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PERMANENCE FOR A TWO-SPECIES LESLIE-TYPE RATIO-DEPENDENT PREDATOR-PREY SYSTEM IN A TWO-PATCH ENVIRONMENT

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Abstract

A two-species Leslie-type ratio-dependent predator-prey periodic system with time delay in a two-patch environment is investigated. By using the comparison theorem of differential equations, we establish sufficient conditions for the permanence of this system. As corollaries, some applications are listed.

1. Introduction

In this paper, we consider the following periodic Leslie-type ratio-dependent predator-prey system with time delay in a two-patch environment:

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$$\begin{cases} x_1'(t) = x_1(t)g_1(t,\ x_1(t)) - x_3(t)\,p\left(t,\ \dfrac{x_1(t)}{x_3(t)}\right) + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) = x_2(t)g_2(t,\ x_2(t)) + d_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) = x_3(t)\left[r(t) - c(t)\dfrac{x_3(t - \tau(t))}{x_1(t - \tau(t))}\right], \end{cases} \tag{1.1}$$

where x_i $(i = 1, 2)$ represents the prey density in the ith patch, and x_3 represents the predator density. The predator is confined to the first patch while the prey can disperse between two patches. d_i denotes the dispersal rate of the prey in the ith patch. The function $g_i(t, x)$ is the growth rate of the prey in the absence of predator. The predator consumes the prey according to the functional response $p(t, x)$ and grows logistically with growth rate $r(t)$ and carrying capacity proportional to the population size of prey.

The form of the predator equation in (1.1) was first proposed by Leslie [1, 2]. Since Leslie, this type of predator-prey model has been studied extensively and seen great progress (see, e.g., [3-9] and the reference cited therein). The ratio-dependent response arises in the situations when predators have to search for food [10, 11]. Many authors [8, 12-17] have studied ratio-dependent predator-prey models, and observed that the ratio-dependent models can exhibit much richer, more complicated and more reasonable or acceptable dynamics. Dispersion takes into account the effect of spatial factors which play a crucial role in the permanence and stability of a population dynamics (see, e.g., [9, 16-25] and the reference cited therein). In fact, because of the ecological effects of human activities and industry, dispersal between patches often occurs in natural ecological environments, and more realistic models should include the dispersal process.

In [16], by using Gaines and Mawhin's continuation theorem of coincidence degree theory, Ding et al. obtained sufficient conditions for the existence of positive periodic solutions in system (1.1). However, to our knowledge, there is no published paper concerned with the permanence of system (1.1).

The aim of the present paper is, by applying the comparison theorem of different equations, to establish sufficient conditions which guarantee the permanence of system (1.1). For the sake of generality and convenience, we always make the following fundamental assumptions for system (1.1):

(A_1) $d_1(t)$, $d_2(t)$, $c(t)$ and $\tau(t)$ are positive continuous periodic functions with a common period $\omega > 0$ and $r(t)$ is a continuous ω-periodic function.

(A_2) $g_1(t, x)$ is a continuous function, ω-periodic in t, continuously differentiable in x, $(\partial g_1/\partial x)(t, x) < 0$ for $t \in \mathbb{R}$, $x > 0$, and there exists a positive constant Γ_1 such that $g_1(t, \Gamma_1) < 0$ for $t \in \mathbb{R}$.

(A_3) $g_2(t, x)$ is a continuous function, ω-periodic in t, continuously differentiable in x, $(\partial g_2/\partial x)(t, x) < 0$ for $t \in \mathbb{R}$, $x > 0$, and there exists a positive constant Γ_2 such that $g_2(t, \Gamma_2) < 0$ for $t \in \mathbb{R}$.

(A_4) $p(t, x)$ is a continuous function, ω-periodic in t, continuously differentiable in x, and there exists a positive continuous ω-periodic function $\alpha(t)$ such that $0 \le p(t, x) \le \alpha(t)x$ for $t \in \mathbb{R}$, $x > 0$.

Readers familiar with predator-prey models may notice that the above assumptions are reasonable for population models. The assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment. By the theory of functional differential equations, it is clear that under the above assumptions system (1.1) with initial condition

$$x_i(s) = \phi_i(s), \quad s \in [-\nu, 0], \quad \phi_i \in C([-\nu, 0], (0, +\infty)), \quad i = 1, 2, 3, \tag{1.2}$$

has a unique positive solution, where $\nu = \max_{t\in[0, \omega]} \tau(t)$.

2. Permanence

In this section, we consider the permanence of system (1.1). First, we state some results which will be useful to establish our main results.

