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FUZZY OPTIMAL PRODUCTION AND SHORTAGE QUANTITY FOR FUZZY PRODUCTION INVENTORY WITH BACKORDER

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Abstract

This study aims at presenting fuzzy optimal production Q^{**} and shortage quantity b^{**} for fuzzy production inventory with backorder when setup, holding, and shortage costs are fuzzy. For this purpose, two different fuzzy models, one of which includes crisp production and crisp shortage quantity, and the other of which involves those that are fuzzy, have been presented by making use of trapezoidal fuzzy numbers. For each model, fuzzy total cost FTC has been attained via function principle. In order to defuzzify the FTC, graded mean integration method has been used, and as to solve inequality constrained problems, extension of the Lagrangean method has been applied.

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Nomenclature

Q^{*}	Optimal production quantity
Q^{**}	Fuzzy optimal production quantity
Q	Production quantity
$\widetilde{Q}$	Fuzzy production quantity
b^{*}	Optimal shortage quantity
b^{**}	Fuzzy optimal shortage quantity
b	Shortage quantity
$\widetilde{b}$	Fuzzy shortage quantity
C_O	Setup cost
$\widetilde{C}_O$	Fuzzy setup cost
C_h	Item holding cost
$\widetilde{C}_h$	Fuzzy item holding cost
π	Item shortage cost
$\widetilde{\pi}$	Fuzzy item shortage cost
P	Production rate
D	Demand rate
$I_{\max}$	Maximum stock level

1. Introduction

In classical inventory models with backorder, orders are granted taking the production capacity into consideration, and also shortage is allowed. In Figure 1, it is shown that inventory increases in P-D rate during T_1 period, that production comes to a halt at the end of T_1, that forthcoming demands are provided from stocks, that stock decreases to zero level at the end of T_2 period, that the demands cannot be met and clients are made to buy on credit until the end of T_3 period, and that production process restarts at the beginning of T_4 period. The assumptions of the model are as follows:

- The demand is crisp and constant in each period.
- Production rate is crisp in each period.
- Shortage is allowed.
- Production rate is greater than demand rate.

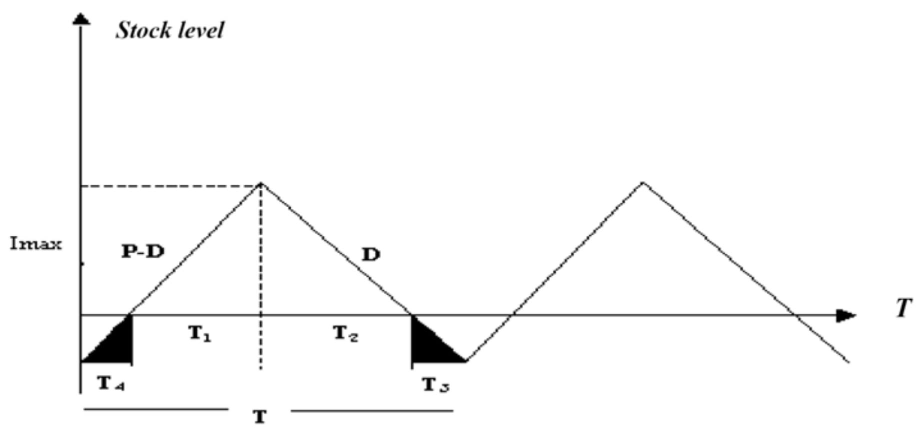

Figure 1. Classical production model with backorder.

In classical production model with backorder, all the parameters are crisp numbers. Here are the optimal production and optimal shortage quantity that minimize the total cost:

$$Q^* = \sqrt{\frac{2C_O D(Ch + \pi)}{Ch\pi\left(1 - \frac{D}{P}\right)}}, \tag{1}$$

$$b^* = \sqrt{\frac{2C_O DCh\left(1 - \frac{D}{P}\right)}{\pi(Ch + \pi)}}. \tag{2}$$

According to the traditional decision making theory, all the parameters affecting the models are either certain or those that involve vagueness are made certain via possibility theory. In accordance with the modern decision making theory, on the other hand, vagueness exists in every aspect of real world. Since the beginning of the early 1950's, uncertainties in inventory systems are fit into a model through possibility theory. Inventory control model is a part of material flow system, which is one of the subsystems of supply chain. Zarandi et al. [1] suggest that it is functional to use fuzzy set theory since all activities and their network contain a high degree of imprecision. It may not be probable to explain all uncertainties, which exist in inventory system as a subsystem of supply chain, by randomness. Therefore, as uncertainties cannot be prevented, models that take the imprecision of the environment into account produce more valid results.

In 1982, Kacpryzk [2] proposed using fuzzy set theory in inventory problems. Making use of the fuzzy set theory in 1987, Park [3] was to first to identify the term fuzzy inventory cost, and he found fuzzy optimal order quantity using extension principle. In 1996, Chen [4] presented fuzzy inventory model with backorder employing function principle. Previously, Chen [5] also introduced second function principle. Similarly in 1996, Yao and Lee asserted inventory model with backorder [6] through extension principle [7]. In 1999, Chang [8] put forward the membership function of FTC for fuzzy production model using the extension method, and attained optimal order quantity via centroid method. Additionally, Chen and Hsieh [9, 10] developed inventory models with trapezoidal fuzzy numbers. Petrovic et al. [11] studied on supply chain in fuzzy environments. In 2002, Chiang and Hsu [12] presented single-period inventory model with fuzzy demand, as Chih [13] presented single-period production model. In 2005, Dutta [14] developed single-period inventory model in an imprecise environment. In 2006, Li and Zhang [15] optimized fuzzy inventory model with backorder.

In this study, fuzzy setup, fuzzy inventory, and fuzzy shortage costs are assumed as fuzzy trapezoidal numbers; and hence, fuzzy optimal order quantity and fuzzy optimal shortage quantity are obtained. In fuzzy arithmetical operations, Chen's function principle [16], and in defuzzification of fuzzy total cost Chen's and Hsieh's graded mean method [17] were used. Besides, extension of Lagrangean method was applied in order to obtain results when production quantity and shortage quantity are fuzzy. Fuzzy production inventory with backorder was explained by an example.

2. Methodology

2.1. Fuzzy arithmetical operation under function principle

For operations with fuzzy trapezoidal numbers, Chen developed function principle [16]. Some basic arithmetical operations with function principle are shown below:

Let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ be two fuzzy trapezoidal numbers. Thus,

1. The *addition* of $\widetilde{A}$ and $\widetilde{B}$; $\widetilde{A} \oplus \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3,$ and $b_4 \in R$.

2. The *multiplication* of $\widetilde{A}$ and $\widetilde{B}$; $\widetilde{A} \otimes \widetilde{B} = (c_1, c_2, c_3, c_4)$, where $T = \{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $T_1 = \{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$; $c_1 = \min T$, $c_2 = \min T_1$, $c_3 = \max T$, $c_4 = \max T_1$. If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are positive real numbers, the multiplication of $\widetilde{A}$ and $\widetilde{B}$ is $\widetilde{A} \otimes \widetilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$.

