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ON THE REDUCTION OF SOME TILED ORDERS

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Abstract

In this paper, a Zavadskij's differentiation algorithm is used to reduce some tiled orders to $(0, 1)$-tiled orders.

1. Introduction

A tiled order over a discrete valuation ring is a Noetherian prime semiperfect semidistributive ring Λ with nonzero Jacobson radical. One of the main problems regarding this kind of rings (i.e., of the integral representation theory) consists of determining the additive category latt Λ of all right Λ-modules, its representation type and the corresponding Auslander-

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Reiten quiver $\Gamma(\Lambda)$. The tiled order is said to be of *finite lattice type* if latt Λ has only finitely many isomorphism classes of indecomposable objects [1-3].

Classification problems of tiled orders can be tackled by using poset representation theory introduced by Nazarova and Roiter in 1972. Given an algebraically closed field k and a poset $\mathcal{P}$, an object U of the additive category rep $\mathcal{P}$ of k-linear representations of $\mathcal{P}$ is a system of k-vector spaces of the form:

$$U = (U_0; U_k \mid x \in \mathcal{P}),$$

where $U_x \subseteq U_y$ provided $x \leq y$.

The main tools in poset representation theory are the algorithms of differentiation, for instance the algorithm of differentiation with respect to a maximal point introduced by Nazarova and Roiter in [5] allowed to Kleiner to classify posets of finite type representation type in [4]. Furthermore, posets of finite growth representation type were classified in [6] by using the algorithm of differentiation with respect to a suitable pair of points *DI* introduced by Zavadskij in [11], defined in such a way that if a poset $(\mathcal{P}, \leq)$ with:

$$\mathcal{P} = a^{\nabla} + b_{\Delta} + C,$$

where $a^{\nabla} = \{x \in \mathcal{P} \mid a \leq x\}$, $b_{\Delta} = \{x \in \mathcal{P} \mid x \leq b\}$ and $C = c_1 < c_2 < \cdots < c_n$ is a chain (eventually empty), then the derived poset $\mathcal{P}'_{(a,b)}$ with respect to the pair of points (a, b) is a subposet of the modular lattice generated by $\mathcal{P}$ such that:

$$\mathcal{P}'_{(a,b)} = a^{\nabla} + C^{-} + C^{+} + b_{\Delta},$$

where $C^{+} = \{c_i^{+} \mid c_i^{+} = a + c_i\}$ and $C^{-} = \{c_i^{-} \mid c_i^{-} = bc_i\}$ are chains with $c_i^{-} < c_i^{+}$ for all $1 \leq i \leq n$.

The derivation functor between the corresponding categories of representations $D_{(a,b)} : \text{rep}\,\mathcal{P} \to \text{rep}\,\mathcal{P}'_{(a,b)}$ is defined as follows [5, 8, 11, 14]:

$U'_0 = U_0,$

$U'_{c_i^+} = U_a + U_{c_i},$

$U'_{c_i^-} = U_b \cap U_{c_i},$

$U'_x = U_x$ for the remaining points $x \in \mathcal{P},$

$\varphi' = \varphi : U_0 \to V_0$ for any linear map-morphism $\varphi \in \mathrm{Hom}_k(U, V).$ (1)

According to Rump [7], the idea of this two point algorithm arose from the Zavadskij's matrix algorithm which forms the essential tool in the characterization of representation-finite tiled orders. Further, Zavadskij and Revitskaya generalized representations of tiled orders and finite posets over a field T by introducing a mixed flat matrix problem over a pair of algebras named algebras of transformation [13]. They proved a bounded module criterion which generalizes criteria of finite representation type for posets and tiled orders. Soon afterwards, Rump generalized in [7] those results obtained by Kirichenko and Zavadskij to general orders to do that, instead of a suitable pair of points (a, b), he considered a monomorphism $u : I \hookrightarrow P$ between Λ-lattices with $kP = kI$. For such u, Rump associated to each Λ-lattice E a pair of Λ-lattices $\partial_u E = \begin{pmatrix} E^+ \\ E^- \end{pmatrix}$ such that $E_- \subset E \subset E^+,$ where E_- is the largest Λ-sublattice with $f(E_-) \subset P$ for each homomorphism $f : E \to I,$ E^+ is defined dually.

