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INDEPENDENCE AMONG RANDOM VARIABLES BETWEEN SEMI-MARTINGALES

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Abstract

Independence is an optimal property among random variables within a semi-martingale and between semi-martingales, and the verification of independence can effectively improve and simplify the applications of semi-martingale processes. This paper investigates the conditions for independence among random variables in a semi-martingale and between semi-martingales. Specifically, we establish the conditions for independence among random variables in a continuous semi-martingale, for independence among random variables in a jump process, and for independence among random variables between a

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jump process and a continuous semi-martingale. Corresponding to each case of the above independence, the theorems and corollaries are stated and proved to demonstrate the conditions for the purpose of identifying independence among random variables in a semi-martingale and between martingales. In the proofs of theorems, Itô's formula is employed. To further detect the conditions for independence, we propose the definitions of local independence and progressive independence. To explore the conditions with respect to independence among random variables in a jump process, the linear decompositions of a jump process are initially utilized to prove the theorems where a jump process is involved. The results regarding independence from the theorems and corollaries are applied to show the independence conditions between Poisson processes in Bertoin [1] and Poisson process and Brownian motion in Shreve [6].

1. Introduction

In stochastic processes, independence is of importance and to characterize the property of independence is a crucial issue. Presently, increment independent processes, such as Brownian motion and Poisson processes, have been extensively analyzed. However, in the literature, only a little study has been fulfilled to effectively investigate independence among random variables in the same sequence of a stochastic process and between distinct stochastic processes. Bertoin [1] obtained that random variables in a Poisson process $N = (N^1, N^2, ..., N^n)$ are independent if and only if for any $i \neq j \in \{1, 2, ..., n\}$, $\delta N^i(t) = N^i(t) - N^i(t-) = 0$ or $\delta N^j(t) = N^j(t) - N^j(t-) = 0$ almost surely. Shreve [6] studied independence between Brownian motion and Poisson processes and concluded that Brownian motion and Poisson processes which are adapted to the same family of filtration are unconditionally independent of each other.

Motivated by the ideas in Shreve [6], this paper investigates the conditions for independence among random variables in a semi-martingale and between semi-martingales. Specifically, we establish the conditions for independence among random variables in a continuous semi-martingale, for independence among random variables in a jump process, and for independence among random variables between a jump process and a continuous semi-martingale. For the purpose of identifying these conditions, the corresponding theorems and corollaries are stated and proved. In the proofs of theorems, Itô's formula is employed. For further investigating the conditions for independence, we propose the definitions of local and progressive independence in a weaker sense. We also explore the conditions with respect to independence among random variables in a jump process with the initial utility of the linear decompositions of a jump process to prove the theorems where a jump process is included. We apply the results regarding independence from the theorems and corollaries to illustrate the independence conditions between Poisson processes in Bertoin [1] and Poisson process and Brownian motion in Shreve [6].

The conclusions in this paper can be realistically extended to a number of fields such as physical and financial processes where a continuous semi-martingale and a jump process are combined. Moreover, based on the theorems and corollaries derived in the paper, we can conclude that continuous semi-martingales and jump processes which are adapted to the same family of filtration are unsurprisingly independent.

The effectiveness of the results in this paper are furthermore illustrated in that Corollary 3.4 is identical to that in Bertoin [1] and in that Corollary 4.3 is one of the results in Shreve [6]. This indicates that the theorems in the paper possess more general consequences with respect to independence among random variables between semi-martingales and simultaneously implies that the developed points in the paper are valuable in applications.

We utilize $(\Omega, \{\mathcal{F}\}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ to denote a filtered complete probability space. All processes thereafter are adapted to a σ-filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Concepts of martingales, local martingales, semi-martingales, pure jump processes, and their quadratic variations and covariances are consistent with those in Karatzas and Shreve [2], Karlin and Taylor [3], and Klebaner [4]. For a semi-martingale X, let $\Delta X = X(t) - X(s)$, $s \leq t$, $\delta X(t) = X(t) - X(t-)$.

Let $\varphi(\theta, t) = E(\exp(\theta X(t)))$ be the characteristic function of X, where $\theta = \sqrt{-1}u$, $u \in \mathbb{R}$. Let $\Delta\overline{\varphi} = \Delta \log \varphi = \log \varphi(\theta, t) - \log \varphi(\theta, s)$, $s \leq t$. We make use of the characteristic function in the derivations of theorems and corollaries.

The paper is organized as follows: Section 2 analyzes independence among random variables in the sequence of a continuous semi-martingale and local independence and progressive independence are defined. Section 3 proposes the linear decompositions of a pure jump process and the independence conditions among random variables in a jump process are studied. In Section 4, we explore the independence conditions among random variables between a continuous semi-martingale and a jump process. Finally, Section 5 concludes the paper.

2. Independence among Random Variables in a Continuous Semi-martingale

2.1. Independence among random variables in a continuous semi-martingale

Let $X = (X^1, X^2, ..., X^m)$ be a semi-martingale of dimension m, where $X^i(0) = 0$, $i = 1, 2, ..., m$, and let $R^+ = [0, \infty)$ be an index set.

