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A NOTE ON UNEQUAL PROBABILITY SAMPLING IN BOOTSTRAP SAMPLING

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Abstract

Bootstrap sampling is gaining popularity as computational tools are becoming more accessible. In bootstrap sampling, usually, samples of equal sizes are selected with replacement and with equal probability in obtaining each element of the original sample at hand. It is the key question in the researchers' mind that whether unequal sampling would provide more accurate results. Here we introduce three alternative unequal probability sampling procedures. A comparison among the three unequal probability choices and the equal probability sampling are conducted using the Monte Carlo simulation.

1. Introduction

Bootstrapping is a general approach to statistical inference based on building a sampling distribution for a statistic by resampling from the data at hand. The term bootstrapping, due to Efron and Tibshirani [2], Efron [1], Hall and Martin [3], Higgins [4], use the sample data as a population from

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which repeated samples are drawn. The approach may seem concentrating only on the sample at hand and overlooking the 'big picture', but has been shown to be sound. It always gives a way to estimate the variability of an estimate when no plausible method is available.

Suppose that we draw a sample $S = \{X_1, X_2, ..., X_n\}$ from a population $P = \{x_1, x_2, ..., x_N\}$. The essential idea of the nonparametric bootstrap is as follows: We proceed to draw a sample of size *n* with replacement from the elements of *S*. Call the resulting bootstrap sample $S_i^* = \{X_{i1}^*, X_{i2}^*, ..., X_{in}^*\}$, $i = 1, 2, ..., L,$ where *L* is the number of bootstrap samples often chosen a fairly large number, say, 10,000. It is necessary to sample with replacement, because we would otherwise simply reproduce the original sample *S*.

Let $T = g(S)$ be an estimate of a parameter θ for which the functional form of the standard error of the estimate is either unavailable or the sample size is so small that the standard error estimate is highly volatile. In such situations, *L* bootstrap samples can produce a reliable estimate of the sampling distribution of the statistic *T* and hence the standard error of the estimate.

2. Motivation

Bootstrap sampling is gaining popularity as computational tools are becoming more accessible. In bootstrap sampling, usually, samples of equal sizes are selected with replacement and with equal probability in obtaining each element of the original sample at hand. It is the key question in the researchers' mind that whether unequal sampling would provide more accurate results. Here we introduce three alternative unequal probability sampling procedures.

Mecatti [5] studied equal and unequal probability sampling in bootstrap sampling. In this study, the author re-enforced the importance of with replacement sampling while bootstrapping. In addition, Davison et al. [8] summarized recent developments in bootstrap sampling.

More recently, Ranalli and Mecatti [6] presented a comparison of recent approaches for bootstrapping in sample surveys. In their study they considered effects on variances for different sampling designs.

3. Bootstrap Sampling

3.1. Equal probability sampling

Each element of the sample $S = \{X_1, X_2, ..., X_n\}$ is selected with probability $p_{ei} = \frac{1}{n}$ with replacement.

3.2. Normal probability sampling

Let us consider the ordered sample $SO = \{X_{(1)}, X_{(2)}, ..., X_{(n)}\}$, ordered from the smallest to the largest. Then each element of the ordered sample *SO* is selected with replacement and with probabilities $p_{g1} = \Phi(Z_{(1)})$, where Φ stands for the standard normal cumulative distribution function (CDF), and $p_{gi} = \Phi(Z_{(i)}) - \Phi(Z_{(i-1)})$ for $i = 2, 3, ..., n$, $Z_{(i)} = \frac{X_{(i)} - \overline{X}}{s_X}$, $\overline{X}$ stands for the sample mean and s_X stands for the sample standard deviation, then the p_{gi}'s are adjusted to ensure the total probability is one, $i = 1, 2, 3, ..., n$.

3.3. Kernel probability sampling

Each element of the ordered sample *SO* is selected with replacement and with probabilities $p_{k1} = G(X_{(1)})$, where G stands for the kernel cumulative distribution function, and $p_{ki} = G(X_{(i)}) - G(X_{(i-1)})$ for $i = 2, 3, ..., n$, where $G(x) = \int_{-\infty}^{x} g(x)dx$, where

$$g(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right),$$

where

$$K(u) = \begin{cases} \frac{3}{4}(1 - u^2), & \text{if } |u| \le 1, \\ 0, & \text{otherwise} \end{cases}$$

is the Epanechnikov kernel and $h = 2.214\sigma n^{-\frac{1}{5}}$ is the smoothing parameter (see Simonoff [7, pp. 40-45]), σ is the population standard deviation and can be replaced by s (the sample standard deviation) whenever necessary. Then the p_{ki}'s are also adjusted to ensure the total probability is one.

