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EXACT SOLUTIONS FOR SEVERAL HIGHER-ORDER NONLINEAR WAVE EQUATIONS

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Abstract

By introducing a transformation containing two trial functions and selecting the appropriate trial functions, we successfully find the exact solutions for several higher-order nonlinear wave equations named the Kuramoto-Sivashinsky equation and the Kawahara equation as well as the KdV-Burgers-Kuramoto equation hard to be solved in usual ways. The technique used herein can also be applied to construct the exact solutions of other nonlinear wave equations.

1. Introduction

As more and more problems in branches of modern mathematics, physics and other interdisciplinary science are described in light of suitable nonlinear models, directly seeking the explicit and exact solutions of nonlinear wave equations (NWEs for short) plays a very important role in the nonlinear science, especially in the nonlinear physics science. In recent years, a lot of simple and direct methods have been presented to construct the explicit and exact solutions to NWEs. Among them are the hyperbolic tangent function

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expansion method [1, 2], the homogeneous balance method [3, 4], the Jacobi elliptic function expansion method [5, 6], the modified variable separated ODE method [7, 8], the combination method [9, 10], and so on. Nevertheless, there is no general rule to solving NWEs. As a result, it is still a very significant work to explore more powerful and efficient methods to solve NWEs.

In the present paper, by introducing a transformation which contains two trial functions and selecting the appropriate trial functions, we successfully obtain the exact solutions to several higher-order nonlinear wave equations which are hard to be solved in usual ways.

2. Summary of the Method

The basic idea of our approach is as follows. Consider the following NWEs:

$$N\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{1}$$

In order to solve equation (1) easily, here we introduce a transformation of the following form:

$$u = u_0 + \frac{\partial v}{\partial x}, \quad v = v(w), \quad w = w(x, t), \tag{2}$$

where u_0 is a undetermined constant, $v(w)$ and $w(x, t)$ are two trial functions.

Firstly, we determine the trial function $w(x, t)$. As is well known, the solutions of NWEs should contain the phase factor $(kx - \omega t)$, i.e., $k(x - ct)$. Therefore, we choose directly the trial function $w(x, t)$ as the following form:

$$w = e^{k(x-ct)} = e^{k\xi}, \tag{3}$$

where k and c are the wave number and wave speed, respectively. However, the other trial function $v(w)$ is not of a given form but varies from equation to equation. It must be flexibly selected according to the specific NWE. After determining the trial functions $v(w)$ and $w(x, t)$, equation (1) can be easily solved. In what follows, we shall make use of the technique stated above to solve several higher-order NWEs, namely the Kuramoto-Sivashinsky equation, the Kawahara equation and the KdV-Burgers-Kuramoto equation, and seek their exact solutions.

3. Applications

3.1. Kuramoto-Sivashinsky equation

The celebrated Kuramoto-Sivashinsky equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} = 0. \tag{4}$$

For equation (4), we select the trial function $v(w)$ as the following form:

$$v = a\ln(b + w^2) + \frac{dw^2}{(b + w^2)^2}, \tag{5}$$

where a, b and d are undetermined constants.

In view of equation (2), equation (3) and equation (5), it is easy to derive that

$$u = u_0 + \frac{2kw^2[ab^2 + bd + (2ab - d)w^2 + aw^4]}{(b + w^2)^3}, \tag{6}$$

$$\frac{\partial u}{\partial t} = -\frac{4ck^2w^2[ab^3 + b^2d + (2ab^2 - 4bd)w^2 + (ab + d)w^4]}{(b + w^2)^4}, \tag{7}$$

$$\frac{\partial u}{\partial x} = \frac{4k^2w^2[ab^3 + b^2d + (2ab^2 - 4bd)w^2 + (ab + d)w^4]}{(b + w^2)^4}, \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{8k^3w^2(b - w^2)[ab^3 + b^2d + (2ab^2 - 10bd)w^2 + (ab + d)w^4]}{(b + w^2)^5}, \tag{9}$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{32k^5 w^2 (b - w^2)}{(b + w^2)^7} [ab^5 + b^4 d - (8ab^4 + 56b^3 d) w^2$$

$$- (18ab^3 - 246b^2 d) w^4 - (8ab^2 + 56bd) w^6 + (ab + d) w^8]. \quad (10)$$

Substituting equations (6)-(10) into equation (4), and collecting the coefficients of powers of w with the aid of Mathematica, then setting each of the obtained coefficients to zero, result in a set of algebraic equations as follows:

$$4ab^6 k^2 u_0 + 4b^5 dk^2 u_0 - ab^6 ck^2 - b^5 cdk^2 + 2ab^6 k^3 \alpha$$

$$+ 2b^5 dk^3 \alpha + 8ab^6 k^5 \gamma + 8b^5 dk^5 \gamma = 0, \quad (11)$$

$$20ab^5 k^2 u_0 - 4b^4 dk^2 u_0 - 5ab^5 ck^2 + b^4 cdk^2 + 2a^2 b^5 k^3 + 4ab^4 dk^3$$

$$+ 2b^3 d^2 k^3 + 6ab^5 k^3 \alpha - 13b^4 dk^3 \alpha - 72ab^5 k^5 \gamma - 456b^4 dk^5 \gamma = 0, \quad (12)$$

$$40ab^4 k^2 u_0 - 32b^3 dk^2 u_0 - 5ab^4 ck^2 + 4b^3 cdk^2 + 4a^2 b^4 k^3 - ab^3 dk^3$$

$$- 5b^2 d^2 k^3 + 2ab^4 k^3 \alpha - 10b^3 dk^3 \alpha - 40ab^4 k^5 \gamma + 1208b^3 dk^5 \gamma = 0, \quad (13)$$

$$40ab^3 k^2 u_0 - 32b^2 dk^2 u_0 - 5ab^3 ck^2 + 4b^2 cdk^2 + 6a^2 b^3 k^3 - 7ab^2 dk^3$$

$$+ 5bd^2 k^3 - 2ab^3 k^3 \alpha + 10b^2 dk^3 \alpha + 40ab^3 k^5 \gamma - 1208b^2 dk^5 \gamma = 0, \quad (14)$$

$$20ab^2 k^2 u_0 - 4bdk^2 u_0 - 5ab^2 ck^2 + bcdk^2 + 8a^2 b^2 k^3 - 6abdk^3$$

$$- 2d^2 k^3 - 6ab^2 k^3 \alpha + 13bdk^3 \alpha + 72ab^2 k^5 \gamma + 456bdk^5 \gamma = 0, \quad (15)$$

$$4abk^2 u_0 + 4dk^2 u_0 - abck^2 - cdk^2 + 2a^2 bk^3 + 2adk^3$$

$$- 2abk^3 \alpha - 2dk^3 \alpha - 8abk^5 \gamma - 8dk^5 \gamma = 0. \quad (16)$$

Solving the above set of algebraic equations with the aid of Mathematica, we obtain

$$a = \frac{60\alpha}{19}, \quad u_0 = c - \frac{60\alpha k}{19}, \quad d = -\frac{60b\alpha}{19}, \quad k = \frac{1}{2}\sqrt{\frac{-\alpha}{19\gamma}}. \tag{17}$$

Plugging equation (17) into equation (6) and considering equation (3), we obtain the general travelling wave solution of the Kuramoto-Sivashinsky equation (4) in the following:

$$u = c + \frac{60\alpha k}{19} - \frac{360b^2\alpha k}{19(b + e^{2k\xi})^2} + \frac{240b^3\alpha k}{19(b + e^{2k\xi})^3}. \tag{18}$$

Making use of the following identity:

$$\frac{e^{2x}}{e^{2x} + 1} = \frac{1}{2}(1 + \tanh x), \tag{19}$$

and setting $b = 1$ in equation (18), we obtain the kink-type solitary wave solution of the Kuramoto-Sivashinsky equation (4) as follows:

$$u = c + \frac{90\alpha k}{19}\tanh k\xi + 120\beta k^3 \tanh^3 k\xi. \tag{20}$$

Similarly, making use of the following identity:

$$\frac{e^{2x}}{e^{2x} - 1} = \frac{1}{2}(1 + \coth x), \tag{21}$$

and setting $b = -1$ in equation (18), we get the singular travelling wave solution to the Kuramoto-Sivashinsky equation (4) as follows:

$$u = c + \frac{90\alpha k}{19}\coth k\xi + 120\beta k^3 \coth^3 k\xi. \tag{22}$$

3.2. Kawahara equation

The Kawahara equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \delta\frac{\partial^5 u}{\partial x^5} = 0 \quad (\beta > 0,\ \delta < 0). \tag{23}$$

For equation (23), we select the trial function $v(w)$ as the following form:

$$v = \frac{aw^2}{b + w^2} + \frac{dw^2 + ew^4}{(b + w^2)^3}, \tag{24}$$

where a, b, d and e are undetermined constants.