Prior to the presentation of the first theorem, we emphasize that in what follows, we assume that all semi-martingales have the decomposition as:

$$X^i = M^i + A^i, \tag{2.1}$$

where M^i is a square integrable continuous martingale and A^i is a process of finite variation.

The first theorem provides the conditions for independence among random variables in a continuous semi-martingale process.

Theorem 2.1. *Let* $X = (X^1, X^2, ..., X^m)$ *be as above, i.e., a continuous semi-martingale of dimension m with zero initial values such that*

$$X^i = M^i + A^i,$$

where M^i *is a square integrable continuous martingale and* A^i *is a process of finite variation. Let* $\varphi_i(\theta_i, t)$ *be the characteristic function of* X^i. *Then, within the martingale,* $X^1, X^2, \ldots, X^m$ *are mutually independent if the following two conditions hold*:

- (A1) $[M^i, M^j] = 0,\ i \neq j;$
- (A2) $E((\theta_i \Delta[M^i] + 2\Delta A^i) \mid \mathcal{F}_s) = 2\Delta\,\overline{\varphi}_i / \theta_i.$

Note that $[M^i]$ is the quadratic variation of M^i, for $i = 1, \ldots, m$; $\Delta[M^i] = [M^i]_t - [M^i]_s$, $s \leq t$; θ_i is the parameter in the characteristic function $\varphi_i(\theta_i, t)$; $\Delta A^i = A^i_t - A^j_s$, for $i, j = 1, \ldots, m$ and $s \leq t$; and $E(* \mid \mathcal{F}_s)$ is the expected value of $*$ with respect to the σ-field $\mathcal{F}_s$. Similarly to Section 1, let $\Delta\overline{\varphi}_i = \Delta \log \varphi_i = \log \varphi_i(\theta_i, t) - \log \varphi_i(\theta_i, s)$, $s \leq t$.

Proof. Utilizing the semi-martingale process along with the characteristic functions, we can define

$$Y = \exp\left(\sum_{i=1}^{m} \theta_i (M^i + A^i) - \log\left(\prod_{i=1}^{m} \varphi_i \right) \right)$$

and it is quite apparent that Y is a semi-martingale. According to Itô's formula of a continuous semi-martingale in Protter [5] and Russo and Vallois [7], we have

$$Y(t) = 1 + \sum_{i=1}^{m} \int_0^t \theta_i Y(s) dM^i(s) + \sum_{i=1}^{m} \int_0^t \theta_i Y(s) dA^i(s)$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \theta_i \theta_j \int_0^t Y(s) d[M^i, M^j](s) - \sum_{i=1}^{m} \int_0^t Y(s) d(\log(\varphi_i(\theta_i, s))).$$

Based on condition (A1), $[M^i, M^j] = 0,\ i \neq j,$ we can have

$$Y(t) = 1 + \sum_{i=1}^{m}\left(\int_0^t \theta_i Y(s)dM^i(s)\right) + \sum_{i=1}^{m}\left(\int_0^t \theta_i Y(s)dA^i(s)\right)$$

$$+ \frac{1}{2}\sum_{i=1}^{m}\left(\theta_i^2\int_0^t Y(s)d[M^i](s)\right) - \sum_{i=1}^{m}\left(\int_0^t Y(s)d(\log(\varphi_i(\theta_i, s)))\right)$$

$$= 1 + \sum_{i=1}^{m}\left(\int_0^t \theta_i Y(s)dM^i(s)\right)$$

$$+ \sum_{i=1}^{m}\left(\frac{1}{2}\theta_i^2\int_0^t Y(s)d[M^i](s) + \int_0^t \theta_i Y(s)dA^i(s)\right.$$

$$\left. - \int_0^t Y(s)d(\log(\varphi_i(\theta_i, s)))\right).$$

We remark that condition (A2), $E(\theta_i\Delta[M^i] + 2\Delta A^i \mid \mathcal{F}_s) = 2\Delta\,\overline{\varphi}_i/\theta_i,$ is equivalent to $\overline{M}^i = \frac{1}{2}\theta_i^2[M^i] + \theta_i A^i - \log\varphi_i,$ which is a continuous martingale. Then, the expression

$$\sum_{i=1}^{m}\left(\frac{1}{2}\theta_i^2\int_0^t Y(s)d[M^i](s) + \int_0^t \theta_i Y(s)dA^i(s) - \int_0^t Y(s)d(\log(\varphi_i(\theta_i, s)))\right)$$

$$= \sum_{i=1}^{m}\left(\int_0^t Y(s)d\overline{M^i}(s)\right),$$

yields

$$Y(t) = 1 + \sum_{i=1}^{m}\left(\int_0^t \theta_i Y(s)dM^i(s)\right) + \sum_{i=1}^{m}\left(\int_0^t Y(s)d\overline{M^i}(s)\right),$$

where $\int_0^t Y(s)dM^i(s)$ and $\int_0^t Y(s)d\overline{M^i}(s)$ are continuous martingales, respectively, because $Y(t)$ is continuous. Therefore, $Y(t)$ is also a continuous martingale and we have $E(Y(t)) = E(Y(0)) = 1,$ which leads to the equation

$$E\left(\exp\left(\sum_{i=1}^{m}\theta_i X^i(t)\right)\right) = \prod_{i=1}^{m}(\varphi_i(\theta_i, t)).$$