If u is such that $\partial_u P = \partial_u I = \begin{pmatrix} I \\ P \end{pmatrix},$ then Λ^+ is an overorder of Λ and $u^* : I^* \hookrightarrow P^*$ induces an overorder Λ^- of Λ. Then it is possible to define the derived order:

$$\partial_u = \begin{pmatrix} \Lambda^+ & \Lambda^+\Lambda^- \\ \Lambda_- & \Lambda^- \end{pmatrix} \subset M_2(A)$$

of Λ and $\partial_u : \mathrm{latt}\,\Lambda \to \partial_u \Lambda$ becomes a functor.

In this paper, we use an algorithm of differentiation introduced by Zavadskij and Kirichenko in [12] for tiled orders in order to prove which $(0, 1, 2)$-tiled orders can be reduced to a $(0, 1)$-tiled order.

This paper is organized as follows: In Section 2, we describe main definitions and notation to be used throughout the work. In Section 3, we give main results describing which $(0, 1, 2)$-tiled orders can be reduced to $(0, 1)$-tiled orders.

2. Preliminaries

In this section, we introduce notation and results to be used throughout the paper [1-3, 12].

A field T is said to be of *discrete norm or discrete valuation* if it is endowed with a surjective map:

$$\nu : T \to \mathbb{Z} \cup \{\infty\},$$

which satisfies the following conditions:

(a) $\nu(x) = \infty$ if and only if $x = 0$,

(b) $\nu(xy) = \nu(x) + \nu(y)$,

(c) $\nu(x + y) \leq \min\{\nu(x), \nu(y)\}$.

We let $\mathbb{O}$ denote the *normalization ring* of T such that

$$\mathbb{O} = \{x \in T \mid \nu(x) \geq 0\}.$$

An element $\pi \in \mathbb{O}$ such that $\nu(\pi) = 1$ is a *prime element* of $\mathbb{O}$. Thus, for each $x \in \mathbb{O}$, we have that:

$$x \in \mathbb{O} \text{ if and only if } x = \varepsilon\pi^m, \text{ for some } m \geq 0,$$

where ε is an unit, i.e., $\varepsilon \in \mathbb{O}^*$. Moreover,

$$x \in T \text{ if and only if } x = \varepsilon\pi^m, \text{ for some } m \in \mathbb{Z} \text{ and } \varepsilon \in \mathbb{O}^*.$$

Ring $\mathbb{O}$ is such that $\mathbb{O} \supset \pi\mathbb{O}$, where $\pi\mathbb{O}$ is the unique maximal ideal, therefore, ideals of $\mathbb{O}$ generate a chain of the form:

$$\mathbb{O} \supset \pi\mathbb{O} \supset \pi^2\mathbb{O} \supset \cdots \supset \pi^m\mathbb{O} \supset \cdots.$$

A tiled order is a subring of the matrix algebra $T^{n\times n}$ with the form:

$$\Lambda = \sum_{i,\, j=1}^{n} e_{ij}\pi^{\lambda_{ij}}\mathbb{O} = \begin{bmatrix} \mathbb{O} & \pi^{\lambda_{12}}\mathbb{O} & \cdots & \pi^{\lambda_{1n}}\mathbb{O} \\ \pi^{\lambda_{21}}\mathbb{O} & \mathbb{O} & \cdots & \pi^{\lambda_{2n}}\mathbb{O} \\ \vdots & \vdots & \vdots & \vdots \\ \pi^{\lambda_{n1}}\mathbb{O} & \pi^{\lambda_{n2}}\mathbb{O} & \cdots & \mathbb{O} \end{bmatrix}.$$

That is, Λ consists of all matrices whose entries ij belong to $\pi^{\lambda_{ij}}\mathbb{O}$, in this case, $e_{ij} \in T^{n\times n}$ are unit matrices such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ ($\delta_{jk} = 1$, if $j = k$, $\delta_{jk} = 0$ otherwise).