3. The *subtraction* of $\widetilde{A}$ and $\widetilde{B}$; $-\widetilde{B} = (-b_4, -b_3, -b_2, -b_1)$, then $\widetilde{A} \ominus \widetilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in R$.

4. $1/\widetilde{B} = \widetilde{B}^{-1} = (1/b_4, 1/b_3, 1/-b_2, 1/-b_1)$, where b_1, b_2, b_3 and b_4 are positive real numbers. If $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are all nonzero positive real numbers, then the division of $\widetilde{A}$ and $\widetilde{B}$ is $\widetilde{A} \oslash \widetilde{B} = (a_1/b_4, a_2/b_3, a_3/b_2, a_4/b_1)$.

5. Let $\alpha \in R$. Then

a. $\alpha \geq 0$, $\alpha \otimes \widetilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4)$,

b. $\alpha < 0$, $\alpha \otimes \widetilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1)$.

2.2. Graded mean integration representation method

In order to defuzzify fuzzy numbers, Chen and Hsieh developed graded mean method [17] in 1999.

Let $\widetilde{A} = (a, b, c, d)$ be a fuzzy trapezoidal number as shown in Figure 2. The real numbers set is defined in R. The specifications of membership function $(\mu_{\widetilde{A}})$ provide the following:

1. $\mu_{\widetilde{A}}$ is a continuous mapping from R to the closed interval $[0, 1]$,

2. $\mu_{\widetilde{A}} = 0, -\infty < x \leq a$,

3. $\mu_{\widetilde{A}} = L(x)$ is strictly increasing on $[a, b]$,

4. $\mu_{\widetilde{A}} = w_A, b \leq x \leq c$,

5. $\mu_{\widetilde{A}} = R(x)$ is strictly decreasing on $[c, d]$,

6. $\mu_{\widetilde{A}} = 0,\ d \leq x < \infty,$ where $0 < w_A \leq 1,$ and a, b, c and d are real numbers.

Fuzzy trapezoidal numbers can generally be expressed as: $\widetilde{A} = (a, b, c, d;\, w_A)_{LR}$. When upper value of membership function $w_A = 1,$ the fuzzy trapezoidal number can be expressed more simply as $\widetilde{A} = (a, b, c, d)$.

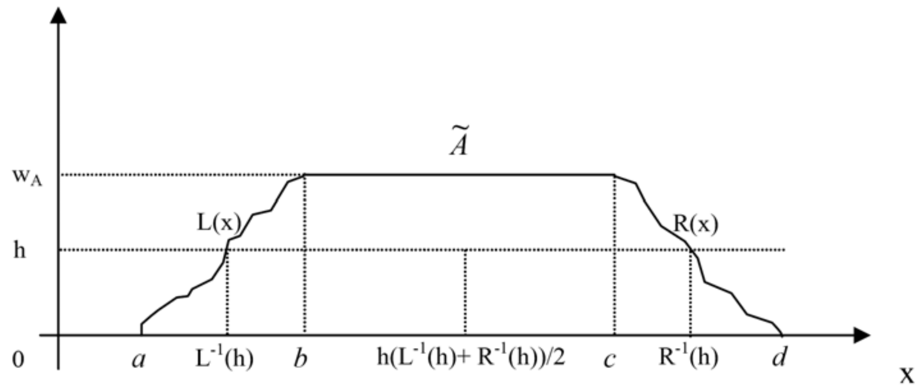

Figure 2. The graded mean of $\widetilde{A} = (a, b, c, d; w_A)_{LR}$.

In Figure 2, it is illustrated that L^{-1} and R^{-1} are inverse functions of L and R, and that the graded mean of $\widetilde{A} = (a, b, c, d; w_A)_{LR}$ is $h(L^{-1}(h) + R^{-1}(h))/2$ in h level. Consequently, the graded mean integration representation of $\widetilde{A}$ $P(\widetilde{A})$ with grade w_A,

$$P(\widetilde{A}) = \int_0^{w_A} h\left(\frac{L^{-1}(h) + R^{-1}(h)}{2}\right)dh \Big/ \int_0^{w_A} hdh, \tag{3}$$

where $0 < h \leq w_A$ and $0 < w_A \leq 1$.

Let $\widetilde{D}$ be a fuzzy trapezoidal number and be shown as $\widetilde{D} = (d_1, d_2, d_3, d_4)$. Therefore, by formula (3), the graded mean of $\widetilde{D}$ is given below:

$$P(\widetilde{D}) = \int_0^1 h\left(\frac{d_1 + d_4 + (d_2 - d_1 - d_4 + d_3)h}{2}\right)dh \Big/ \int_0^1 hdh,$$

$$= \frac{d_1 + 2d_2 + 2d_3 + d_4}{6}. \tag{4}$$

2.3. Extension of the Lagrangean method

Taha [18] presented how non-linear models involving equality constraints can be solved by Lagrangean method. This method is broadly based upon the fact that optimal solution is on bounded constraint(s).

Let the model be:

$$\text{Min } y = f(x)$$

$$g_i(x) \geq 0,\ i = 1, 2, \ldots, m$$

$$x \geq 0.$$

The steps of Lagrangean method are given below:

Step 1. Solve the unconstrained problem

$$\text{Min} = f(x).$$

If optimal solution meets all constraints, stop. All constraints are redundant. Otherwise, set $k = 1$ and go to Step 2.

Step 2. Activate any k constraints (i.e., convert them into equality) and optimize $f(x)$ subject to the k active constraints by the Lagrangean method. If the resulting solution is feasible with respect to the remaining constraints, stop; it is a local optimum. Otherwise, activate another set of k constraints and repeat the step. If all sets of active constraints taken k at a time are considered without encountering a feasible solution, go to Step 3.

Step 3. If $k = m$, stop. There is no feasible solution. If not, set $k = k + 1$ and go to Step 2.

3. Fuzzy Inventory Model with Fuzzy Setup, Fuzzy Inventory, and Fuzzy Shortage Cost

$\widetilde{C}_O = (C_{O1}, C_{O2}, C_{O3}, C_{O4})$, $\widetilde{C}_h = (C_{h1}, C_{h2}, C_{h3}, C_{h4})$, $\widetilde{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ are positive trapezoidal fuzzy numbers. $\oplus$, $\otimes$, $\ominus$ and $\oslash$ sequentially symbolize operations of fuzzy addition, fuzzy multiplication, fuzzy subtraction and fuzzy division.