Based upon the properties of characteristic functions, we obtain that $X^1, X^2, \ldots, X^m$ are mutually independent. □

Corollary 2.2. *Let* $X = (X^1, X^2, \ldots, X^m)$ *be a continuous sub-martingale of dimension m with zero initial values such that* X^i *has the following Doob-Meyer decomposition*

$$X^i = M^i + A^i,$$

where M^i *is a uniformly integrable continuous martingale and* A^i *is a predictable increasing process. Let* $\varphi_i(\theta_i, t)$ *be the characteristic function of* X^i. *Then,* $X^1, X^2, \ldots, X^m$ *are mutually independent if the following conditions hold* (*with obvious notation*):

- (B1) $[M^i, M^j] = 0, \; i \neq j;$

- (B2) $E(\theta_i\Delta[M^i] + 2\Delta A^i \,|\, \mathcal{F}_s) = 2\Delta\,\overline{\varphi}_i/\theta_i.$

Proof. Note that sub-martingales in Corollary 2.2 are a special type of semi-martingales in Theorem 2.1. Although the decomposition of Doob-Meyer is different from that of semi-martingales, by decomposing M^i again, we can change the decomposition of Doob-Meyer to that of semi-martingales, where M^i is a square integrable continuous martingale. Then, Corollary 2.2 follows from Theorem 2.1. □

Corollary 2.3. *Let* $M = (M^1, M^2, ..., M^m)$ *be a square integrable continuous martingale of dimension m with zero initial values. Then* $M^1, M^2, ..., M^m$ *are mutually independent if the following conditions hold*:

- (C1) $[M^i, M^j] = 0,\ i \neq j;$
- (C2) $E(\Delta[M^i] | \mathcal{F}_s) = 2\Delta\overline{\varphi}_i / \theta_i^2.$

Proof. Following the proof of Theorem 2.1, it is easy to prove Corollary 2.3. □

Corollary 2.4. *Let* $W = (W^1, W^2, ..., W^m)$ *be a normal Brownian motion. Then, it is easily seen that* $W^1, W^2, ..., W^m$ *are mutually independent if and only if the following condition holds* (*with obvious notation*): $[W^i, W^j] = 0,$ *where* $i \neq j$.

Proof. For a standard Brownian motion, we have $[W^i](t) = t$ and $\varphi_i(\theta_i, t) = \exp\left(\frac{1}{2}\theta_i^2 t\right)$. Then, we get the expression $\Delta[W^i] = t - s,\ s \leq t$ and $2\Delta\varphi_i / \theta_i^2 = 2\log\left(\exp\left(\frac{1}{2}\theta_i^2(t - s)\right)\right)/\theta_i^2 = t - s.$ Condition (C2) in Corollary 2.3 is therefore characterized by a standard Brownian motion. By Theorem 2.1 and Corollary 2.3, all we need is Condition (C1) in Corollary 2.3, that is, $W^1, W^2, ..., W^m$ are mutually independent if and only if $[W^i, W^j] = 0,$ $i \neq j.$ □

Corollary 2.5. *Let* $X = (X^1, X^2, ..., X^m)$ *be an Itô process with* $X^i(0) = 0$ *such that*

$$X^i(t) = \int_0^t \sigma_i(X^i(s), s)dW^i(s) + \int_0^t b_i(X^i(s), s)ds,$$

where σ *and* b *are cádlág predictable integrable processes which satisfy* *integrable conditions and* $(W^i)_{1\le i\le m}$ *is a standard Brownian motion. Then,* $X^1, X^2, \ldots, X^m$ *are mutually independent if the following conditions hold*:

- (D1) $\left[\int_0^t \sigma_i dW^i(s), \int_0^t \sigma_i dW^j(s)\right] = 0,\ i \neq j;$

- (D2) $E\left(\int_s^t (\theta_i \sigma_i^2 + 2b_i) ds \,|\, \mathcal{F}_s\right) = 2\Delta\overline{\varphi}_i / \theta_i.$

We need to notice that we have a standard Brownian motion $(W^i)_{1\le i\le m}$, as mentioned in Corollary 2.3.

Proof. Following Theorem 2.1, we get trivially Corollary 2.5. □

We comment that if σ_i and b_i in Corollary 2.5 are constants, then plainly the random variables $X^1, X^2, \ldots, X^m$ in Itô process are mutually independent if and only if $W^1, W^2, \ldots, W^m$ are mutually independent, i.e., $[W^i, W^j] = 0,\ i \neq j,$ as shown in Corollary 2.4.

2.2. Local independence and progressive independence

Let M^i be a square integrable continuous local martingale. Since local stopping time series have influences on a semi-martingale $X^1, X^2, \ldots, X^m$, the conditions in Theorem 2.1 cannot be applied to infer that $X^1, X^2, \ldots, X^m$ are mutually independent. However, to employ the essential ideas in Theorem 2.1, we can define the following concepts of local and progressive independence for utilizing the weaker conditions for independence.