Based on equation (2), equation (3) and equation (24), it is easy to derive that

$$u = u_0 + \frac{2kw^2[ab^3 + bd + 2(ab^2 - d + be)w^2 + (ab - e)w^4]}{(b + w^2)^4}, \tag{25}$$

$$\frac{\partial u}{\partial t} = \frac{\begin{array}{l} 4ck^2w^2[-ab^4 - b^2d + (-ab^3 + 7bd - 4b^2e)w^2 \\ \quad + (ab^2 - 4d + 7be)w^4 + (ab - e)w^6] \end{array}}{(b + w^2)^5}, \tag{26}$$

$$\frac{\partial u}{\partial x} = -\frac{\begin{array}{l} 4k^2w^2[-ab^4 - b^2d + (-ab^3 + 7bd - 4b^2e)w^2 \\ \quad + (ab^2 - 4d + 7be)w^4 + (ab - e)w^6] \end{array}}{(b + w^2)^5}, \tag{27}$$

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3} = \frac{16k^4w^2}{(b + w^2)^7}[ab^6 + b^4d - (9ab^5 + 41b^3d - 16b^4e)w^2 \\ + (10ab^4 + 171b^2d - 131b^3e)w^4 + (10ab^3 - 131bd \\ + 171b^2e)w^6 - (9ab^2 - 16d + 41be)w^8 - (ab - e)w^{10}], \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial^5 u}{\partial x^5} = \frac{64k^6w^2}{(b + w^2)^9}[-(55ab^7 + 183b^5d - 64b^6e)w^2 + (189ab^6 \\ + 2682b^4d - 1611b^5e)w^4 + (245ab^5 - 8422b^3d + 7197b^4e)w^6 \\ - (245ab^4 - 7197b^2d + 8422b^3e)w^8 + ab^8 + b^6d - (189ab^3 + 1611bd \\ - 2682b^2e)w^{10} + (55ab^2 + 64d - 183be)w^{12} - (ab - e)w^{14}]. \end{aligned} \tag{29}$$

Substituting equations (25)-(29) into equation (23), and collecting the coefficients of powers of w with the aid of Mathematica, then setting each of the obtained coefficients to zero, result in a set of algebraic equations as follows:

$$4ab^8k^2u_0 + 4b^6dk^2u_0 - 4ab^8ck^2 - 4b^6cdk^2 + 16ab^8k^4\beta$$

$$+ 16b^6dk^4\beta + 64ab^8k^6\delta + 64b^6dk^6\delta = 0, \tag{30}$$

$$20ab^7k^2u_0 + 16b^6ek^2u - 12b^5dk^2u_0 - 20ab^7ck^2 + 12b^5cdk^2 - 16b^6cek^2$$

$$+ 8a^2b^7k^3 + 16ab^5dk^3 + 8b^3d^2k^3 - 112ab^7k^4\beta - 624b^5dk^4\beta$$

$$+ 256b^6ek^4\beta - 3520ab^7k^6\delta - 11712b^5dk^6\delta + 4096b^6ek^6\delta = 0, \tag{31}$$

$$36ab^6k^2u_0 - 72b^4dk^2u_0 + 36b^5ek^2u_0 - 36ab^6ck^2 + 72b^4cdk^2$$

$$- 36b^5cek^2 + 24a^2b^6k^3 - 48ab^4dk^3 - 72b^2d^2k^3 + 48ab^5ek^3$$

$$+ 48b^3dek^3 - 432ab^6k^4\beta + 1440b^4dk^4\beta - 1584b^5ek^4\beta$$

$$+ 12096ab^6k^6\delta + 171648b^4dk^6\delta - 103104b^5ek^6\delta = 0, \tag{32}$$

$$20ab^5k^2u_0 - 88b^3dk^2u_0 - 12b^4ek^2u_0 - 20ab^5ck^2 + 88b^3cdk^2$$

$$+ 12b^4cek^2 + 16a^2b^5k^3 - 96ab^3dk^3 + 144bd^2k^3 + 16ab^4ek^3$$

$$- 240b^2dek^3 + 64b^3e^2k^3 - 304ab^5k^4\beta + 2720b^3dk^4\beta - 1200b^4ek^4\beta$$

$$+ 15680ab^5k^6\delta - 539008b^3dk^6\delta + 40608b^4ek^6\delta = 0, \tag{33}$$