Numbers λ_{ij} are rational integers which satisfy the following conditions:

(1) $\lambda_{ii} = 0$, for each i,

(2) $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$ for all i, j, k.

An order Λ is said to be *Morita reduced* or *reduced* if it satisfies the additional condition:

(3) $\lambda_{ij} + \lambda_{ji} > 0$, for each $i \neq j$. In this case, projective modules are pairwise non-isomorphic, that is, in the decomposition of $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ via projective modules (i.e., the rows of Λ) all summands indecomposable projectives are pairwise not isomorphic, i.e., $P_i \not\simeq P_j$ provided $i \neq j$.

Henceforth, we will assume that tiled orders satisfy conditions (1), (2) and (3). According to Kirichenko et al., it means that the matrix Λ is an *exponent reduced matrix* [2].

We denote $\Lambda = (\lambda_{ij})_{i, j=1, \ldots, n}$, furthermore, note that $\Lambda \subset T^{n\times n} = Q = \Lambda \otimes_{\mathbb{O}} T$, where Q is the rational hull of Λ, $\operatorname{Rad} Q = 0$ and Λ has a unique simple right module (up to isomorphism) denoted $S_R = (T, T, \ldots, T) = \sum_{i=1}^{n} e_i T$, where $\{e_i \,|\, 1 \leq i \leq n\}$ is the standard basis such that $e_i e_{jk} = \delta_{ij} e_k$. We assume the notation $S_L = (T, T, \ldots, T)^t$ for left modules.

The *main problem* in this situation consists of describing all finitely generated Λ-modules without $\mathbb{O}$-torsion which are called *admissible modules*.

A Λ-admissible right module (not null) is said to be *irreducible* if it is a submodule of the unique simple module (up to isomorphism), $S_R = (T, T, \ldots, T)$. For instance, any module P_i indecomposable projective is a tiled order Λ. Thus,

$$P_i = (\pi^{\lambda_{i1}}\mathbb{O}, \pi^{\lambda_{i2}}\mathbb{O}, \ldots, \pi^{\lambda_{in}}\mathbb{O})$$

is a finitely generated irreducible module without $\mathbb{O}$-torsión. Actually, any irreducible right module A has the form:

$$A = (\pi^{\alpha_1}\mathbb{O}, \pi^{\alpha_2}\mathbb{O}, \ldots, \pi^{\alpha_n}\mathbb{O}),$$

where $\alpha_i + \lambda_{ij} \geq \alpha_j$, $\alpha_i \in \mathbb{Z}$, $1 \leq i \leq n$.

If A is a A left module, then we have that $\lambda_{ij} + \alpha_j \geq \alpha_i$. Henceforth, we write $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for a right (left) module $((\alpha_1, \alpha_2, \ldots, \alpha_n)^t$, respectively).

Note that $A \simeq A'$ if and only if $\alpha_i = \alpha_i' + k$, for some $k \in \mathbb{Z}$ and any $1 \leq i \leq n$.

Irreducible right modules which are contained in a Q-simple module of a Λ-order constitute a lattice $\mathfrak{L}(\Lambda)(\subseteq, \bigcup, \bigcap) = \mathfrak{L}(\Lambda) = \mathfrak{L}_R(\Lambda)$. The corresponding lattice of irreducible left modules $\mathfrak{L}_L(\Lambda)$ is antiisomorphic to $\mathfrak{L}_R(\Lambda)$, by the correspondence $\sigma : \mathfrak{L}_R(\Lambda) \to \mathfrak{L}_L(\Lambda)$ given by the formula:

$$\sigma(\alpha_1, \ldots, \alpha_n) = (-\alpha_1, \ldots, -\alpha_n)^t.$$

Let $\mathcal{P}(\Lambda) = \mathcal{P}_R(\Lambda)$ be the subposet of $\mathfrak{L}(\Lambda)$ of irreducible projective modules, if $\mathcal{P}(\Lambda) = \mathcal{P}_R(\Lambda)$, then projective modules P_i are called *principals* where:

$$P_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}) = P_i^0, \quad P_i^0 \in \mathcal{P}_R(\Lambda).$$