Definition 2.6. Let $X = X^1, X^2, \ldots, X^m$ be a semi-martingale of dimension m with zero initial values, and let $\varphi_i(\theta_i, t)$ be the characteristic function of X^i which has a decomposition of (2.1) and in which the local stopping time series $\tau_k \uparrow \infty.$ We assume that

$$Y_k = \exp\left(\sum_{i=1}^{m} \theta_i (M^i)^{\tau_k} + \sum_{i=1}^{m} \theta_i A^i - \log\left(\prod_{i=1}^{m} \varphi_i\right)\right).$$

They have $X^1, X^2, ..., X^m$ are mutually progressive independent if they have $\lim_{k\to\infty} E(Y_k) = 1$ and $X^1, X^2, ..., X^m$ are mutually local independent if $E(Y_k) = 1$ for an arbitrary natural number *k*.

The following theorem aims to provide the weaker conditions for the previously defined local and progressive independence.

Theorem 2.7. *Let X be as in Definition* 2.6. *Then*, $X^1, X^2, ..., X^m$ *are mutually local and progressive independent if the following conditions hold*:

- (E1) $[M^i, M^j] = 0, i \neq j$;
- (E2) $E(\theta_i \Delta[M^i]^{\tau_k} + 2\Delta A^i \mid \mathcal{F}_s) = 2\Delta\overline{\varphi}_i/\theta_i$.

Proof. It is similar to that of Theorem 2.1. The key point of the proof here is that we can apply Itô's formula to the equation

$$Y_k = \exp\left(\sum_{i=1}^{m} \theta_i (M^i)^{\tau_k} + \sum_{i=1}^{m} \theta_i A^i - \log\left(\prod_{i=1}^{m} \varphi_i\right)\right).$$

Now, by conditions (E1) and (E2), we obtain that the process *Y* is a continuous martingale. We therefore have $E(Y_k) = 1$, which leads to the result written as

$$E\left(\exp\left(\sum_{i=1}^{m} \theta_i (M^i(t))^{\tau_k} + \sum_{i=1}^{m} \theta_i A^i(t)\right)\right) = \prod_{i=1}^{m} (\varphi_i(\theta_i, t)). \tag{2.2}$$

By the previous definition, we conclude that $X^1, X^2, ..., X^m$ are mutually local and progressive independent. □

We emphasize that by equation (2.2), we can easily infer that if X^1, $X^2, ..., X^m$ are mutually independent, then $X^1, X^2, ..., X^m$ are mutually local and progressive independent. However, the converse proposition is not necessarily true, indicating that the conditions for mutually local and progressive independence are weaker than those for independence.

Corollary 2.8. *Let* $M = (M^1, M^2, \ldots, M^m)$ *be as above, and the local stopping time series* $\tau_k \uparrow \infty$. *Then* $M^1, M^2, \ldots, M^m$ *are mutually local independent and progressive independent if the following conditions hold*:

- (F1) $[M^i, M^j]^{\tau_k} = 0,\ i \neq j$;

- (F2) $E(\theta_i \Delta [M^i]^{\tau_k} \mid \mathcal{F}_s) = 2\Delta\overline{\varphi}_i / \theta_i$.

Proof. This corollary follows from Theorem 2.7. □

For the sake of simplicity, this paper concentrates on independence among the random variables in the process $X = (X^1, X^2, \ldots, X^m)$ instead of local and progressive independence. However, the framework of studying local and progressive independence is analogous to that of independence among the random variables in the process $X = (X^1, X^2, \ldots, X^m)$.

3. Independence among Random Variables in a Jump Process

3.1. The linear decompositions of a jump process

Let $N = (N^1, N^2, \ldots, N^n)$ be an n-dimensional pure jump (counting) process with $N^i(0) = 0,\ i = 1, 2, \ldots, n.$ A finite number of jump points exist for every process N^i in the interval $[0, t]$. We denote by S_k the set which is composed of all combinations of every k elements in $\{1, 2, \ldots, n\}$.

We also define $S_0 = \phi$ and consider different arrangements of $j_1, j_2, \ldots, j_k \in S_k$, where $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, n\}$. Let $j_1, j_2, \ldots, j_k = \sigma(k) \in S_k$. Then, it is easy to see that the number of elements in S_k is $\frac{n!}{k!(n-k)!}$ ($S_0 = \phi$ excluded). Let $\delta N^{j_i}(s) = N^{j_i}(s) - N^{j_i}(s-), i = 1, 2, \ldots, k,$

and let $B^k = \left\{\omega \in \Omega : \sum_{i=1}^{n} \delta N^i(s) = k\right\}$. Then, it is also apparent to figure out that B^k is $\mathcal{F}$-measurable set and $\{B^0, B^1, ..., B^n\}$ is a partition of the sample space Ω.