The membership functions of fuzzy setup cost $\widetilde{C}_O$, fuzzy holding cost

$\widetilde{C}_h$ and fuzzy shortage cost $\widetilde{\pi}$ are as follows:

$$\mu_{\widetilde{C}_O}(C_O) = \begin{cases} 0, & C_O \le C_{O1}, \\ \dfrac{C_O - C_{O1}}{C_{O2} - C_{O1}}, & C_{O1} \le C_O \le C_{O2}, \\ 1, & C_{O2} \le C_O \le C_{O3}, \\ \dfrac{C_{O4} - C_O}{C_{O4} - C_{O3}}, & C_{O3} \le C_O \le C_{O4}, \\ 0, & C_{O4} \le C_O. \end{cases}$$

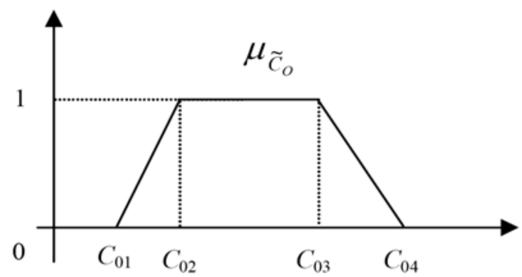

Figure 3. Fuzzy setup cost $\widetilde{C}_O$.

$$\mu_{\widetilde{C}_h}(C_h) = \begin{cases} 0, & C_h \le C_{h1}, \\ \dfrac{C_h - C_{h1}}{C_{h2} - C_{h1}}, & C_{h1} \le C_h \le C_{h2}, \\ 1, & C_{h2} \le C_h \le C_{h3}, \\ \dfrac{C_{h4} - C_h}{C_{h4} - C_{h3}}, & C_{h3} \le C_h \le C_{h4}, \\ 0, & C_{h4} \le C_h. \end{cases}$$

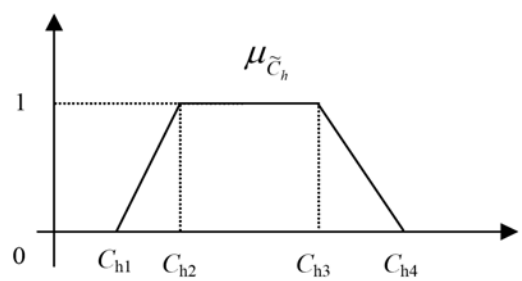

Figure 4. Fuzzy holding cost $\widetilde{C}_h$.

$$\mu_{\tilde{\pi}}(\pi) = \begin{cases} 0, & \pi \le \pi_1, \\ \dfrac{\pi - \pi_1}{\pi_2 - \pi_1}, & \pi_1 \le \pi \le \pi_2, \\ 1, & \pi_2 \le \pi \le \pi_3, \\ \dfrac{\pi_4 - \pi}{\pi_4 - \pi_3}, & \pi_3 \le \pi \le \pi_4, \\ 0, & \pi_4 \le \pi. \end{cases}$$

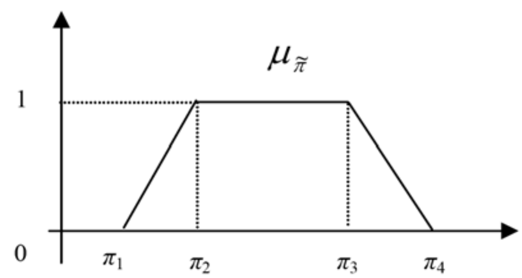

Figure 5. Fuzzy shortage cost $\tilde{\pi}$.

In this model, the annual FTC is:

$$\tilde{F} = \left[\tilde{C}_O \frac{D}{Q}\right] \oplus \left[\frac{\tilde{C}_h D}{2Q}\left[Q\left(1 - \frac{D}{P}\right) - b\right]^2 \left(\frac{1}{P - D} + \frac{1}{D}\right)\right]$$

$$\oplus \left[\tilde{\pi} \frac{Db^2}{2Q}\left(\frac{1}{P - D} + \frac{1}{D}\right)\right]. \tag{5}$$

As a result of arithmetical operations by function principle [16], FTC will be a positive fuzzy trapezoidal number as given below:

$$\tilde{F} = \left[\frac{C_{O1}D}{Q} + \frac{C_{h1}D}{2Q}\left[Q\left(1 - \frac{D}{P}\right) - b\right]^2 \left(\frac{1}{P - D} + \frac{1}{D}\right)\right.$$

$$+ \frac{\pi_1 Db^2}{2Q}\left(\frac{1}{P - D} + \frac{1}{D}\right), \frac{C_{O2}D}{Q} + \frac{C_{h2}D}{2Q}\left[Q\left(1 - \frac{D}{P}\right) - b\right]^2 \left(\frac{1}{P - D} + \frac{1}{D}\right)$$

$$+ \frac{\pi_2 Db^2}{2Q}\left(\frac{1}{P - D} + \frac{1}{D}\right), \frac{C_{O3}D}{Q} + \frac{C_{h3}D}{2Q}\left[Q\left(1 - \frac{D}{P}\right) - b\right]^2 \left(\frac{1}{P - D} + \frac{1}{D}\right)$$

$$+\frac{\pi_3 Db^2}{2Q}\left(\frac{1}{P-D}+\frac{1}{D}\right), \frac{C_{O4}D}{Q}+\frac{C_{h4}D}{2Q}\left[Q\left(1-\frac{D}{P}\right)-b\right]^2\left(\frac{1}{P-D}+\frac{1}{D}\right)$$

$$+\frac{\pi_4 Db^2}{2Q}\left(\frac{1}{P-D}+\frac{1}{D}\right)\Bigg]. \tag{6}$$

Then, when FTC function is defuzzified through graded mean method [17], graded mean of FTC will be:

$$P(\widetilde{F}_1)=\frac{1}{6}\Bigg[\frac{C_{O1}D}{Q}+\frac{C_{h1}D}{2Q}\left[Q\left(1-\frac{D}{P}\right)-b\right]^2\left(\frac{1}{P-D}+\frac{1}{D}\right)$$

$$+\frac{\pi_1 Db^2}{2Q}\left(\frac{1}{P-D}+\frac{1}{D}\right)+\frac{2C_{O2}D}{Q}$$

$$+\frac{C_{h2}D}{Q}\left[Q\left(1-\frac{D}{P}\right)-b\right]^2\left(\frac{1}{P-D}+\frac{1}{D}\right)+\frac{\pi_2 Db^2}{Q}\left(\frac{1}{P-D}+\frac{1}{D}\right)$$

$$+\frac{2C_{O3}D}{Q}+\frac{C_{h3}D}{Q}\left[Q\left(1-\frac{D}{P}\right)-b\right]^2\left(\frac{1}{P-D}+\frac{1}{D}\right)+\frac{\pi_3 Db^2}{Q}\left(\frac{1}{P-D}+\frac{1}{D}\right)$$

$$+\frac{C_{O4}D}{Q}+\frac{C_{h4}D}{2Q}\left[Q\left(1-\frac{D}{P}\right)-b\right]^2\left(\frac{1}{P-D}+\frac{1}{D}\right)$$

$$+\frac{\pi_4 Db^2}{2Q}\left(\frac{1}{P-D}+\frac{1}{D}\right)\Bigg]. \tag{7}$$

The production quantity that minimizes $P(\widetilde{F}_1)$ is optimal. If $P(\widetilde{F}_1)$ is derived with respect to Q and is equalized to zero in order to find optimal production quantity, the following equation is obtained:

$$\frac{\partial P(\widetilde{F})}{\partial Q}=\frac{1}{6}\Bigg[-(C_{O1}+2C_{O2}+2C_{O3}+C_{O4})\frac{D}{Q^2}$$

$$+ (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})\frac{\left(1 - \frac{D}{P}\right)}{2} - (C_{h1} + 2C_{h2}$$

$$\left. + 2C_{h3} + C_{h4} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)\frac{b^2}{2Q^2\left(1 - \frac{D}{P}\right)}\right] = 0. \quad (8)$$