3.4. Empirical unequal probability sampling

Each element of the ordered sample SO is selected with replacement and with probabilities $p_{u1} = \frac{U(2)}{2}$ and $p_{ui} = U(i)$ for $i = 2, 3, ..., n,$ where $U(i) = \frac{X_{(i)} - X_{(1)}}{X_{(n)} - X_{(1)}}$, then the p_{ui}'s are adjusted to ensure the total probability is one, note that $U(1)$ is zero and $U(n)$ is one.

4. Monte Carlo Simulation

One thousand samples are selected from each of the standard normal distribution $(N(0, 1)),$ the standard exponential distribution $(E(1)),$ a normal mixture distribution $\left(\frac{3}{4}N(0, 1) + \frac{1}{4}N\left(\frac{3}{2}, \frac{1}{3}\right)\right)$ which is a skewed bimodal distribution), and the standard uniform $(U(0, 1))$ distribution. Samples are selected of sizes 10, 50, and 100. From each sample mean $(\mu),$ standard deviation $(\sigma),$ the first quartile $(Q_1),$ and the third quartile (Q_3) are computed. Then these parameters are recomputed using bootstrap sampling $(L = 10{,}000$ samples in each case) are performed for all four sampling plans mentioned above. The total values of the absolute error as

$$\text{absolute} = \sum_{i=1}^{L} |\hat{\theta}_i - \theta|$$

and the squared error as

$$\text{squared} = \sum_{i=1}^{L} (\hat{\theta}_i - \theta)^2,$$

where $\hat{\theta}$ is the bootstrap estimate of the parameter θ (which are $\mu, \sigma, Q_1,$ and Q_3) are computed and stored.

All computations are performed using the MATLAB computational software and the codes are available on request from the first author.

In Tables 1-4, the means (mean) and the standard deviations (SD) for both the absolute and the squared distance measures are displayed. The ranks of the total values are also displayed to have a quick impression of the comparisons.

Overall values are not clearly distinguished but the rankings showed that the equal probability sampling should be preferred.

In conclusion, the equal probability sampling should be the choice in sampling for bootstrap sampling.