$$-20ab^4k^2u_0 - 12b^2dk^2u_0 - 88b^3ek^2u_0 + 20ab^4ck^2 + 12b^2cdk^2$$

$$+ 88b^3cek^2 - 16a^2b^4k^3 + 16ab^2dk^3 - 64d^2k^3 - 96ab^3ek^3 + 240bdek^3$$

$$- 144b^2e^2k^3 + 304ab^4k^4\beta - 1200b^2dk^4\beta + 2720b^3ek^4\beta$$

$$- 15680ab^4k^6\delta - 539008b^3ek^6\delta + 460808b^2dk^6\delta = 0, \tag{34}$$

$$-36ab^3k^2u_0 + 36bdk^2u_0 - 72b^2ek^2u_0 + 36ab^3ck^2 - 36bcdk^2$$

$$+ 72b^2cek^2 - 24a^2b^3k^3 + 48abdk^3 - 48ab^2ek^3 - 48dek^3 + 72be^2k^3$$

$$+ 432ab^3k^4\beta - 1584bdk^4\beta + 1440b^2ek^4\beta - 12096ab^3k^6\delta$$

$$- 103104bdk^6\delta + 171648b^2ek^6\delta = 0, \tag{35}$$

$$-20ab^2k^2u_0 + 16dk^2u_0 - 12bek^2u_0 + 20ab^2ck^2 - 16cdk^2$$

$$+ 12bcek^2 - 8a^2b^2k^3 + 16abek^3 + 8e^2k^3 + 112ab^2k^4\beta + 256dk^4\beta$$

$$- 624bek^4\beta + 3520ab^2k^6\delta + 4096dk^6\delta - 11712bek^6\delta = 0, \tag{36}$$

$$-4abk^2u_0 + 4ek^2u_0 + 4abck^2 - 4cek^2 - 16abk^4\beta$$

$$+ 16ek^4\beta - 64abk^6\delta + 64ek^6\delta = 0. \tag{37}$$

Solving the above set of algebraic equations with the aid of Mathematica, we obtain

$$a = \frac{280\beta^{3/2}}{13\sqrt{-13\delta}}, \quad d = \frac{280b^2\beta^{3/2}}{13\sqrt{-13\delta}}, \quad e = \frac{280b\beta^{3/2}}{13\sqrt{-13\delta}},$$

$$u_0 = c + \frac{36\beta^2}{169\delta}, \quad k = \frac{\sqrt{\beta}}{2\sqrt{-13\delta}}. \tag{38}$$

Substituting equation (38) into equation (25) and considering equation (3), we acquire the general travelling wave solution of the Kawahara equation (23) in the following:

$$u = c + \frac{36\beta^2}{169\delta} - \frac{16805b^2\beta^2e^{4k\xi}}{169\delta(b + e^{2k\xi})^4}. \tag{39}$$

Making use of the following identity:

$$\frac{e^x}{e^{2x} + 1} = \frac{1}{2}\operatorname{sech} x, \tag{40}$$

and setting $b = 1$ in equation (39), we possess the bell-type solitary wave solution to the Kawahara equation (23) as follows:

$$u = c + \frac{36\beta^2}{169\delta} - \frac{105\beta^2}{169\delta}\operatorname{sech}^4 k\xi. \tag{41}$$

Similarly, making use of the following identity:

$$\frac{e^x}{e^{2x} - 1} = \frac{1}{2}\operatorname{csch} x, \tag{42}$$

and setting $b = -1$ in equation (39), we obtain the singular travelling wave solution to the Kawahara equation (23) as follows:

$$u = c + \frac{36\beta^2}{169\delta} - \frac{105\beta^2}{169\delta}\operatorname{csch}^4 k\xi. \tag{43}$$

3.3. KdV-Burgers-Kuramoto equation

The famous KdV-Burgers-Kuramoto equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0. \tag{44}$$

For equation (44), we select the trial function $v(w)$ as the following form:

$$v = a\ln(b + w^2) + \frac{dw^2}{b + w^2} + \frac{ew^2}{(b + w^2)^2}, \tag{45}$$

where a, b, d and e are undetermined constants.