Let

$$B_{\sigma(k)}^{k} = \left\{\omega \in \Omega : \sum_{i=1}^{k} \delta N^{j_i}(s) = k,\ \delta N^{p}(s) = 0,\ p \neq j_1, j_2, ..., j_k\right\},$$

$k = 1, 2, ..., n$ and $B_{\sigma(0)}^{0} = B^0$. Then, $B^k = \bigcup_{\sigma(k) \in S_k} B_{\sigma(k)}^{k}$, which means $B_{\sigma(k)}^{k}$ with $\sigma(k) \in S_k$ is a partition of B^k. We define a pure jump process $N_{\sigma(k)}^{k}$ in a measurable set $B_{\sigma(k)}^{k}$ which satisfies

$$\delta N_{\sigma(k)}^{k} = \begin{cases} 1, & \omega \in B_{\sigma(k)}^{k}, \\ 0, & \omega \notin B_{\sigma(k)}^{k}. \end{cases}$$

Clearly, $\delta N_{\sigma(0)}^{0} \equiv 0$ and $\delta N_{\sigma(1)}^{1} = \delta N^i(s)$, $i = 1, 2, ..., n$.

Theorem 3.1. *Let $f : R \to R$ be a linear function with $f(0) = 0$, and let N be as above. $Y(t) = f\left(\sum_{i=1}^{n} \theta_i N^i(t)\right)$, and let $\theta_{\sigma(k)} = \sum_{i=1}^{k} \theta_{j_i}$. If at least one of processes $N^1, N^2, ..., N^n$ produces a jump at the point s, then we have*

$$Y(s) = f\left(\sum_{i=1}^{n} \theta_i N^i(s)\right) = Y(s-) + \sum_{k=1}^{n} \sum_{\sigma(k) \in S_k} (f(\theta_{\sigma(k)}) \delta N_{\sigma(k)}^{k}).$$

Proof. First, we develop $Y(s)$ as follows:

$$Y(s) = f\left(\sum_{i=1}^{n} \theta_i N^i(s)\right) = f\left(\sum_{i=1}^{n} \theta_i N^i(s-) + \sum_{i=1}^{n} \theta_i \delta N^i(s)\right)$$

$$= Y(s-) + f\left(\sum_{i=1}^{n} \theta_i \delta N^i(s)\right) = Y(s-) + \sum_{k=0}^{n}\left(f\left(\sum_{i=1}^{n} \theta_i \delta N^i(s)\right) I_{B^k}\right)$$

$$= Y(s-) + \sum_{\sigma(k)\in S_k}\left(\sum_{k=0}^{n} f\left(\sum_{i=1}^{n} \theta_i \delta N^i(s)\right) I_{B^k_{\sigma(k)}}\right). \tag{3.1}$$

Then, since $f(0) = 0$, $\delta N^0_{\sigma(0)} \equiv 0$ and $\delta N^1_{\sigma(1)} = \delta N^i(s)$, $i = 1, 2, ..., n$, equation (3.1) can be re-written as

$$Y(s) = Y(s-) + \sum_{k=1}^{n} \sum_{\sigma(k)\in S_k} (f(\theta_{\sigma(k)} \delta N^k_{\sigma(k)}))$$

$$= Y(s-) + \left(\sum_{i=1}^{n} f(\theta_i)\delta N^i + \sum_{\sigma(2)\in S_2} \delta N^2_{\sigma(2)} + \cdots + f\left(\sum_{i=1}^{n} \theta_i\right) \delta N^n_{\sigma(n)}\right)$$

$$= Y(s-) + \sum_{k=1}^{n} \sum_{\sigma(k)\in S_k} (f(\theta_{\sigma(k)})\delta N^k_{\sigma(k)}). \tag{3.2}$$

Equation (3.2) therefore closes the proof of Theorem 3.1. □

Theorem 3.2. *Assume that a continuous function $g : R \to R$ possesses the exponential nature, i.e.,*

$$g(x + y) = g(x)g(y), \quad \forall x,\ y \in R, \quad f(0) = 1.$$

Let $N = (N^1, N^2, ..., N^n)$ be an n-dimensional pure jump (counting) process with $N^i(0) = 0$, $i = 1, 2, ..., n$. The finite number of jump points exists for every process N^i in the interval $[0, t]$. Let $Y(t) = g\left(\sum_{i=1}^{n} \theta_i N^i\right)$ and $\theta_{\sigma(k)} = \sum_{i=1}^{k} \theta_{j_i}$. At least one of processes $N^1, N^2, ..., N^n$ produces a jump at the point s. Then

$$Y(s) = Y(s-) + Y(s-)\sum_{k=1}^{n}\sum_{\sigma(k)\in S_k}((g(\theta_{\sigma(k)}) - 1)\delta N_{\sigma(k)}^{k}).$$

Proof. First, we have

$$Y(s) = g\left(\sum_{i=1}^{n}\theta_i N^i(s-) + \sum_{i=1}^{n}\theta_i \delta N^i(s)\right) = g\left(\sum_{i=1}^{n}\theta_i N^i(s-)\right) g\left(\sum_{i=1}^{n}\theta_i \delta N^i(s)\right)$$

$$= Y(s-) g\left(\sum_{i=1}^{n}\theta_i \delta N^i(s)\right) = Y(s-)\sum_{k=0}^{n}\left(g\left(\sum_{i=1}^{n}\delta N^i(s)\right) I_{B^k}\right)$$

$$= Y(s-)\sum_{\sigma(k)\in S_k}\left(\sum_{k=0}^{n} g\left(\sum_{i=1}^{n}\delta N^i(s)\right) I_{B_{\sigma(k)}^k}\right). \tag{3.3}$$