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Table 1. Normal samples

Parameter	Distance	Statistic	p_e	p_g	p_k	p_u	p_e	p_g	p_k	p_u
			Total values				Ranks			
Standard Normal samples of size $n = 10$										
μ	Absolute	Mean	2324.95	2620.89	6037.85	3924.41	1	2	4	3
μ	Absolute	SD	1613.65	2035.58	2273.29	1753.85	1	3	4	2
σ	Absolute	Mean	3141.28	3566.37	7447.62	4648.56	1	2	4	3
σ	Absolute	SD	3162.59	4004.86	7362.37	5631.03	1	2	4	3
Q_1	Absolute	Mean	894.11	1200.46	5168.78	2626.94	1	2	4	3
Q_1	Absolute	SD	471.49	666.44	861.15	520.95	1	3	4	2
Q_3	Absolute	Mean	2166.42	2419.90	7939.13	4263.29	1	2	4	3
Q_3	Absolute	SD	2179.82	3446.26	9093.12	5902.10	1	2	4	3
μ	Squared	Mean	572.84	1043.39	2278.12	2100.45	1	2	4	3
μ	Squared	SD	533.89	828.41	1064.44	781.63	1	3	4	2
σ	Squared	Mean	1366.27	1656.90	4324.40	2279.56	1	2	4	3
σ	Squared	SD	1348.55	2598.13	2300.37	3932.07	1	3	2	4
Q_1	Squared	Mean	440.98	1056.70	3689.96	3064.32	1	2	4	3
Q_1	Squared	SD	327.35	540.13	825.52	479.49	1	3	4	2
Q_3	Squared	Mean	1705.28	2191.62	9391.57	4412.58	1	2	4	3
Q_3	Squared	SD	1762.84	5178.00	5786.80	8539.18	1	2	3	4
Standard Normal samples of size $n = 50$										
μ	Absolute	Mean	1114.63	1181.39	6115.67	3155.04	1	2	4	3
μ	Absolute	SD	776.77	925.43	5532.10	3636.07	1	2	4	3
σ	Absolute	Mean	1512.20	1709.51	13875.10	5409.05	1	2	4	3
σ	Absolute	SD	1526.91	1843.86	4886.56	6069.15	1	2	3	4
Q_1	Absolute	Mean	297.20	223.52	4679.60	1626.81	1	2	4	3
Q_1	Absolute	SD	97.24	133.16	3524.55	1721.87	1	2	4	3
Q_3	Absolute	Mean	431.39	482.64	20692.09	4361.80	1	2	4	3
Q_3	Absolute	SD	440.71	575.33	4333.70	6582.03	1	2	3	4
μ	Squared	Mean	114.32	207.07	2239.54	1891.98	1	2	4	3
μ	Squared	SD	140.33	240.91	1882.28	1783.41	1	2	4	3
σ	Squared	Mean	457.73	457.96	2894.08	3064.43	1	2	3	4
σ	Squared	SD	485.88	531.71	2502.51	4256.91	1	2	3	4
Q_1	Squared	Mean	40.14	84.26	2868.18	1942.38	1	2	4	3
Q_1	Squared	SD	35.66	83.23	2514.24	1691.71	1	2	4	3
Q_3	Squared	Mean	229.10	254.57	8376.50	4505.72	1	2	4	3
Q_3	Squared	SD	248.67	366.38	6398.61	10136.61	1	2	3	4
Standard Normal samples of size $n = 100$										
μ	Absolute	Mean	793.12	822.31	5520.35	2898.65	1	2	4	3
μ	Absolute	SD	554.62	627.06	6964.70	5026.92	1	2	4	3
σ	Absolute	Mean	1063.84	1244.80	14997.23	6592.19	1	2	4	3
σ	Absolute	SD	1096.16	1296.40	4333.45	6287.41	1	2	3	4
Q_1	Absolute	Mean	99.36	106.74	3754.43	1385.54	1	2	4	3
Q_1	Absolute	SD	49.03	60.89	5209.67	2879.97	1	2	4	3
Q_3	Absolute	Mean	209.79	250.44	23401.49	6002.92	1	2	4	3
Q_3	Absolute	SD	217.00	271.78	3270.57	6076.71	1	2	3	4
μ	Squared	Mean	56.39	84.48	209.51	1930.11	1	2	4	3
μ	Squared	SD	75.54	117.38	1733.73	1744.68	1	2	3	4
σ	Squared	Mean	305.58	295.23	2197.85	3575.69	2	1	3	4
σ	Squared	SD	209.96	307.54	2578.34	3800.54	1	2	3	4
Q_1	Squared	Mean	14.10	21.84	2388.94	1873.64	1	2	4	3
Q_1	Squared	SD	13.61	21.85	2661.92	1998.84	1	2	4	3
Q_3	Squared	Mean	108.46	112.65	6694.60	5842.87	1	2	4	3
Q_3	Squared	SD	102.07	129.22	4601.50	7908.41	1	2	3	4