In what follows, we use the notations:

$$f^u = \max_{t\in[0,\omega]} f(t), \quad f^l = \min_{t\in[0,\omega]} f(t),$$

$$\hat{f} = \frac{1}{\omega}\int_0^\omega f(t)dt, \quad \tilde{g}(x) = \frac{1}{\omega}\int_0^\omega g(t, x)dt,$$

where $f(t)$ is a continuous ω-periodic function, and $g(t, x)$ is a continuous function and is ω-periodic in t.

Definition 2.1. System (1.1) is said to be *permanent* if there exist positive constants $M_i, m_i, i = 1, 2, 3,$ which are independent of the solution of system (1.1), such that

$$m_i \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le M_i, \quad i = 1, 2, 3,$$

for any solution $(x_1(t), x_2(t), x_3(t))$ of system (1.1) with the initial condition (1.2).

Consider the following logistic equation:

$$x'(t) = x(t)[a(t) - b(t)x(t)], \tag{2.1}$$

where $a(t)$ and $b(t)$ are ω-periodic continuous functions. By Lemma 2.2 of [26], we have the following result.

Lemma 2.1. *If* $\hat{a} > 0$ *and* $b(t) > 0$ *for all* $t \in \mathbb{R}$, *then equation* (2.1) *admits a unique positive* ω-*periodic solution which is global asymptotically stable*.

Next consider the following single species dispersion system:

$$\begin{cases} x_1'(t) = x_1(t)g_1(t, x_1(t)) + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) = x_2(t)g_2(t, x_2(t)) + d_2(t)(x_1(t) - x_2(t)). \end{cases} \tag{2.2}$$

By Lemma 3.1 of [22], we have the following result.

Lemma 2.2. *In addition to* (A_1)-(A_3), *suppose further that* $\tilde{g}_1(0) > \hat{d}_1$ *or* $\tilde{g}_2(0) > \hat{d}_2$ *hold. Then system* (2.2) *admits a unique positive* ω*-periodic solution which is globally asymptotically stable.*

We are now in a position to state our result on the permanence of system (1.1).

Theorem 2.1. *In addition to* (A_1)-(A_4), *suppose further that the following hold*:

(A_5) $\tilde{g}_1(0) > \hat{\alpha} + \hat{d}_1$ *or* $\tilde{g}_2(0) > \hat{d}_2$,

(A_6) $\hat{r} > 0$.

Then system (1.1) *with initial condition* (1.2) *is permanent.*

Proof. For any positive solution $(x_1(t), x_2(t), x_3(t))$ of system (1.1), we get from the first two equations of system (1.1) that

$$\begin{cases} x_1'(t) \le x_1(t)g_1(t, x_1(t)) + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) \le x_2(t)g_2(t, x_2(t)) + d_2(t)(x_1(t) - x_2(t)). \end{cases}$$

By Lemma 2.2 and (A_5), the following auxiliary system

$$\begin{cases} u_1'(t) = u_1(t)g_1(t, u_1(t)) + d_1(t)(u_2(t) - u_1(t)), \\ u_2'(t) = u_2(t)g_2(t, u_2(t)) + d_2(t)(u_1(t) - u_2(t)), \end{cases} \tag{2.3}$$

admits a globally asymptotically stable positive ω-periodic solution $(\bar{u}_1(t), \bar{u}_2(t))$. Let $(u_1(t), u_2(t))$ be the solution of (2.3) with $(u_1(0), u_2(0)) = (x_1(0), x_2(0))$. It follows from the comparison theorem that, for all $t \ge 0$,

$$x_i(t) \le u_i(t), \quad i = 1, 2.$$

The global stability of $(\bar{u}_1(t), \bar{u}_2(t))$ gives that

$$\limsup_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} u_i(t) \le \bar{u}_i^u, \quad i = 1, 2.$$

Thus there exists $T_1 > 0$ such that for all $t \geq T_1$,

$$x_i(t) \leq \frac{3}{2}\bar{u}_i^u := \eta_i, \quad i = 1, 2. \tag{2.4}$$

Again from the first two equations of system (1.1), we get that

$$\begin{cases} x_1'(t) \geq x_1(t)(g_1(t, x_1(t)) - \alpha(t)) + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) \geq x_2(t)g_2(t, x_2(t)) + d_2(t)(x_1(t) - x_2(t)). \end{cases}$$