The shortage quantity that minimizes $P(\widetilde{F}_1)$ is optimal. If $P(\widetilde{F}_1)$ is derived with respect to b and is equalized to zero in order to find optimal shortage quantity, the following equation is obtained:

$$\frac{\partial P(\widetilde{F})}{\partial Q} = \frac{1}{6}\left[-(C_{O1} + 2C_{O2} + 2C_{O3} + C_{O4})\frac{D}{Q^2}\right.$$

$$+ (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})\frac{\left(1 - \frac{D}{P}\right)}{2} - (C_{h1} + 2C_{h2} + 2C_{h3}$$

$$\left. + C_{h4} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)\frac{b^2}{2Q^2\left(1 - \frac{D}{P}\right)}\right] = 0,$$

$$b = \left[\frac{(C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})}{(C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)}\right]Q\left(1 - \frac{D}{P}\right). \quad (9)$$

When b stated in equation (9) is placed in equation (8), fuzzy optimal production quantity Q^{**} is

$$Q^{**} = \sqrt{\frac{2D(C_{O1} + 2C_{O2} + 2C_{O3} + C_{O4}) \cdot (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)P}{(C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})(\pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)(P - D)}}. \quad (10)$$

When Q^{**} stated in equation (10) is placed in equation (9), fuzzy optimal shortage quantity b^{**} is

$$b^{**} = \sqrt{\frac{2D(C_{O1} + 2C_{O2} + 2C_{O3} + C_{O4}) \cdot (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})(P - D)}{(\pi_1 + 2\pi_2 + 2\pi_3 + \pi_4) \cdot (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)P}}. \quad (11)$$

4. Fuzzy Inventory Model with Fuzzy Production and Fuzzy Shortage Quantity

In this model, in addition to the assumptions of the previous model, production quantity and shortage quantity are taken as fuzzy numbers, and hence fuzzy optimal production quantity $Q^{**}(q_1^*, q_2^*, q_3^*, q_4^*)$ and fuzzy optimal shortage quantity $b^{**}(b_1^*, b_2^*, b_3^*, b_4^*)$ are tried to be found accordingly.

$$\widetilde{Q} = (q_1, q_2, q_3, q_4), \ \widetilde{b} = (b_1, b_2, b_3, b_4), \ \widetilde{C}_O = (C_{O1}, C_{O2}, C_{O3}, C_{O4}),$$

$$\widetilde{C}_h = (C_{h1}, C_{h2}, C_{h3}, C_{h4}) \text{ and } \widetilde{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$$

are positive trapezoidal fuzzy numbers.

Subject to $0 < b_1 \le b_2 \le b_3 \le b_4 \le q_1 \le q_2 \le q_3 \le q_4$, FTC function is:

$$\widetilde{F}_2 = \left[\frac{C_{O1}D}{q_4} + \frac{C_{h1}D}{2q_4}\left[q_4\left(1 - \frac{D}{P}\right) - b_1\right]^2\left(\frac{1}{P - D} + \frac{1}{D}\right)\right.$$

$$\left. + \frac{\pi_1 D b_1^2}{2q_4}\left(\frac{1}{P - D} + \frac{1}{D}\right), \right.$$

$$\frac{C_{O2}D}{q_3} + \frac{C_{h2}D}{2q_3}\left[q_3\left(1 - \frac{D}{P}\right) - b_2\right]^2\left(\frac{1}{P - D} + \frac{1}{D}\right) + \frac{\pi_2 D b_2^2}{2q_3}\left(\frac{1}{P - D} + \frac{1}{D}\right),$$

$$\frac{C_{O3}D}{q_2} + \frac{C_{h3}D}{2q_2}\left[q_2\left(1 - \frac{D}{P}\right) - b_3\right]^2\left(\frac{1}{P - D} + \frac{1}{D}\right) + \frac{\pi_3 D b_3^2}{2q_2}\left(\frac{1}{P - D} + \frac{1}{D}\right),$$

$$\frac{C_{O4}D}{q_1} + \frac{C_{h4}D}{2q_1}\left[q_1\left(1 - \frac{D}{P}\right) - b_4\right]^2\left(\frac{1}{P-D} + \frac{1}{D}\right)$$

$$+ \frac{\pi_4 D b_4^2}{2q_1}\left(\frac{1}{P-D} + \frac{1}{D}\right)\Bigg]. \tag{12}$$

Then, when FTC function is defuzzified through graded mean method [17], graded mean of FTC will be:

$$P(\widetilde{F}_2) = \frac{1}{6}\left[\frac{C_{O1}D}{q_4} + \frac{C_{h1}D}{2q_4}\left[q_4\left(1 - \frac{D}{P}\right) - b_1\right]^2\left(\frac{1}{P-D} + \frac{1}{D}\right)\right.$$

$$+ \frac{\pi_1 D b_1^2}{2q_4}\left(\frac{1}{P-D} + \frac{1}{D}\right) + \frac{2C_{O2}D}{q_3}$$

$$+ \frac{C_{h2}D}{q_3}\left[q_3\left(1 - \frac{D}{P}\right) - b_2\right]^2\left(\frac{1}{P-D} + \frac{1}{D}\right) + \frac{\pi_2 D b_2^2}{q_3}\left(\frac{1}{P-D} + \frac{1}{D}\right)$$

$$+ \frac{2C_{O3}D}{q_2} + \frac{C_{h3}D}{q_2}\left[q_2\left(1 - \frac{D}{P}\right) - b_3\right]^2\left(\frac{1}{P-D} + \frac{1}{D}\right)$$

$$+ \frac{\pi_3 D b_3^2}{q_2}\left(\frac{1}{P-D} + \frac{1}{D}\right) + \frac{C_{O4}D}{q_1} + \frac{C_{h4}D}{2q_1}\left[q_1\left(1 - \frac{D}{P}\right) - b_4\right]^2$$

$$\left.\cdot\left(\frac{1}{P-D} + \frac{1}{D}\right) + \frac{\pi_4 D b_4^2}{2q_1}\left(\frac{1}{P-D} + \frac{1}{D}\right)\right]. \tag{13}$$

$0 < b_1 \leq b_2 \leq b_3 \leq b_4 \leq q_1 \leq q_2 \leq q_3 \leq q_4$ inequality can be converted into: $b_2 - b_1 \geq 0$, $b_3 - b_2 \geq 0$, $b_4 - b_3 \geq 0$, $q_1 - b_4 > 0$, $q_2 - q_1 \geq 0$, $q_3 - q_2 \geq 0$, $q_4 - q_3 \geq 0$ and $b_1 > 0$.

