \leq p$,

$$K^4\mathfrak{D}(fp, gq) \leq \frac{M_s(p, q) + N_s(p, q)}{2}, \tag{4.33}$$

where

$M_s(p, q)$

$$= \max\left\{\psi(\mathfrak{D}(p, q)), \psi(\mathfrak{D}(p, fp)), \psi(\mathfrak{D}(q, gq)), \psi\left(\frac{\mathfrak{D}(p, gq) + \mathfrak{D}(q, fp)}{6K}\right)\right\},$$

$M_s(p, q)$

$$= \max\left\{\psi((p+q)^2), \psi\left(\left(p + \frac{p}{9}\right)^2\right), \psi\left(\left(q + \frac{q}{7}\right)^2\right), \psi\left(\frac{\left(p + \frac{q}{7}\right)^2 + \left(q + \frac{p}{9}\right)^2}{6K}\right)\right\}.$$

We have the following cases:

Case 1. If $\frac{q}{7} \leq \frac{p}{9}$, then we have

$$M_s(p, q)$$

$$\leq \max\left\{\psi\left(\left(p+\frac{7p}{9}\right)^2\right), \psi\left(\left(\frac{10p}{9}\right)^2\right), \psi\left(\left(\frac{8p}{9}\right)^2\right), \psi\left(\frac{\left(p+\frac{p}{9}\right)^2+\left(q+\frac{p}{9}\right)^2}{6K}\right)\right\},$$

$$M_s(p, q)$$

$$\leq \max\left\{\psi\left(\left(\frac{16p}{9}\right)^2\right), \psi\left(\left(\frac{10p}{9}\right)^2\right), \psi\left(\left(\frac{8p}{9}\right)^2\right), \psi\left(\frac{164p^2}{972}\right)\right\}.$$

Clearly, $M_s(p, q) = \frac{1}{2}\left(\frac{16p}{9}\right)^2$ and

$$N_s(p, q)$$

$$= \min\{\psi(\mathfrak{D}^s(p, fp)), \psi(\mathfrak{D}^s(q, gq)), \psi(\mathfrak{D}^s(p, gq)), \psi(\mathfrak{D}^s(q, fp))\},$$

$$N_s(p, q)$$

$$= \min\left\{\psi\left(\mathfrak{D}^s\left(p, \frac{p}{9}\right)\right), \psi\left(\mathfrak{D}^s\left(q, \frac{q}{7}\right)\right), \psi\left(\mathfrak{D}^s\left(p, \frac{q}{7}\right)\right), \psi\left(\mathfrak{D}^s\left(q, \frac{p}{9}\right)\right)\right\},$$

$$\mathfrak{D}^s\left(p, \frac{q}{7}\right) \leq \mathfrak{D}^s(p, p) \quad \left(\because \frac{q}{7} \leq \frac{p}{9} \leq p\right)$$

$$= |\, 2\mathfrak{D}(p, p) - \mathfrak{D}(p, p) - \mathfrak{D}(p, p)\,|$$

$$= 0.$$

Clearly, $N_s(p, q) = 0.$ Then, by (4.33),

$$16\mathfrak{D}\left(\frac{p}{9}, \frac{q}{7}\right) \leq 16\left(\frac{p}{9}+\frac{p}{9}\right)^2 \leq 16\left(\frac{2p}{9}\right)^2 = \frac{16\times 16\times p^2}{81\times 4}.$$

Clearly, $16\mathfrak{D}\left(\frac{p}{9}, \frac{q}{7}\right) \leq \frac{M_s(p, q) + N_s(p, q)}{2}$.

Case 2. If $\frac{p}{9} < \frac{q}{7}$, then we have

$$M_s(p, q)$$

$$\leq \max\left\{\psi\left(\left(q + \frac{9q}{7}\right)^2\right), \psi\left(\left(\frac{10q}{7}\right)^2\right), \psi\left(\left(\frac{8q}{7}\right)^2\right), \psi\left(\frac{\left(\frac{9q}{7} + \frac{q}{7}\right)^2 + \left(q + \frac{q}{7}\right)^2}{12}\right)\right\},$$

$$M_s(p, q)$$

$$\leq \max\left\{\psi\left(\left(\frac{16q}{7}\right)^2\right), \psi\left(\left(\frac{10q}{7}\right)^2\right), \psi\left(\left(\frac{8q}{7}\right)^2\right), \psi\left(\frac{164q^2}{588}\right)\right\}.$$

Clearly, $M_s(p, q) = \frac{1}{2}\left(\frac{16q}{7}\right)^2$ and

$$N_s(p, q)$$

$$= \min\{\psi(\mathfrak{D}^s(p, fp)), \psi(\mathfrak{D}^s(q, gq)), \psi(\mathfrak{D}^s(p, gq)), \psi(\mathfrak{D}^s(q, fp))\},$$

$$N_s(p, q)$$

$$= \min\left\{\psi\left(\mathfrak{D}^s\left(p, \frac{p}{9}\right)\right), \psi\left(\mathfrak{D}^s\left(q, \frac{q}{7}\right)\right), \psi\left(\mathfrak{D}^s\left(p, \frac{q}{7}\right)\right), \psi\left(\mathfrak{D}^s\left(q, \frac{p}{9}\right)\right)\right\},$$

$$\mathfrak{D}^s\left(q, \frac{p}{9}\right) \leq \mathfrak{D}^s(q, q) \quad \left(\because \frac{p}{9} \leq \frac{q}{7} \leq q\right)$$

$$= |\, 2\mathfrak{D}(q, q) - \mathfrak{D}(q, q) - \mathfrak{D}(q, q)\,|$$

$$= 0.$$

Clearly, $N_s(p, q) = 0.$ Then, by (4.33),

$$16\mathfrak{D}\left(\frac{p}{9}, \frac{q}{7}\right) \leq 16\left(\frac{q}{7} + \frac{q}{7}\right)^2 \leq 16\left(\frac{2p}{7}\right)^2 = \frac{16 \times 16 \times p^2}{49 \times 4}.$$

Clearly, $16\mathfrak{D}\left(\frac{p}{9}, \frac{q}{7}\right) \leq \frac{M_s(p, q) + N_s(p, q)}{2}.$

By combining all cases together, we conclude that *f*, *g* and ψ satisfy all the hypotheses of Theorem 4.1 and hence *f* and *g* have a common fixed point. Indeed, 0 is the unique fixed point of *f* and *g*. □

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