In this poset, there are so many projective modules as infinite chains. In such a case, modules of the form:

$$P_i^k = (\lambda_{i1} + k, \ldots, \lambda_{in} + k), \quad k \in \mathbb{Z}$$

are projective modules isomorphic to P_i^0. Therefore,

$$\mathcal{P}(\Lambda) = \{P_i^k \mid 1 \le i \le n,\ k \in \mathbb{Z}\},$$

where

$$P_i^k \le P_j^l \text{ if and only if } \begin{cases} k - l \ge \lambda_{ij}, & \mathcal{P}_L(\Lambda), \\ k - l \ge \lambda_{ji}, & \mathcal{P}_R(\Lambda). \end{cases}$$

Thus, the poset $\mathcal{P}(\Lambda)$ is infinite, periodic and the sum of *n* chains with the form $\{P_i^k \mid 1 \le i \le n,\ k \in \mathbb{Z}\}$, with width $w(\mathcal{P}(\Lambda)) \le n$.

The map $\sigma : \mathcal{P}_R(\Lambda) \to \mathcal{P}_L(\Lambda)$, given by $\sigma(P_i^k) = P_i^{-k}$ is a natural poset antiisomorphism, thus the pair $\{\mathbb{O}, \mathcal{P}(\Lambda)\}$ defines the tiled order Λ up to isomorphism, in the sense that

$$\Lambda \simeq \Lambda' \text{ if and only if pairs } \{\mathbb{O}, \mathcal{P}(\Lambda)\} \simeq \{\mathbb{O}', \mathcal{P}(\Lambda')\}.$$

That is, $\mathbb{O} \simeq \mathbb{O}'$ and $\mathcal{P}(\Lambda) \simeq \mathcal{P}(\Lambda')$ which means that there exists a poset isomorphism $\varphi : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda')$ such that

$$A \simeq B \text{ if and only if } \varphi(A) \simeq \varphi(B),$$

φ preserves isomorphisms, thus Λ and Λ' are Morita-equivalents.

In particular, we have the following result proved by Zavadskij in [10]:

Theorem 1. *Two orders* Λ *and* Λ' *are isomorphic if the corresponding exponent matrices* (λ_{ij}) *and* (λ'_{ij}) *can be turned into each other with the help of the following admissible t-transformations*:

(1) *To add an integer n to each entry of a given row i and simultaneously subtract n to each entry of the column i.*

(2) *To transpose simultaneously rows i and j and columns i and j.*

The following is the finite type representation type for tiled orders introduced by Zavadskij and Kirichenko in [12].

Theorem 2. *A tiled order* Λ *is of finite representation type if and only if* $\mathcal{P}(\Lambda) \not\supset K_1, ..., K_5,$ *where*

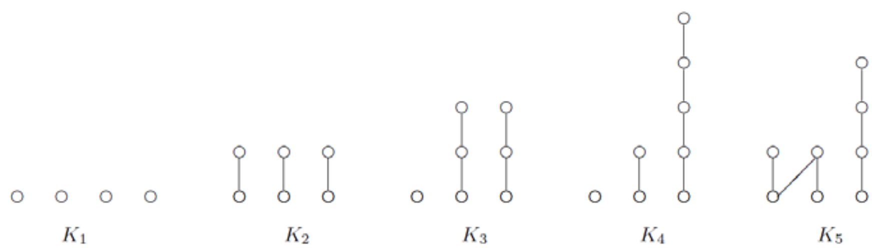

Figure 1

Often, posets $K_1, ..., K_5$ are called the *Kleiner's critical*.