Table 2. Exponential samples

Parameter	Distance	Statistic	p_e	p_g	p_k	p_u	p_e	p_g	p_k	p_u
			Total values				Ranks			
Standard Exponential samples of size $n = 10$										
μ	Absolute	Mean	2216.89	4211.93	3844.53	9174.39	1	3	2	4
μ	Absolute	SD	1947.30	2073.78	2862.87	1920.14	2	3	4	1
σ	Absolute	Mean	1341.11	3198.02	2051.06	8202.62	1	3	2	4
σ	Absolute	SD	4289.04	7990.40	9840.52	11987.23	1	2	3	4
Q_1	Absolute	Mean	898.20	4023.70	2929.33	15829.64	1	3	2	4
Q_1	Absolute	SD	980.72	961.21	1611.52	771.24	3	2	4	1
Q_3	Absolute	Mean	493.26	2410.76	733.03	15865.46	1	3	2	4
Q_3	Absolute	SD	4778.84	17926.85	20811.17	28254.81	1	2	3	4
μ	Squared	Mean	914.60	3533.00	2332.03	7937.31	1	3	2	4
μ	Squared	SD	1357.57	1878.32	2230.35	1352.42	2	3	4	1
σ	Squared	Mean	680.63	1341.52	1356.34	6732.61	1	2	3	4
σ	Squared	SD	2210.83	9157.29	7984.13	11415.77	1	3	2	4
Q_1	Squared	Mean	798.76	9214.32	5179.85	32999.36	1	3	2	4
Q_1	Squared	SD	1745.43	2244.56	2789.74	1433.34	2	3	4	1
Q_3	Squared	Mean	506.07	2829.42	956.69	37500.57	1	3	2	4
Q_3	Squared	SD	5670.84	46754.44	42327.09	56843.55	1	3	2	4
Standard Exponential samples of size $n = 50$										
μ	Absolute	Mean	1108.37	1961.80	5937.03	16026.79	1	2	3	4
μ	Absolute	SD	1289.91	1150.84	7517.02	5358.59	2	1	4	3
σ	Absolute	Mean	642.31	1686.76	2702.84	9501.48	1	2	3	4
σ	Absolute	SD	1953.94	4047.77	18071.06	26611.85	1	2	3	4
Q_1	Absolute	Mean	200.20	641.22	6157.00	33975.55	1	2	3	4
Q_1	Absolute	SD	315.08	345.11	7735.08	4568.28	1	2	4	3
Q_3	Absolute	Mean	82.57	444.47	791.33	12541.25	1	2	3	4
Q_3	Absolute	SD	755.32	2946.32	52275.23	92989.57	1	2	3	4
μ	Squared	Mean	215.79	964.37	4486.51	8815.95	1	2	3	4
μ	Squared	SD	605.79	1158.11	4404.95	4022.38	1	2	4	3
σ	Squared	Mean	220.27	350.43	765.54	4547.35	1	2	3	4
σ	Squared	SD	661.36	2516.76	11750.52	14442.30	1	2	3	4
Q_1	Squared	Mean	80.34	1391.42	10064.44	47547.57	1	2	3	4
Q_1	Squared	SD	375.45	2443.00	10824.33	7541.66	1	2	4	3
Q_3	Squared	Mean	52.63	183.97	456.84	28953.08	1	2	3	4
Q_3	Squared	SD	486.47	7940.34	77848.64	110561.00	1	2	3	4
Standard Exponential samples of size $n = 100$										
μ	Absolute	Mean	790.75	1388.29	6810.01	18929.62	1	2	3	4
μ	Absolute	SD	989.07	942.62	8594.90	7258.52	2	1	4	3
σ	Absolute	Mean	449.27	1243.21	2787.89	10689.46	1	2	3	4
σ	Absolute	SD	1381.48	3376.81	18359.78	31691.09	1	2	3	4
Q_1	Absolute	Mean	100.15	275.70	5906.05	42531.17	1	2	3	4
Q_1	Absolute	SD	170.14	130.58	8518.55	7129.86	2	1	4	3
Q_3	Absolute	Mean	38.16	239.31	809.16	13158.47	1	2	3	4
Q_3	Absolute	SD	354.20	1598.35	40723.39	121442.79	1	2	3	4
μ	Squared	Mean	110.95	355.16	3019.73	7988.28	1	2	3	4
μ	Squared	SD	353.31	254.10	3212.77	4258.21	2	1	3	4
σ	Squared	Mean	128.60	213.43	563.70	3360.46	1	2	3	4
σ	Squared	SD	419.20	1534.48	6323.87	14031.70	1	2	3	4
Q_1	Squared	Mean	28.53	140.24	4793.16	40481.15	1	2	4	3
Q_1	Squared	SD	136.11	87.57	6959.29	9111.13	2	1	3	4
Q_3	Squared	Mean	20.32	677.34	331.35	8815.68	1	2	3	4
Q_3	Squared	SD	203.18	1343.66	31219.29	115353.79	1	2	3	4