Finding b_1, b_2, b_3, b_4, q_1, q_2, q_3, q_4 that minimize the $P(\widetilde{F}_2)$ by Lagrangean method [18] is explained below:

Unconstrained problem is solved. When the function is derived with respect to q_1, q_2, q_3, q_4 and is equalized to zero in order to find the production quantity that minimizes $P(\widetilde{F}_2)$, the following equalities hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h4}+\pi_4)b_4^2}{2q_1^2\left(1-\frac{D}{P}\right)} = 0, \tag{14}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1-\frac{D}{P}\right) - \frac{(C_{h3}+\pi_3)b_3^2}{q_2^2\left(1-\frac{D}{P}\right)} = 0, \tag{15}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1-\frac{D}{P}\right) - \frac{(C_{h2}+\pi_2)b_2^2}{q_3^2\left(1-\frac{D}{P}\right)} = 0, \tag{16}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h1}+\pi_1)b_1^2}{2q_4^2\left(1-\frac{D}{P}\right)} = 0. \tag{17}$$

When the function is derived with respect to b_1, b_2, b_3, b_4 and is equalized to zero in order to find the production quantity that minimizes $P(\widetilde{F}_2)$, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } b_1 = \frac{C_{h1}q_4\left(1-\frac{D}{P}\right)}{C_{h1}+\pi_1}, \tag{18}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } b_2 = \frac{C_{h2}q_3\left(1-\frac{D}{P}\right)}{C_{h2}+\pi_2}, \tag{19}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } b_3 = \frac{C_{h3}q_2\left(1-\frac{D}{P}\right)}{C_{h3}+\pi_3}, \tag{20}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } b_4 = \frac{C_{h4}q_1\left(1-\frac{D}{P}\right)}{C_{h4}+\pi_4}. \tag{21}$$

When b_4 in equation (21) is placed in equation (14),

$$q_1 = \sqrt{\frac{2C_{O4}D(C_{h4} + \pi_4)}{C_{h4}\pi_4\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{22}$$

When b_3 in equation (20) is placed in equation (15),

$$q_2 = \sqrt{\frac{2C_{O3}D(C_{h3} + \pi_3)}{C_{h3}\pi_3\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{23}$$

When b_2 in equation (19) is placed in equation (16),

$$q_3 = \sqrt{\frac{2C_{O2}D(C_{h2} + \pi_2)}{C_{h2}\pi_2\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{24}$$

When b_1 in equation (18) is placed in equation (17),

$$q_4 = \sqrt{\frac{2C_{O1}D(C_{h1} + \pi_1)}{C_{h1}\pi_1\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{25}$$

When q_4 in equation (25) is placed in equation (18),

$$b_1 = \sqrt{\frac{2C_{O1}DC_{h1}(P - D)}{\pi_1(C_{h1} + \pi_1)P}} \text{ holds.} \tag{26}$$

When q_3 in equation (24) is placed in equation (19),

$$b_2 = \sqrt{\frac{2C_{O2}DC_{h2}(P - D)}{\pi_2(C_{h2} + \pi_2)P}} \text{ holds.} \tag{27}$$

When q_2 in equation (23) is placed in equation (20),

$$b_3 = \sqrt{\frac{2C_{O3}DC_{h3}(P - D)}{\pi_3(C_{h3} + \pi_3)P}} \text{ holds.} \tag{28}$$

When q_1 in equation (22) is placed in equation (21),

$$b_4 = \sqrt{\frac{2C_{O4}DC_{h4}(P-D)}{\pi_4(C_{h4}+\pi_4)P}} \text{ holds.} \tag{29}$$

These solutions show the possibility of $0 > b_1 > b_2 > b_3 > b_4 > q_1 > q_2 > q_3 > q_4$ and they do not satisfy the $0 < b_1 \le b_2 \le b_3 \le b_4 \le q_1 \le q_2 \le q_3 \le q_4$ constraint. $P(\widetilde{F}_2)$ has to be optimized subject to $q_2 - q_1 = 0$. Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda) = P(\widetilde{F}_2) - \lambda(q_2 - q_1).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda)$ is derived with respect to q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4 and λ, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h4}+\pi_4)b_4^2}{2q_1^2\left(1-\frac{D}{P}\right)} + \lambda = 0, \tag{30}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1-\frac{D}{P}\right) - \frac{(C_{h3}+\pi_3)b_3^2}{q_2^2\left(1-\frac{D}{P}\right)} - \lambda = 0, \tag{31}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1-\frac{D}{P}\right) - \frac{(C_{h2}+\pi_2)b_2^2}{q_3^2\left(1-\frac{D}{P}\right)} = 0, \tag{32}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h1}+\pi_1)b_1^2}{2q_4^2\left(1-\frac{D}{P}\right)} = 0, \tag{33}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } b_1 = \frac{C_{h1}q_4\left(1-\frac{D}{P}\right)}{C_{h1}+\pi_1}, \tag{34}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } b_2 = \frac{C_{h2} q_3 \left(1 - \frac{D}{P}\right)}{C_{h2} + \pi_2}, \tag{35}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } b_3 = \frac{C_{h3} q_2 \left(1 - \frac{D}{P}\right)}{C_{h3} + \pi_3}, \tag{36}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } b_4 = \frac{C_{h4} q_1 \left(1 - \frac{D}{P}\right)}{C_{h4} + \pi_4}, \tag{37}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda} = 0, \text{ then } q_1 = q_2. \tag{38}$$

When b_4 in equation (37) is placed in equation (30),

$$-\frac{C_{O4} D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{C_{h4}^2 \left(1 - \frac{D}{P}\right)}{2(C_{h4} + \pi_4)} + \lambda = 0 \text{ holds.} \tag{39}$$

When b_3 in equation (36) is placed in equation (31),

$$-\frac{2C_{O3} D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{C_{h3}^2 \left(1 - \frac{D}{P}\right)}{(C_{h3} + \pi_3)} - \lambda = 0 \text{ holds.} \tag{40}$$

If equations (39) and (40) are added and rearranged,

$$q_1 = q_2 = \sqrt{\frac{2D(2C_{O3} + C_{O4})(2C_{h3} + C_{h4} + 2\pi_3 + \pi_4)}{(2C_{h3} + C_{h4})(2\pi_3 + \pi_4)\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{41}$$

When b_2 in equation (35) is placed in equation (32),

$$q_3 = \sqrt{\frac{2DC_{O2}(C_{h2} + \pi_2)}{C_{h2}\pi_2\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{42}$$

When b_1 in equation (34) is placed in equation (33),

$$q_4 = \sqrt{\frac{2DC_{O1}(C_{h1} + \pi_1)}{C_{h1}\pi_1\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{43}$$