By Lemma 2.2 and (A_5), the following auxiliary system

$$\begin{cases} v_1'(t) = v_1(t)(g_1(t, v_1(t)) - \alpha(t)) + d_1(t)(v_2(t) - v_1(t)), \\ v_2'(t) = v_2(t)g_2(t, v_2(t)) + d_2(t)(v_1(t) - v_2(t)), \end{cases} \tag{2.5}$$

also admits a globally asymptotically stable positive ω-periodic solution $(\bar{v}_1(t), \bar{v}_2(t))$. Let $(v_1(t), v_2(t))$ be the solution of (2.5) with $(v_1(0), v_2(0)) = (x_1(0), x_2(0))$. It follows from the comparison theorem that, for all $t \geq 0$,

$$x_i(t) \geq v_i(t), \quad i = 1, 2.$$

The global stability of $(\bar{v}_1(t), \bar{v}_2(t))$ gives that

$$\liminf_{t\to+\infty} x_i(t) \geq \liminf_{t\to+\infty} v_i(t) \geq \bar{v}_i^l, \quad i = 1, 2.$$

Thus there exists $T_2 > 0$ such that for all $t \geq T_2$,

$$x_i(t) \geq \frac{1}{2}\bar{v}_i^l := \xi_i, \quad i = 1, 2. \tag{2.6}$$

For any positive solution $(x_1(t), x_2(t), x_3(t))$ of system (1.1), we get from the third equation of (1.1) that

$$x_3'(t) \leq r(t)x_3(t),$$

that is,

$$\frac{x_3'(t)}{x_3(t)} \leq r(t).$$

Integrating both sides of the above inequality over the interval $[t - \tau(t), t]$ leads to

$$x_3(t) \le x_3(t - \tau(t))e^{\int_{t-\tau(t)}^{t} r(s)\,ds} \le x_3(t - \tau(t))e^{\nu r^u}. \tag{2.7}$$

It follows from the third equation of (1.1), (2.4) and (2.7), that for all $t \ge T_1 + \nu$,

$$x_3'(t) \le x_3(t)\left[r(t) - \frac{c(t)}{\eta_1} e^{-\nu r^u} x_3(t)\right].$$

By Lemma 2.1 and (A_6), the following auxiliary equation

$$w'(t) = w(t)\left[r(t) - \frac{c(t)}{\eta_1} e^{-\nu r^u} w(t)\right] \tag{2.8}$$

admits a globally asymptotically stable positive ω-periodic solution $\overline{w}(t)$. Let $w(t)$ be the solution of (2.8) with $w(T_1 + \nu) = x_3(T_1 + \nu)$. It follows from the comparison theorem that, for all $t \ge T_1 + \nu$,

$$x_3(t) \le w(t).$$

The global stability of $\overline{w}(t)$ gives that

$$\limsup_{t\to+\infty} x_3(t) \le \limsup_{t\to+\infty} w(t) \le \overline{w}^u.$$

Thus there exists $T_3 > 0$ such that for all $t \ge T_3$,

$$x_3(t) \le \frac{3}{2}\overline{w}^u := \eta_3. \tag{2.9}$$

In view of the third equation of (1.1), (2.6) and (2.9), we find that, for $t \ge T_2 + T_3 + \nu$,

$$x_3'(t) \ge \left[r(t) - \frac{\eta_3}{\xi_1} c(t)\right] x_3(t),$$

that is,

$$\frac{x_3'(t)}{x_3(t)} \geq r(t) - \frac{\eta_3}{\xi_1} c(t).$$

Integrating both sides of the above inequality over the interval $[t - \tau(t), t]$ leads to

$$x_3(t) \geq x_3(t - \tau(t)) e^{\int_{t-\tau(t)}^{t} \left[r(s) - \frac{\eta_3}{\xi_1} c(s) \right] ds} \geq x_3(t - \tau(t)) e^{-\nu \left[|r|^u + \frac{\eta_3}{\xi_1} c^u \right]}. \tag{2.10}$$

It follows from the third equation of (1.1), (2.6) and (2.10) that, for all $t \geq T_2 + T_3 + \nu$,

$$x_3'(t) \geq x_3(t) \left[r(t) - \frac{c(t)}{\xi_1} e^{\nu \left[|r|^u + \frac{\eta_3}{\xi_1} c^u \right]} x_3(t) \right].$$

By Lemma 2.1 and (A_6), the following auxiliary equation

$$z'(t) = z(t) \left[r(t) - \frac{c(t)}{\xi_1} e^{\nu \left[|r|^u + \frac{\eta_3}{\xi_1} c^u \right]} z(t) \right] \tag{2.11}$$

also admits a globally asymptotically stable positive ω-periodic solution $\bar{z}(t)$. Let $w(t)$ be the solution of (2.11) with

$$z(T_2 + T_3 + \nu) = x_3(T_2 + T_3 + \nu).$$

It follows from the comparison theorem that, for all $t \geq T_2 + T_3 + \nu$,

$$x_3(t) \geq z(t).$$

The global stability of $\bar{z}(t)$ gives that

$$\liminf_{t \to +\infty} x_3(t) \geq \liminf_{t \to +\infty} z(t) \geq \bar{z}^l.$$

The proof is complete. □

By Theorem 2 of [27], we can get the following existent result about positive periodic solutions of system (1.1).