By $f(0) = 1,\ \delta N_{\sigma(0)}^0 \equiv 0$ and $\delta N_{\sigma(1)}^1 = \delta N^i(s),\ i = 1, 2, ..., n,$ equation (3.3) can be re-written as

$$Y(s) = Y(s-) + Y(s-)\sum_{k=1}^{n}\sum_{\sigma(k)\in S_k}(g(\theta_{\sigma(k)}\delta N_{\sigma(k)}^{k}) - 1)$$

$$= Y(s-) + Y(s-)\sum_{k=1}^{n}\sum_{\sigma(k)\in S_k}(g(\theta_{\sigma(k)} - 1)\delta N_{\sigma(k)}^{k}). \tag{3.4}$$

Equation (3.4) therefore ends the proof of Theorem 3.2. □

Prior to the further study of independence for jump processes, we comment on the utility of Theorems 3.1 and 3.2. Based upon equations (3.2) or (3.4), we will apply the stochastic integral of a pure jump family $\{N_{\sigma(k)}^{k}\}$ constructed by the finite number of pure jump processes to express the jump process Y in order to decipher the conditions for independence among random variables in a pure jump process.

3.2. Independence among random variables in a pure jump process

Theorem 3.3. *Let* $N = (N^1, N^2, \ldots, N^n)$ *be an n-dimensional pure jump* (*counting*) *process with* $N^i(0) = 0$. *Then, there exists a finite number of jump points for every process* N^i *in the interval* $[0, t]$ *and processes* $N^1, N^2, \ldots, N^n$ *are mutually independent if the following conditions hold*:

- (G1) $E(N^i(t) - N^i(s) | \mathcal{F}_s) = \dfrac{\Delta\overline{\varphi}_i}{\exp\theta_i - 1}, \quad s \leq t;$

- (G2) $P(B^2) = 0.$

Proof. Let $Y(t) = \exp\left(\sum_{i=1}^{n}(\theta_i N^i(t)) - \log\left(\prod_{i=1}^{n}\varphi(\theta_i, t)\right)\right)$. Then by the Itô's formula of the jump process (semi-martingale) Protter [5], Russo and Vallois [7], and Williams [8], we have

$$Y(t) = 1 + \sum_{0<s\leq t}(Y(s) - Y(s-)) - \sum_{i=1}^{n}\left(\int_0^t Y(s)d(\log\varphi_i(\theta_i, s))\right). \tag{3.5}$$

We also assume that at least one process of $N^1, N^2, \ldots, N^n$ produces a jump at the point s (otherwise $Y(s) - Y(s-) = 0$). By Theorem 3.2, we get

$$\begin{aligned} Y(s) - Y(s-) &= Y(s-)\exp\left(\sum_{i=1}^{n}\theta_i\delta N^i(s) - 1\right) \\ &= Y(s-)\sum_{k=1}^{n}\sum_{\sigma(k)\in S}((\exp(\theta_{\sigma(k)}\delta N_{\sigma(k)}^k) - 1) \\ &= Y(s-)\sum_{k=1}^{n}\sum_{\sigma(k)\in S}((\exp\theta_{\sigma(k)} - 1)\delta N_{\sigma(k)}^k). \end{aligned} \tag{3.6}$$

According to equations (3.5) and (3.6), we further obtain a linear decomposition of Y which takes account of the pure jump family $\{N_{\sigma(k)}^k\}$, and it can be written as

$$Y(t) = 1 + \sum_{0<s\le t}(Y(s) - Y(s-)) - \sum_{i=1}^{n}\left(\int_0^t Y(s)d(\log\varphi_i(\theta_i,\, s))\right)$$

$$= 1 + \sum_{0<s\le t}\left(\sum_{k=1}^{n}\sum_{\sigma(k)\in S_k}(\exp\theta_{\sigma(k)} - 1)Y(s-)\delta N_{\sigma(k)}^{k}\right)$$

$$- \sum_{i=1}^{n}\left(\int_0^t Y(s-)d(\log\varphi_i(\theta_i,\, s))\right)$$

$$= 1 + \sum_{k=1}^{n}\sum_{\sigma(k)\in S_k}\left((\exp\theta_{\sigma(k)} - 1)\int_0^t Y(s-)dN_{\sigma(k)}^{k}(s)\right)$$

$$- \sum_{i=1}^{n}\left(\int_0^t Y(s-)d(\log\varphi_i(\theta_i,\, s))\right)$$

$$= 1 + \sum_{i=1}^{n}\left((\exp\theta_i - 1)\int_0^t Y(s-)dN^i(s)\right) - \sum_{i=1}^{n}\left(\int_0^t Y(s-)d(\log\varphi_i(\theta_i,\, s))\right)$$

$$+ \sum_{k=2}^{n}\sum_{\sigma(k)\in S_k}\left((\exp\theta_{\sigma(k)} - 1)\int_0^t Y(s-)dN_{\sigma(k)}^{k}(s)\right). \tag{3.7}$$