These solutions show the possibility of $q_1 = q_2 > q_3 > q_4$ and they do not satisfy the $q_1 \leq q_2 \leq q_3 \leq q_4$ constraint. $P(\widetilde{F}_2)$ has to be optimized subject to $q_2 - q_1 = 0$ and $q_3 - q_2 = 0$. The new Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2)$$

$$= P(\widetilde{F}_2) - \lambda_1(q_2 - q_1) - \lambda_2(q_3 - q_2).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2)$ is derived with respect to q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, λ_1 and λ_2, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h4} + \pi_4)b_4^2}{2q_1^2\left(1 - \frac{D}{P}\right)} + \lambda_1 = 0, \tag{44}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{(C_{h3} + \pi_3)b_3^2}{q_2^2\left(1 - \frac{D}{P}\right)} - \lambda_1 + \lambda_2 = 0, \tag{45}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1 - \frac{D}{P}\right) - \frac{(C_{h2} + \pi_2)b_2^2}{q_3^2\left(1 - \frac{D}{P}\right)} - \lambda_2 = 0, \tag{46}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h1} + \pi_1)b_1^2}{2q_4^2\left(1 - \frac{D}{P}\right)} = 0, \tag{47}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } b_1 = \frac{C_{h1}q_4\left(1 - \frac{D}{P}\right)}{C_{h1} + \pi_1}, \tag{48}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } b_2 = \frac{C_{h2} q_3 \left(1 - \frac{D}{P}\right)}{C_{h2} + \pi_2}, \tag{49}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } b_3 = \frac{C_{h3} q_2 \left(1 - \frac{D}{P}\right)}{C_{h3} + \pi_3}, \tag{50}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } b_4 = \frac{C_{h4} q_1 \left(1 - \frac{D}{P}\right)}{C_{h4} + \pi_4}, \tag{51}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_1} = 0, \text{ then } q_1 = q_2, \tag{52}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_2} = 0, \text{ then } q_2 = q_3. \tag{53}$$

When b_4 in equation (51) is placed in equation (44),

$$-\frac{C_{O4} D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{C_{h4}^2\left(1 - \frac{D}{P}\right)}{2(C_{h4} + \pi_4)} + \lambda_1 = 0 \text{ holds.} \tag{54}$$

When b_3 in equation (50) is placed in equation (45),

$$-\frac{2C_{O3} D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{C_{h3}^2\left(1 - \frac{D}{P}\right)}{(C_{h3} + \pi_3)} - \lambda_1 + \lambda_2 = 0 \text{ holds.} \tag{55}$$

When b_2 in equation (48) is placed in equation (47),

$$-\frac{2C_{O2} D}{q_3^2} + C_{h2}\left(1 - \frac{D}{P}\right) - \frac{C_{h2}^2\left(1 - \frac{D}{P}\right)}{(C_{h2} + \pi_2)} - \lambda_2 = 0 \text{ holds.} \tag{56}$$

If equations (54), (55) and (56) are added and rearranged,

$$q_1 = q_2 = q_3 = \sqrt{\frac{2D(2C_{O2} + 2C_{O3} + C_{O4}) \cdot (2C_{h2} + 2C_{h3} + C_{h4} + 2\pi_2 + 2\pi_3 + \pi_4)}{(2C_{h2} + 2C_{h3} + C_{h4})(2\pi_2 + 2\pi_3 + \pi_4)\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{57}$$

When b_1 in equation (49) is placed in equation (48),

$$q_4 = \sqrt{\frac{2DC_{O1}(C_{h1} + \pi_1)}{C_{h1}\pi_1\left(1 - \frac{D}{P}\right)}} \text{ holds.} \tag{58}$$

These solutions show the possibility of $q_1 = q_2 = q_3 > q_4$ and they do not satisfy the $q_1 \leq q_2 \leq q_3 \leq q_4$ constraint. $P(\widetilde{F}_2)$ has to be optimized subject to $q_2 - q_1 = 0$, $q_3 - q_2 = 0$ and $q_4 - q_3 = 0$. The new Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2, \lambda_3)$$
$$= P(\widetilde{F}_2) - \lambda_1(q_2 - q_1) - \lambda_2(q_3 - q_2) - \lambda_3(q_4 - q_3).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2, \lambda_3)$ is derived with respect to q_1, $q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2$ and λ_3, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h4} + \pi_4)b_4^2}{2q_1^2\left(1 - \frac{D}{P}\right)} + \lambda_1 = 0, \tag{59}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{(C_{h3} + \pi_3)b_3^2}{q_2^2\left(1 - \frac{D}{P}\right)} - \lambda_1 + \lambda_2 = 0, \tag{60}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1 - \frac{D}{P}\right) - \frac{(C_{h2} + \pi_2)b_2^2}{q_3^2\left(1 - \frac{D}{P}\right)} - \lambda_2 + \lambda_3 = 0, \tag{61}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h1} + \pi_1)b_1^2}{2q_4^2\left(1 - \frac{D}{P}\right)} - \lambda_3 = 0, \tag{62}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } b_1 = \frac{C_{h1}q_4\left(1 - \frac{D}{P}\right)}{C_{h1} + \pi_1}, \tag{63}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } b_2 = \frac{C_{h2} q_3 \left(1 - \frac{D}{P}\right)}{C_{h2} + \pi_2}, \tag{64}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } b_3 = \frac{C_{h3} q_2 \left(1 - \frac{D}{P}\right)}{C_{h3} + \pi_3}, \tag{65}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } b_4 = \frac{C_{h4} q_1 \left(1 - \frac{D}{P}\right)}{C_{h4} + \pi_4}, \tag{66}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_1} = 0, \text{ then } q_1 = q_2, \tag{67}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_2} = 0, \text{ then } q_2 = q_3, \tag{68}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_3} = 0, \text{ then } q_3 = q_4. \tag{69}$$

When b_4 in equation (66) is placed in equation (59),

$$-\frac{C_{O4} D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{C_{h4}^2 \left(1 - \frac{D}{P}\right)}{2(C_{h4} + \pi_4)} + \lambda_1 = 0 \text{ holds.} \tag{70}$$

When b_3 in equation (65) is placed in equation (60),

$$-\frac{2C_{O3} D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{C_{h3}^2 \left(1 - \frac{D}{P}\right)}{(C_{h3} + \pi_3)} - \lambda_1 + \lambda_2 = 0 \text{ holds.} \tag{71}$$

When b_2 in equation (64) is placed in equation (61),

$$-\frac{2C_{O2} D}{q_3^2} + C_{h2}\left(1 - \frac{D}{P}\right) - \frac{C_{h2}^2 \left(1 - \frac{D}{P}\right)}{(C_{h2} + \pi_2)} - \lambda_2 + \lambda_3 = 0 \text{ holds.} \tag{72}$$

When b_1 in equation (63) is placed in equation (62),

$$-\frac{C_{O1}D}{q_4^2}+\frac{C_{h1}\left(1-\frac{D}{P}\right)}{2}-\frac{C_{h1}^2\left(1-\frac{D}{P}\right)}{2(C_{h1}+\pi_1)}-\lambda_3=0 \text{ holds.} \tag{73}$$

If equations (70), (71), (72) and (73) are added and rearranged, $Q^{**}(q_1^*, q_2^*, q_3^*, q_4^*)$ holds as follows:

$$q_1^* = q_2^* = q_3^* = q_4^* = Q^{**}$$

$$= \sqrt{\frac{\begin{array}{l}2D(C_{O1}+2C_{O2}+2C_{O3}+C_{O4})\\ \quad\cdot(C_{h1}+2C_{h2}+2C_{h3}+C_{h4}+\pi_1+2\pi_2+2\pi_3+\pi_4)P\end{array}}{(C_{h1}+2C_{h2}+2C_{h3}+C_{h4})(\pi_1+2\pi_2+2\pi_3+\pi_4)(P-D)}}. \tag{74}$$