Table 3. Normal mixture samples

Parameter	Distance	Statistic	p_e	p_g	p_k	p_u	p_e	p_g	p_k	p_u
			Total values				Ranks			
Normal Mixture $\frac{3}{4}N(0,1)+\frac{1}{4}N\left(\frac{3}{2},\frac{1}{3}\right)$ samples of size $n=10$										
μ	Absolute	Mean	2589.98	2512.38	7690.49	3733.50	2	1	4	3
μ	Absolute	SD	1665.74	2555.89	2053.22	1952.45	1	4	3	2
σ	Absolute	Mean	3750.54	4325.35	8542.48	5224.88	1	2	4	3
σ	Absolute	SD	2852.11	3487.20	8405.67	4560.48	1	2	4	3
Q_1	Absolute	Mean	1089.45	1038.75	7888.37	2303.64	2	1	4	3
Q_1	Absolute	SD	511.68	1018.51	702.94	635.97	1	4	3	2
Q_3	Absolute	Mean	3154.25	3562.36	10411.80	5404.57	1	2	4	3
Q_3	Absolute	SD	1912.45	2297.75	11779.97	3857.78	1	2	4	3
μ	Squared	Mean	542.92	717.14	2709.45	1685.83	1	2	4	3
μ	Squared	SD	539.31	1060.20	667.24	828.21	1	4	2	3
σ	Squared	Mean	1607.74	2165.57	4942.85	2743.60	1	2	4	3
σ	Squared	SD	1240.24	1511.07	2266.61	2446.10	1	2	3	4
Q_1	Squared	Mean	454.64	620.54	5196.36	2161.25	1	2	4	3
Q_1	Squared	SD	371.83	854.48	436.85	590.80	1	4	2	3
Q_3	Squared	Mean	2532.90	3592.33	12340.99	5947.04	1	2	4	3
Q_3	Squared	SD	1469.59	2185.00	6106.68	4145.65	1	2	4	3
Normal Mixture $\frac{3}{4}N(0,1)+\frac{1}{4}N\left(\frac{3}{2},\frac{1}{3}\right)$ samples of size $n=50$										
μ	Absolute	Mean	1218.47	1199.76	9146.80	3491.25	2	1	4	3
μ	Absolute	SD	719.32	1337.94	3978.27	2215.04	1	2	4	3
σ	Absolute	Mean	1764.56	2068.12	15072.03	5820.36	1	2	4	3
σ	Absolute	SD	1346.28	2595.73	6574.89	3653.91	1	2	4	3
Q_1	Absolute	Mean	234.71	229.66	9324.61	1928.62	2	1	4	3
Q_1	Absolute	SD	83.52	248.79	1950.08	772.65	1	2	4	3
Q_3	Absolute	Mean	594.92	700.57	24215.25	4882.70	1	2	4	3
Q_3	Absolute	SD	368.65	985.23	6351.48	2627.96	1	2	4	3
μ	Squared	Mean	107.44	181.52	2270.59	2109.38	1	2	4	3
μ	Squared	SD	127.95	377.84	1621.18	1462.53	1	2	4	3
σ	Squared	Mean	535.73	683.04	3172.90	3138.63	1	2	4	3
σ	Squared	SD	489.63	802.87	2173.60	2465.54	1	2	3	4
Q_1	Squared	Mean	41.43	74.30	4070.97	2280.02	1	2	4	3
Q_1	Squared	SD	31.06	123.07	1715.52	1025.07	1	2	4	3
Q_3	Squared	Mean	333.85	439.10	10126.18	4843.59	1	2	4	3
Q_3	Squared	SD	244.30	532.49	3990.74	5643.74	1	2	3	4
Normal Mixture $\frac{3}{4}N(0,1)+\frac{1}{4}N\left(\frac{3}{2},\frac{1}{3}\right)$ samples of size $n=100$										
μ	Absolute	Mean	866.36	911.71	9167.00	3851.10	1	2	4	3
μ	Absolute	SD	516.50	999.77	5359.63	3503.12	1	2	4	3
σ	Absolute	Mean	1256.11	1502.00	16798.81	7716.90	1	2	4	3
σ	Absolute	SD	943.43	2426.68	5386.80	3221.30	1	2	4	3
Q_1	Absolute	Mean	118.30	131.18	9166.96	2194.15	2	1	4	3
Q_1	Absolute	SD	42.61	139.14	3238.08	1586.56	1	2	4	3
Q_3	Absolute	Mean	288.41	362.28	29252.35	7742.21	1	2	4	3
Q_3	Absolute	SD	170.90	794.04	4186.48	2056.82	1	2	4	3
μ	Squared	Mean	51.41	132.72	2233.75	2303.72	1	2	3	4
μ	Squared	SD	72.41	295.81	1755.13	1773.46	1	2	3	4
σ	Squared	Mean	326.97	468.05	2496.84	3754.27	1	2	3	4
σ	Squared	SD	288.90	720.17	2084.45	2305.90	1	2	3	4
Q_1	Squared	Mean	13.98	37.50	3798.25	2464.01	1	2	4	3
Q_1	Squared	SD	12.24	69.49	2282.44	1602.84	1	2	4	3
Q_3	Squared	Mean	136.97	213.77	8449.81	6806.28	1	2	4	3
Q_3	Squared	SD	99.17	401.79	3009.85	4588.14	1	2	3	4