Theorem 2.2. *Assume that* (A_1)-(A_6) *hold*. *Then system* (1.1) *admits at least one positive* ω-*periodic solution*.

We remark that Theorems 2.1 and 2.2 remain valid for the following system

$$\begin{cases} x_1'(t) = x_1(t)g_1(t,\ x_1(t)) - x_3(t)p\left(t,\ \dfrac{x_1(t)}{x_3(t)}\right) + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) = x_2(t)g_2(t,\ x_2(t)) + d_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) = x_3(t)\left[r(t) - c(t)\dfrac{x_3(t)}{x_1(t-\tau(t))}\right], \end{cases} \tag{2.12}$$

and Theorem 2.2 supplements Theorem 2.1 of [16].

3. Applications

In this section, we list some applications of our above results.

Example 3.1. Consider the following Leslie-type ratio-dependent predator-prey dispersion system with Holling-type II functional response and time delay

$$\begin{cases} x_1'(t) = x_1(t)(r_1(t) - a_1(t)x_1(t)) \\ \qquad\qquad - \dfrac{b(t)x_3(t)x_1(t)}{m(t)x_3(t) + x_1(t)} + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) = x_2(t)(r_2(t) - a_2(t)x_2(t)) + d_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) = x_3(t)\left[r_3(t) - a_3(t)\dfrac{x_3(t-\tau(t))}{x_1(t-\tau(t))}\right], \end{cases} \tag{3.1}$$

where $a_1(t)$, $a_2(t)$, $a_3(t)$, $d_1(t)$, $d_2(t)$, $b(t)$, $m(t)$ and $\tau(t)$ are positive continuous ω-periodic functions, and $r_1(t)$, $r_2(t)$ and $r_3(t)$ are continuous ω-periodic functions. Let

$$g_1(t, x) = r_1(t) - a_1(t)x, \quad g_2(t, x) = r_2(t) - a_2(t)x,$$

$$p(t, x) = \frac{b(t)x}{m(t) + x}, \quad r(t) = r_3(t), \ \ c(t) = a_3(t),$$

in system (1.1). By Theorems 2.1 and 2.2, we have the following result.

Theorem 3.1. *Suppose that*

(1) $\hat{r}_1 > \hat{d}_1 + \widehat{(b/m)}$ *or* $\hat{r}_2 > \hat{d}_2$,

(2) $\hat{r}_3 > 0$,

hold, then system (3.1) *is permanent and admits at least one positive* ω*-periodic solution*.

Example 3.2. Consider the following delayed Leslie-type ratio-dependent predator-prey dispersion system with Monod-Haldane functional response

$$\begin{cases} x_1'(t) = x_1(t)(r_1(t) - a_1(t)x_1(t)) - \dfrac{b(t)x_3^2(t)x_1(t)}{m(t)x_3^2(t) + x_1^2(t)} \\ \qquad\qquad + d_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) = x_2(t)(r_2(t) - a_2(t)x_2(t)) + d_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) = x_3(t)\left[r_3(t) - a_3(t)\dfrac{x_3(t)}{x_1(t - \tau(t))}\right], \end{cases} \tag{3.2}$$

where all functions are defined as above. Let

$$g_1(t, x) = r_1(t) - a_1(t)x, \quad g_2(t, x) = r_2(t) - a_2(t)x,$$

$$p(t, x) = \frac{b(t)x}{m(t) + x^2}, \quad r(t) = r_3(t), \ \ c(t) = a_3(t),$$

in system (1.1). By Theorems 2.1 and 2.2, we have the following result.

Theorem 3.2. *Suppose that*

(1) $\hat{r}_1 > \hat{d}_1 + \widehat{(b/m)}$ *or* $\hat{r}_2 > \hat{d}_2$,

(2) $\hat{r}_3 > 0$,

hold, then system (3.2) *is permanent and admits at least one positive* ω-*periodic solution*.

4. Conclusion

In this paper, we investigate a delayed two-species ratio-dependent Leslie-type predator-prey system with the prey dispersal in a two-patch environment. By using the comparison theorem of differential equations, we establish sufficient conditions for the permanence of this system. In the proof of our results, the monotonicity of the prey growth plays a crucial role. Therefore, our method is not adapted to the non-monotonic case for the prey growth. We leave these for our future work.

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