It is concluded that the feasible solution converges in a single point. Additionally, shortage quantity that minimizes $P(\widetilde{F}_2)$ has to be found. b_1, b_2, b_3 and b_4 in equations (26), (27), (28) and (29) do not satisfy $b_1 \le b_2 \le b_3 \le b_4$ constraint. For this reason, $P(\widetilde{F}_2)$ has to be optimized subject to $b_2 - b_1 = 0$ constraint. The Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda) = P(\widetilde{F}_2) - \lambda(b_2 - b_1).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda)$ is derived with respect to $q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4$ and λ, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2}+\frac{C_{h4}\left(1-\frac{D}{P}\right)}{2}-\frac{(C_{h4}+\pi_4)b_4^2}{2q_1^2\left(1-\frac{D}{P}\right)}=0, \tag{75}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2}+C_{h3}\left(1-\frac{D}{P}\right)-\frac{(C_{h3}+\pi_3)b_3^2}{q_2^2\left(1-\frac{D}{P}\right)}=0, \tag{76}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1-\frac{D}{P}\right) - \frac{(C_{h2}+\pi_2)b_2^2}{q_3^2\left(1-\frac{D}{P}\right)} = 0, \tag{77}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h1}+\pi_1)b_1^2}{2q_4^2\left(1-\frac{D}{P}\right)} = 0, \tag{78}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } -C_{h1} + \frac{C_{h1}b_1}{q_4\left(1-\frac{D}{P}\right)} + \frac{\pi_1 b_1}{q_4\left(1-\frac{D}{P}\right)} + \lambda_1 = 0, \tag{79}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } -2C_{h2} + \frac{2C_{h2}b_2}{q_3\left(1-\frac{D}{P}\right)} + \frac{\pi_2 b_2}{q_3\left(1-\frac{D}{P}\right)} - \lambda_1 = 0, \tag{80}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } -2C_{h3} + \frac{2C_{h3}b_3}{q_2\left(1-\frac{D}{P}\right)} + \frac{2\pi_3 b_3}{q_2\left(1-\frac{D}{P}\right)} = 0, \tag{81}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } -C_{h4} + \frac{2C_{h4}b_4}{q_1\left(1-\frac{D}{P}\right)} + \frac{\pi_4 b_4}{q_1\left(1-\frac{D}{P}\right)} = 0, \tag{82}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_1} = 0, \text{ then } b_1 = b_2. \tag{83}$$

When q_4 in equation (78) is placed in equation (79), q_3 in equation (77) is placed in equation (80), and these two obtained equations added and rearranged,

$$b_1 = b_2 = \sqrt{\frac{2(C_{O1}+2C_{O2})D(C_{h1}+2C_{h2})(P-D)}{(\pi_1+2\pi_2)(C_{h1}+2C_{h2}+\pi_1+2\pi_2)P}} \text{ holds.} \tag{84}$$

When q_2 in equation (76) is placed in equation (81),

$$b_3 = \sqrt{\frac{2C_{O3}DC_{h3}(P-D)}{\pi_3(C_{h3}+\pi_3)P}} \text{ holds.} \tag{85}$$

When q_1 in equation (75) is placed in equation (82),

$$b_4 = \sqrt{\frac{2C_{O4}DC_{h4}(P-D)}{\pi_4(C_{h4}+\pi_4)P}} \text{ holds.} \tag{86}$$

These solutions show the possibility of $b_1 = b_2 > b_3 > b_4$ and they do not satisfy the $b_1 \le b_2 \le b_3 \le b_4$ constraint. $P(\widetilde{F}_2)$ has to be optimized subject to $b_2 - b_1 = 0$ and $b_3 - b_2 = 0$. The new Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2)$$

$$= P(\widetilde{F}_2) - \lambda_1(b_2 - b_1) - \lambda_2(b_3 - b_2).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2)$ is derived with respect to $q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1$ and λ_2, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h4}+\pi_4)b_4^2}{2q_1^2\left(1-\frac{D}{P}\right)} = 0, \tag{87}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1-\frac{D}{P}\right) - \frac{(C_{h3}+\pi_3)b_3^2}{q_2^2\left(1-\frac{D}{P}\right)} = 0, \tag{88}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1-\frac{D}{P}\right) - \frac{(C_{h2}+\pi_2)b_2^2}{q_3^2\left(1-\frac{D}{P}\right)} = 0, \tag{89}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1-\frac{D}{P}\right)}{2} - \frac{(C_{h1}+\pi_1)b_1^2}{2q_4^2\left(1-\frac{D}{P}\right)} = 0, \tag{90}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } -C_{h1} + \frac{C_{h1}b_1}{q_4\left(1-\frac{D}{P}\right)} + \frac{\pi_1 b_1}{q_4\left(1-\frac{D}{P}\right)} + \lambda_1 = 0, \tag{91}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } -2C_{h2} + \frac{2C_{h2}b_2}{q_3\left(1-\frac{D}{P}\right)} + \frac{\pi_2 b_2}{q_3\left(1-\frac{D}{P}\right)} - \lambda_1 + \lambda_2 = 0, \quad (92)$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } -2C_{h3} + \frac{2C_{h3}b_3}{q_2\left(1-\frac{D}{P}\right)} + \frac{2\pi_3 b_3}{q_2\left(1-\frac{D}{P}\right)} - \lambda_2 = 0, \quad (93)$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } -C_{h4} + \frac{2C_{h4}b_4}{q_1\left(1-\frac{D}{P}\right)} + \frac{\pi_4 b_4}{q_1\left(1-\frac{D}{P}\right)} = 0, \quad (94)$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_1} = 0, \text{ then } b_1 = b_2, \quad (95)$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_2} = 0, \text{ then } b_2 = b_3. \quad (96)$$

When q_4 in equation (90) is placed in equation (91), q_3 in equation (89) is placed in equation (92), and q_2 in equation (88) is placed in equation (93) and these three obtained equations added and rearranged,

$$b_1 = b_2 = b_3 = \sqrt{\frac{2(C_{O1} + 2C_{O2} + 2C_{O3})D(C_{h1} + 2C_{h2} + 2C_{h3})(P-D)}{(\pi_1 + 2\pi_2 + 2\pi_3)(C_{h1} + 2C_{h2} + 2C_{h3} + \pi_1 + 2\pi_2 + 2\pi_3)P}} \text{ holds.} \quad (97)$$

When q_1 in equation (87) is placed in equation (94),

$$b_4 = \sqrt{\frac{2C_{O4}DC_{h4}(P-D)}{\pi_4(C_{h4} + \pi_4)P}} \text{ holds.} \quad (98)$$