From condition (G1) of Theorem 3.3, we obtain that the equation $E(N(t) - N(s)|\mathcal{F}_s) = \frac{\Delta\overline{\varphi}_i}{\exp\theta_i - 1}$ is equivalent to the fact that the process

$$\overline{N^i}(t) = (\exp\theta_i - 1)N^i(t) - \log\varphi_i(\theta_i,\, t)$$

is a martingale. We therefore get that the process

$$\overline{\overline{N^i}}(t) = \sum_{i=1}^{n}\left((\exp\theta_i - 1)\int_0^t Y(s-)dN^i(s)\right) - \sum_{i=1}^{n}\left(\int_0^t Y(s-)d(\log\varphi_i(\theta_i,\, s))\right)$$

$$= \sum_{i=1}^{n}\int_0^t Y(s-)d\overline{N^i}(s) \tag{3.8}$$

is a martingale and $\overline{\overline{N^i}}(0) = 0.$

By the fact that $P(B^2) = 0 \Rightarrow P(B^k) = 0,\ k \geq 2 \Leftrightarrow \delta N^k_{\sigma(k)} = 0,$ almost surely, with $(k \geq 2),$ we have

$$E\left(\sum_{k=2}^{n} \sum_{\sigma(k)\in S_k} (\exp\theta_k - 1)\int_0^t Y(s-)dN^k_{\sigma(k)}(s)\right) = 0. \tag{3.9}$$

According to equations (3.7), (3.8) and (3.9), we conclude that $E(Y(t)) = 1,$ i.e., $E\left(\exp\left(\sum_{i=1}^n \theta_i N^i(t)\right)\right) = \prod_{i=1}^n (\varphi_i(\theta_i, t)).$ Again, by the properties of characteristic functions, $N^1, N^2, \ldots, N^n$ are mutually independent. □

Corollary 3.4 (Bertoin [1]). *Let $N = (N^1, N^2, \ldots, N^n)$ be an n-dimensional Poisson process. Then, the random variables $N^1, N^2, \ldots, N^n$ are mutually independent if and only if $\forall i, j \in \{1, 2, \ldots, n\}, i \neq j,$*

$$\delta N^i = N^i(s) - N^i(s-) = 0, \textit{ or } \delta N^j = N^j(s) - N^j(s-) = 0, \ a.s.$$

Proof. For a Poisson process $N^i,$ the moment generating function $\varphi_i(\theta_i, t) = \exp(\lambda_i t(e^{\theta_i} - 1)).$ Obviously, the process

$$\overline{N^i}(t) = N^i(t) - \log\varphi_i(\theta_i, t)/(\exp\theta_i - 1) = N_i(t) - \lambda_i t$$

is a martingale. Thus, it is quite apparent that condition (G1) of Theorem 3.3 holds. The condition (G2) from Theorem 3.3 here is equivalent to $\delta N^i = N^i(s) - N^i(s-) = 0$ or $\delta N^j = N^j(s) - N^j(s-) = 0,\ i \neq j,$ almost surely. Therefore, we conclude Corollary 3.4 trivially.

Compared with the condition for Corollary 3.4, the conditions of Theorem 3.3 are evidently weaker.

Note that Corollary 3.4 is one of the important results in Bertoin [1], which demonstrates that Theorem 3.3 is a quite general settlement for identifying independence among random variables in a jump process. Moreover, the values of the linear decompositions of jump processes and Theorem 3.3 are verified.

4. Independence among Random Variables between a Continuous Semi-martingale and a Pure Jump Process

Let $X = (X^1, X^2, ..., X^m)$ be an m-dimensional continuous semi-martingale with zero initial values, and $N = (N^1, N^2, ..., N^n)$ be an n-dimensional pure jump process with zero initial values. Then, we have the following theorem.

Theorem 4.1. *Processes* $X^1, X^2, ..., X^m, N^1, N^2, ..., N^n$ *are mutually independent if the following conditions hold*:

- (H1) $[M^i, M^j] = 0,\ i \neq j$;
- (H2) $E(\theta_i \Delta [M^i] + 2\Delta A^i \,|\, \mathcal{F}_s) = 2\Delta\, \overline{\varphi}_i / \theta_i$;
- (H3) $E(N^i(t) - N^i(s) \,|\, \mathcal{F}_s) = \Delta\, \overline{\varphi}_i / (\exp\theta_i - 1),\ s \leq t$;
- (H4) $P(B^2) = 0$.

Proof. First, we write

$$Y(t) = \exp\left(\sum_{i=1}^{m}(\theta_i X^i(t)) + \sum_{j=1}^{n}(\gamma_j N^j(t)) - \log\left(\prod_{i=1}^{m}\varphi_i(\theta_i, t)\right) - \log\left(\prod_{j=1}^{n}\varphi_j(\gamma_j, t)\right)\right),$$

where $\varphi_i(\theta_i, t)$ and $\varphi_j(\gamma_j, t)$ are the characteristic functions of processes X^i and N^i, respectively.