According to equation (2), equation (3) and equation (45), it is easy to obtain that

$$u = u_0 + \frac{2kw^2[ab^2 + b^2d + be + (2ab + bd - e)w^2 + aw^4]}{(b + w^2)^3}, \tag{46}$$

$$\frac{\partial u}{\partial t} = -\frac{4ck^2w^2[ab^3 + b^3d + b^2e + (2ab^2 - 4be)w^2 + (ab - bd + e)w^4]}{(b + w^2)^4}, \tag{47}$$

$$\frac{\partial u}{\partial x}=\frac{4k^2w^2[ab^3+b^3d+b^2e+(2ab^2-4be)w^2+(ab-bd+e)w^4]}{(b+w^2)^4}, \tag{48}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{8k^3w^2}{(b+w^2)^5}[ab^4+b^4d+b^3e+(ab^3-3b^3d-11b^2e)w^2$$

$$-(ab^2+3b^2d-11be)w^4-(ab-bd+e)w^6], \tag{49}$$

$$\frac{\partial^3 u}{\partial x^3}=\frac{16k^4w^2}{(b+w^2)^6}[ab^5+b^5d+b^4e-(2ab^4+10b^4d+26b^3e)w^2$$

$$-(6ab^3-66b^2e)w^4-(2ab^2-10b^2d+26be)w^6$$

$$+(ab-bd+e)w^8], \tag{50}$$

$$\frac{\partial^4 u}{\partial x^4}=\frac{32k^5w^2}{(b+w^2)^7}[ab^6+b^6d+b^5e-(9ab^5+25b^5d+57b^4e)w^2$$

$$-(10ab^4-40b^4d-302b^3e)w^4+(10ab^3+40b^3d-302b^2e)w^6$$

$$+(9ab^2-25b^2d+57be)w^8-(ab-bd+e)w^{10}]. \tag{51}$$

Substituting equations (46)-(51) into equation (44), and collecting the coefficients of powers of w with the aid of Mathematica, then setting each of the obtained coefficients to zero, result in a set of algebraic equations as follows:

$$4ab^6k^2u_0+4b^6dk^2u_0+4b^5ek^2u_0-4ab^6ck^2-4b^6cdk^2$$

$$+4b^5cek^2+8ab^6k^3\alpha+8b^6dk^3\alpha+8b^5ek^3\alpha+16ab^6k^4\beta$$

$$+16b^6dk^4\beta+16b^5ek^4\beta+8ab^6k^5\gamma+8b^6dk^5\gamma+8b^5ek^5\gamma=0, \tag{52}$$

$$20ab^5k^2u_0+12b^5dk^2u_0-4b^4ek^2u_0-20ab^5ck^2-12b^5cdk^2+4b^4cek^2$$

$$+8a^2b^5k^3+16ab^5dk^3+8b^5d^2k^3+6ab^4ek^3+16b^4dek^3+8b^3e^2k^3$$

$$+\,24ab^5k^3\alpha - 8b^5dk^3\alpha - 72b^4ek^3\alpha - 16ab^5k^4\beta - 144b^5dk^4\beta$$

$$-\,400b^4ek^4\beta - 288ab^5k^5\gamma - 800b^5dk^5\gamma - 1824b^4ek^5\gamma = 0, \tag{53}$$

$$40ab^4k^2u_0 + 8b^4dk^2u_0 - 32b^3ek^2u_0 - 40ab^4ck^2 - 8b^4cdk^2 + 32b^3cek^2$$

$$+\,32a^2b^4k^3 + 40ab^4dk^3 + 8b^4d^2k^3 - 8ab^3ek^3 - 32b^3dek^3 - 40b^2e^2k^3$$

$$+\,16ab^4k^3\alpha - 64b^4dk^3\alpha - 80b^3ek^3\alpha - 128ab^4k^4\beta - 160b^4dk^4\beta$$

$$+\,640b^3ek^4\beta - 320ab^4k^5\gamma + 1280b^4dk^5\gamma + 9664b^3ek^5\gamma = 0, \tag{54}$$

$$40ab^3k^2u_0 - 8b^3dk^2u_0 - 32b^2ek^2u_0 - 40ab^3ck^2 + 8b^3cdk^2 + 32b^2cek^2$$

$$+\,48a^2b^3k^3 + 24ab^3dk^3 - 8b^3d^2k^3 - 56ab^2ek^3 - 32b^2dek^3 + 40be^2k^3$$

$$-\,16ab^3k^3\alpha - 64b^3dk^3\alpha + 80b^2ek^3\alpha - 128ab^3k^4\beta + 160b^3dk^4\beta$$

$$+\,640b^2ek^4\beta + 320ab^3k^5\gamma + 1280b^3dk^5\gamma - 9664b^2ek^5\gamma = 0, \tag{55}$$