These solutions show the possibility of $b_1 = b_2 = b_3 > b_4$ and they do not satisfy the $b_1 \le b_2 \le b_3 \le b_4$ constraint. $P(\widetilde{F}_2)$ has to be optimized subject to $b_2 - b_1 = 0,$ $b_3 - b_2 = 0$ and $b_4 - b_3 = 0.$ The new Lagrangean function is as follows:

$$L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2, \lambda_3)$$

$$= P(\widetilde{F}_2) - \lambda_1(b_2 - b_1) - \lambda_2(b_3 - b_2) - \lambda_3(b_4 - b_3).$$

If $L(q_1, q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2, \lambda_3)$ is derived with respect to q_1, $q_2, q_3, q_4, b_1, b_2, b_3, b_4, \lambda_1, \lambda_2$ and λ_3, and is equalized to zero in order to be minimized, the following equations hold:

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_1} = -\frac{C_{O4}D}{q_1^2} + \frac{C_{h4}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h4} + \pi_4)b_4^2}{2q_1^2\left(1 - \frac{D}{P}\right)} = 0, \tag{99}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_2} = -\frac{2C_{O3}D}{q_2^2} + C_{h3}\left(1 - \frac{D}{P}\right) - \frac{(C_{h3} + \pi_3)b_3^2}{q_2^2\left(1 - \frac{D}{P}\right)} = 0, \tag{100}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_3} = -\frac{2C_{O2}D}{q_3^2} + C_{h2}\left(1 - \frac{D}{P}\right) - \frac{(C_{h2} + \pi_2)b_2^2}{q_3^2\left(1 - \frac{D}{P}\right)} = 0, \tag{101}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial q_4} = -\frac{C_{O1}D}{q_4^2} + \frac{C_{h1}\left(1 - \frac{D}{P}\right)}{2} - \frac{(C_{h1} + \pi_1)b_1^2}{2q_4^2\left(1 - \frac{D}{P}\right)} = 0, \tag{102}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_1} = 0, \text{ then } -C_{h1} + \frac{C_{h1}b_1}{q_4\left(1 - \frac{D}{P}\right)} + \frac{\pi_1 b_1}{q_4\left(1 - \frac{D}{P}\right)} + \lambda_1 = 0, \tag{103}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_2} = 0, \text{ then } -2C_{h2} + \frac{2C_{h2}b_2}{q_3\left(1 - \frac{D}{P}\right)} + \frac{\pi_2 b_2}{q_3\left(1 - \frac{D}{P}\right)} - \lambda_1 + \lambda_2 = 0, \tag{104}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_3} = 0, \text{ then } -2C_{h3} + \frac{2C_{h3}b_3}{q_2\left(1 - \frac{D}{P}\right)} + \frac{2\pi_3 b_3}{q_2\left(1 - \frac{D}{P}\right)} - \lambda_2 + \lambda_3 = 0, \tag{105}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial b_4} = 0, \text{ then } -C_{h4} + \frac{2C_{h4}b_4}{q_1\left(1 - \frac{D}{P}\right)} + \frac{\pi_4 b_4}{q_1\left(1 - \frac{D}{P}\right)} - \lambda_3 = 0, \tag{106}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_1} = 0, \text{ then } b_1 = b_2, \tag{107}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_2} = 0, \text{ then } b_2 = b_3, \tag{108}$$

$$\frac{\partial P(\widetilde{F}_2)}{\partial \lambda_3} = 0, \text{ then } b_3 = b_4. \tag{109}$$

When q_4 that provides equation (102) is placed in equation (103), q_3 that provides equation (101) is placed in equation (104), q_2 that provides equation (100) is placed in equation (105), q_1 that provides equation (99) is placed in equation (106), and these four obtained equations are added to each other and reorganized, fuzzy optimal shortage quantity $b^{**}(b_1^*, b_2^*, b_3^*, b_4^*)$ holds as follows:

$$b_1^* = b_2^* = b_3^* = b_4^* = b^{**}$$

$$= \sqrt{\frac{\begin{array}{c}2D(C_{O1} + 2C_{O2} + 2C_{O3} + C_{O4}) \\ \cdot (C_{h1} + 2C_{h2} + 2C_{h3} + C_{h4})(P - D)\end{array}}{\begin{array}{c}(\pi_1 + 2\pi_2 + 2\pi_3 + \pi_4) \\ \cdot (C_{h1} + 2C_{h2} + 2C_{h3} + \pi_1 + 2\pi_2 + 2\pi_3 + \pi_4)P\end{array}}}. \tag{110}$$

It is concluded that the optimal result occurs at a single point. Moreover, it has been observed that Q^{**} and b^{**} are the same.

5. Example

In this example, an item whose annual demand is 3650 and whose daily production rate is 20 is given. As shown in Table I, the item's inventory costs are assumed crisp and fuzzy. In this regard, Q^*, Q^{**}, b^*, b^{**} are calculated and given in Table II, and so are total cost (TC) and fuzzy total cost (FTC) in Table III.

Table I. Crisp and fuzzy inventory costs

PLAN 1 Crisp inventory costs		PLAN 2 Fuzzy inventory costs	
C_O (\$)	1000	$\widetilde{C}_O$ (\$)	(900, 950, 1100, 1200)
C_h (\$/item year)	12	$\widetilde{C}_h$ (\$/item year)	(8, 9, 11, 13)
π (\$/item year)	55	$\widetilde{\pi}$ (\$/item year)	(40, 45, 55, 60)

Table II. Optimal production and shortage quantities

PLAN 1		PLAN 2	
Q^*	1217, 42	Q^{**}	1336, 28
b^*	109, 02	b^{**}	112, 90

Table III. Crisp and fuzzy total costs

	Total setup cost (\$)	Total holding cost (\$/item year)	Total shortage cost (\$/item year)	Annual total cost (\$)	Graded annual total cost (\$)
	C_O; $\widetilde{C}_O(\widetilde{C}_{O1}, \widetilde{C}_{O2}, \widetilde{C}_{O3}, \widetilde{C}_{O4})$	C_h; $\widetilde{C}_h(\widetilde{C}_{h1}, \widetilde{C}_{h2}, \widetilde{C}_{h3}, \widetilde{C}_{h4})$	π; $\widetilde{\pi}(\widetilde{\pi}_1, \widetilde{\pi}_2, \widetilde{\pi}_3, \widetilde{\pi}_4)$		
PLAN 1	2998, 14	2461, 17	536, 95	5.996, 26	**5.996, 26**
PLAN 2	(2698.33, 2848.24, 3297.95, 3597.77)	(1640.78, 1845.88, 2256.08, 2666.27)	(390.51, 439.32, 536.95, 585.76)	(4729.62, 5133.44, 6090.98, 6849.80)	**5.671, 38**

6. Conclusion

Decision makers may not be sure of the factors that affect their decisions. In such cases, it is practical and useful to make use of basics and principles of fuzzy logic. In this study, two cases in which fuzzy logic is applied in inventory models that are subsystems of supply chain are presented. The same results are obtained in both cases. Additionally, an example is given in order to clarify the models. Consequently, it has been observed that outputs, attained through models with fuzzy parameters instead of those that are crisp, are closer to real-world since the models include vagueness.

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