Table 4. Uniform samples

Parameter	Distance	Statistic	p_e	p_g	p_k	p_u	p_e	p_g	p_k	p_u
			Total values				Ranks			
Standard Uniform samples of size $n = 10$										
μ	Absolute	Mean	690.51	713.47	1773.59	1034.63	1	2	4	3
μ	Absolute	SD	396.77	610.48	524.57	357.59	1	4	2	3
σ	Absolute	Mean	873.92	1293.43	1514.71	1508.51	1	2	4	3
σ	Absolute	SD	883.16	965.97	2289.51	1174.40	1	2	4	3
Q_1	Absolute	Mean	76.14	82.66	418.60	171.31	1	2	4	3
Q_1	Absolute	SD	27.96	58.27	47.58	51.33	1	4	2	3
Q_3	Absolute	Mean	172.10	300.71	325.49	402.10	1	2	3	4
Q_3	Absolute	SD	175.27	175.42	872.39	252.82	1	2	4	3
μ	Squared	Mean	114.64	194.73	559.38	483.14	1	2	4	3
μ	Squared	SD	77.44	217.30	134.26	268.84	1	3	2	4
σ	Squared	Mean	362.50	557.69	912.11	741.91	1	2	4	3
σ	Squared	SD	351.47	423.34	571.78	694.12	1	2	3	4
Q_1	Squared	Mean	24.30	46.70	227.99	157.43	1	2	4	3
Q_1	Squared	SD	11.00	39.30	22.54	50.56	1	3	2	4
Q_3	Squared	Mean	124.40	249.31	370.52	351.19	1	2	4	3
Q_3	Squared	SD	122.84	169.52	421.42	304.09	1	2	4	3
Standard Uniform samples of size $n = 50$										
μ	Absolute	Mean	321.08	294.20	2010.81	469.07	2	1	4	3
μ	Absolute	SD	152.41	406.47	414.43	209.17	1	3	4	2
σ	Absolute	Mean	459.81	694.41	2316.68	675.49	1	3	4	2
σ	Absolute	SD	464.54	737.87	1840.17	658.63	1	2	3	4
Q_1	Absolute	Mean	16.21	13.66	438.71	34.61	2	1	4	3
Q_1	Absolute	SD	3.69	20.88	23.58	6.88	1	3	4	2
Q_3	Absolute	Mean	40.28	74.67	570.15	74.14	1	3	4	2
Q_3	Absolute	SD	41.27	79.62	472.84	71.03	1	3	4	2
μ	Squared	Mean	20.52	26.13	363.76	185.21	1	2	4	3
μ	Squared	SD	10.25	88.89	145.19	76.92	1	3	4	2
σ	Squared	Mean	145.95	245.71	576.99	261.91	1	2	4	3
σ	Squared	SD	149.47	248.87	533.53	265.50	1	2	4	3
Q_1	Squared	Mean	2.06	2.48	145.78	27.50	1	2	4	3
Q_1	Squared	SD	0.50	7.90	14.69	4.97	1	3	4	2
Q_3	Squared	Mean	23.28	46.98	279.72	53.95	1	2	4	3
Q_3	Squared	SD	24.56	46.80	227.06	57.56	1	2	4	3
Standard Uniform samples of size $n = 100$										
μ	Absolute	Mean	229.22	235.53	2078.30	322.83	1	2	4	3
μ	Absolute	SD	105.59	392.03	409.13	149.37	1	3	4	2
σ	Absolute	Mean	344.78	581.54	2408.40	476.30	1	3	4	2
σ	Absolute	SD	339.62	760.64	1829.24	479.85	1	3	4	2
Q_1	Absolute	Mean	8.26	8.71	449.59	16.32	1	2	4	3
Q_1	Absolute	SD	1.76	17.74	20.98	3.49	1	3	4	2
Q_3	Absolute	Mean	21.45	48.65	596.52	36.09	1	3	4	2
Q_3	Absolute	SD	20.91	75.19	413.69	36.69	1	3	4	2
μ	Squared	Mean	10.74	32.72	261.29	114.67	1	2	4	3
μ	Squared	SD	4.71	67.33	127.87	54.73	1	3	4	2
σ	Squared	Mean	94.33	185.74	406.05	168.82	1	3	4	2
σ	Squared	SD	90.77	241.62	463.65	177.40	1	3	4	2
Q_1	Squared	Mean	0.77	2.31	107.93	11.38	1	3	4	2
Q_1	Squared	SD	0.16	5.45	11.33	2.47	1	3	4	2
Q_3	Squared	Mean	10.63	826.40	199.03	24.30	1	3	4	2
Q_3	Squared	SD	9.78	41.10	177.87	26.72	1	3	4	2