Applying Itô's formula for a semi-martingale and using conditions (H1), (H2) and (H3), we have

$$Y(t) = 1 + \sum_{i=1}^{m}\left(\int_0^t Y(s-)dM^i(s) + \int_0^t Y(s-)d\overline{M^i}(s)\right)$$

$$+ \sum_{j=1}^{n}\int_0^t Y(s-)d\overline{N^j}(s) + \sum_{k=2}^{n}\sum_{\sigma(k)\in S_k}\left((\exp\gamma_{\sigma(k)} - 1)\int_0^t Y(s-)dN^k_{\sigma(k)}(s)\right),$$

where M^i, $\overline{M^i}$ and $\overline{N^j}$ are all martingales (see Theorems 2.1 and 3.3). Due to condition (H4), we have $E[Y(t)] = 1.$ Therefore, we arrive at

$$\exp\left(\sum_{i=1}^{m}(\theta_i X^i(t)) + \sum_{j=1}^{n}(\gamma_j N^j(t))\right) = \prod_{i=1}^{m}\varphi_i(\theta_i,\ t)\cdot\prod_{j=1}^{n}\varphi_j(\gamma_j,\ t)$$

$$= \prod_{i=1}^{m}\prod_{j=1}^{n}(\varphi_i(\theta_i,\ t)\cdot\varphi_j(\gamma_j,\ t)),$$

which means that processes $X^1,\ X^2,\ ...,\ X^m,\ N^1,\ N^2,\ ...,\ N^n$ are mutually independent.

Corollary 4.2. *Let M be a one-dimensional square integrable martingale with a zero initial value and N be a one-dimensional pure jump process with a zero initial value. Then M and N are independent of each other if the following conditions hold*:

- (I1) $E[\Delta[M] \mid \mathcal{F}_s] = 2\Delta\overline{\varphi}_M/\theta_M^2;$
- (I2) $E[N(t) - N(s) \mid \mathcal{F}_s] = \Delta\overline{\varphi}_N/(\exp\theta_N - 1).$

Corollary 4.2 serves as a direct proposition of Theorem 4.1.

Corollary 4.3 (Shreve [6])**.** *A one-dimension standard Brownian motion W and a one-dimensional Poisson process N which are both defined in the same filtered complete probability space* $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_{t\geq 0}\},\ P)$ *are independent of each other*.

Corollary 4.3 can be derived from Corollary 4.2. For a standard Brownian motion *W* and a Poisson process *N*, the two conditions of Corollary 4.2 are effortlessly fulfilled. So, it is intriguing to emphasize the fact that processes *W* and *N* are unconditionally independent of each other.

The unconditional independence provides numerous opportunities for applications of Corollary 4.3. For example, if we assume that the price processes of two stocks are driven by a Brownian motion *W* and a Poisson process N, respectively, when we build up a model for stocks, the price processes of these two stocks are independent of each other.

Corollary 4.4. *Let W be an m-dimensional standard Brownian motion defined in a filtered complete probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_{t\geq 0}\}, P)$ *and let N be an n-dimensional. Poisson process defined in the same space, then the random variables in W and N are mutually independent.*

Corollary 4.5. *Let* $W = (W^1, W^2, ..., W^m)$ *be an m-dimensional standard Brownian motion, and let* $N = (N^1, N^2, ..., N^n)$ *be an n-dimensional Poisson process. Then*

$$W^1, W^2, ..., W^m, \quad N^1, N^2, ..., N^n$$

are mutually independent if and only if the following conditions hold:

- (J1) $[W^i, W^j] = 0,\ i \neq j;$
- (J2) $P(B^2) = 0.$

We note that Corollaries 4.4 and 4.5 are all quite obvious to obtain from Theorem 4.1.

5. Concluding Remarks

We first detect the conditions for independence among random variables in a continuous semi-martingale, and then local independence and progressive independence are defined and the corresponding independence conditions are explored. The linear decompositions of a jump process are presented for further proofs regarding the independence conditions for random variables in a jump process and between a continuous semi-martingale and a pure jump process.

More importantly, the results in this paper accommodate certain classic results such as Corollary 3.4 and Corollary 4.3, which implies that the theorems presented here require weaker conditions for independence. With the weaker conditions, the corresponding independence is easier to be satisfied and the theorems can be more smoothly applied.

The concepts of local independence and progressive independence are initially defined in the paper, and they are original and efficient tools for studying processes where independence conditions are weaker than those for independence defined by the traditional way, as discussed in Subsection 2.2.

The linear decompositions of a jump process are proposed. Based on the linear decompositions, the independence conditions for random variables in a jump process are investigated. The independence conditions for random variables between a continuous semi-martingale and a pure jump process are also inspected.

Between semi-martingales used in practice, if independence exists among random variables (which is the optimal case), the utility of these stochastic processes will be quite unproblematic and less complicated.

References

[1] J. Bertoin, Lévy Processes, Cambridge Press, 1996.

[2] I. Karatzas and S. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer-Verlag, 1991.

[3] S. Karlin and H. M. Taylor, A Second Course in Stochastic Processes, Elsevier Pte Ltd., 2009, pp. 433-441.

[4] F. C. Klebaner, Introductions to Stochastic Calculus with Applications, Imperial College Press, 2005.

[5] P. E. Protter, Stochastic Integration and Differential Equations, 2nd ed., Springer-Verlag, 2005.

[6] S. E. Shreve, Stochastic Calculus for Finance, Springer-Verlag, 2004.

[7] F. Russo and P. Vallois, Itô formula for C^1 functions of semimartingales, Probab. Theory and Related Fields 104 (1996), 27-41.

[8] D. Williams, Probability with Martingales, Cambridge Press, 1991.