$$20ab^2k^2u_0 - 12b^2dk^2u_0 - 4bek^2u_0 - 20ab^2ck^2 + 12b^2cdk^2 + 4bcek^2$$

$$+\,32a^2b^2k^3 - 8ab^2dk^3 - 8b^2d^2k^3 - 24abek + 16bdek^3 - 8e^2k^3$$

$$-\,24ab^2k^3\alpha - 8b^2dk^3\alpha + 72bek^3\alpha - 16ab^2k^4\beta + 144b^2dk^4\beta$$

$$-\,400bek^4\beta + 288ab^2k^5\gamma - 800b^2dk^5\gamma + 1824bek^5\gamma = 0, \tag{56}$$

$$4abk^2u_0 - 4bdk^2u_0 + 4ek^2u_0 - 4abck^2 + 4bcdk^2 - 4cek^2$$

$$+\,8a^2bk^3 - 8abdk^3 + 8aek^3 - 8abk^3\alpha + 8bdk^3\alpha - 8ek^3\alpha + 16abk^4\beta$$

$$-\,16bdk^4\beta + 16ek^4\beta - 32abk^5\gamma + 32bdk^5\gamma - 32ek^5\gamma = 0. \tag{57}$$

Solving the above system of algebraic equations with the aid of Mathematica, we obtain the following results:

Case 1. $\beta > 0,\ \alpha = \dfrac{47\beta^2}{144\gamma}.$

$$a = 480\gamma k^2, \quad u_0 = c + 20\beta k^2, \quad d = 720\gamma k^2,$$

$$e = \frac{720}{3} b\gamma k^2, \quad k = \frac{\beta}{24\gamma}. \tag{58}$$

Substituting equation (58) into equation (46) and considering equation (3), we acquire the general travelling wave solution of the KdV-Burgers-Kuramoto equation (44) in the following:

$$u = c + 20\beta k^2 - \frac{960 b^3 \gamma k^3}{(b + e^{2k\xi})^3}. \tag{59}$$

Setting $b = 1$ in equation (59) and using the previous identity (19), we obtain the kink-type solitary wave solution to the KdV-Burgers-Kuramoto equation (44) as follows:

$$u = c + 15\beta k^2 + \frac{90}{47}\alpha k \tanh k\xi - 15\beta k^2 \tanh^2 k\xi + 120\gamma k^3 \tanh^3 k\xi. \tag{60}$$

Similarly, taking $b = -1$ in equation (59) and using the previous identity (21), we get the singular travelling wave solution to the KdV-Burgers-Kuramoto equation (44) as follows:

$$u = c + 15\beta k^2 + \frac{90}{47}\alpha k \coth k\xi - 15\beta k^2 \coth^2 k\xi + 120\gamma k^3 \coth^3 k\xi. \tag{61}$$

Case 2. $\beta < 0,\ \alpha = \dfrac{\beta^2}{16\gamma}.$

$$a = 0, \quad u_0 = c - 6\beta k^2, \quad d = 240\gamma k^2, \quad e = 240 b\gamma k^2, \quad k = \frac{\beta}{8\gamma}. \tag{62}$$

Substituting equation (62) into equation (46) and considering equation (3), we get the general travelling wave solution of the KdV-Burgers-Kuramoto equation (44) in the following:

$$u = c - 6\beta k^2 - \frac{960 b^3 \gamma k^3}{(b + e^{2k\xi})^3} + \frac{960 b^2 \gamma k^3}{(b + e^{2k\xi})^2}. \tag{63}$$

Setting $b = 1$ in equation (63) and using the previous identity (19), we obtain the kink-type solitary wave solution to the KdV-Burgers-Kuramoto equation (44) as follows:

$$u = c + 9\beta k^2 - 30\alpha k \tanh k\xi - 15\beta k^2 \tanh^2 k\xi + 120\gamma k^3 \tanh^3 k\xi. \tag{64}$$

Similarly, taking $b = -1$ in equation (63) and using the previous identity (21), we get the singular travelling wave solution to the KdV-Burgers-Kuramoto equation (44) as follows:

$$u = c + 9\beta k^2 - 30\alpha k \coth k\xi - 15\beta k^2 \coth^2 k\xi + 120\gamma k^3 \coth^3 k\xi. \tag{65}$$

4. Conclusions

In summary, the exact solutions to several higher-order NWEs called the Kuramoto-Sivashinsky equation and the Kawahara equation as well as the KdV-Burgers-Kuramoto equation are successfully presented by introducing a transformation containing two trial functions and selecting the appropriate trial functions. It is not difficult to see that the technique used herein is particularly simple.

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