For $m \geq 1,$ a $(0, 1, 2, ..., m)$*-tiled order* is a tiled order $\Lambda = (\lambda_{ij}),$ $1 \leq i, j \leq n,$ where $\lambda_{ij} \in \{0, 1, 2, ..., m\}.$ In particular,

If $\Lambda = (\lambda_{ij})$ is a $(0, m)$-tiled order, then Λ has associated a finite poset $(\mathfrak{R}, \leq) = \mathfrak{R}(\Lambda) = (\{1, 2, ..., n\}, \leq),$ where

$$i \leq j \text{ if and only if } \lambda_{ij} = 0.$$

We let $\Lambda(m, Q)$ denote the unique $(0, m)$-tiled order $\Lambda = (\lambda_{ij})$ such that $\Re(\Lambda(m, Q)) = Q$, where Q is a finite poset, that is, if $Q = \{1, 2, ..., n\}$, then

$$\lambda_{ij} = \begin{cases} 0, & \text{if } i \leq j, \\ m, & \text{if } j < i. \end{cases}$$

In [12], it is proved that there is a bijective correspondence between isomorphism classes of representations of a finite poset Q over a quotient ring $\overline{\mathcal{O}_m} = \mathcal{O}/\pi^m\mathcal{O}$ (where $\mathcal{O}$ is a ring of discrete valuation, π is a prime element) and isomorphism classes of admissible modules over a tiled order $\Lambda = \Lambda(m, Q \cup \{*\})$, where $*$ is an additional maximal point with $x < *$ for all $x \in Q$. Moreover, we have the following result.

Theorem 3. *For a finite poset Q, the following identities hold*:

(a) *Q is of finite representation type over the ring $\overline{\mathcal{O}_m}$, $m \geq 1$.*

(b) *The corresponding tiled order $\Lambda(m, Q \cup *)$ is of finite representation type.*

(c) *The infinite periodic poset $\mathcal{P}(\Lambda)$ does not contain the critical $K_1, ..., K_5$.*

(d) *Q has not as a subposet one of the following lists.*

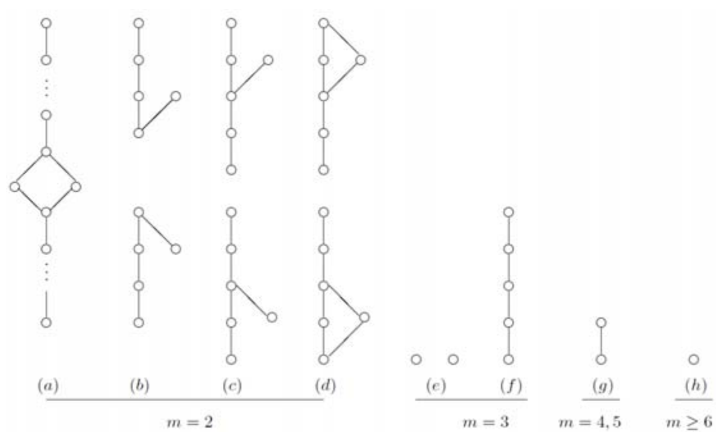

Figure 2

Theorem 2 was proved by Zavadskij and Kirichenko in [12] by using an algorithm of differentiation (for tiled orders) with respect to a suitable pair of points introduced by Zavadskij. The following lemma allows to define such an algorithm.

Lemma 4. *If* Λ *is a* $(0, 1, 2, ..., n)$*-tiled order, and* $\mathcal{P}(\Lambda)$ *does not contain as a subposet the critical posets* $(1, 1, 1, 1)$ *and* $(2, 2, 2)$, *then there is a pair of indices* (k, l) *which satisfies either condition* (a) *or condition* (b):

(a) $\lambda_{ki} + \lambda_{il} = \lambda_{kl}$ *for all* i.

(b) *There exists an index* m *such that* $\lambda_{ki} + \lambda_{il} = \lambda_{kl}$ *and* $\lambda_{km} + \lambda_{ml} = \lambda_{kl} + 1$ for all $m \neq i$.

Note that, if $B = (b_{ij}) \in M_n(\mathbb{Z})$ is such that $b_{ii} = 0$ for any i and for values $i_1, i_2, ..., i_n \in \{1, 2, ..., n\}$, we have that

$$b_{i_1 i_2} + b_{i_2 i_3} + \cdots + b_{i_n i_1} \geq 0. \tag{2}$$

Matrix $\Lambda = (\lambda_{ij})$ generated by B is a tiled order such that:

$$\lambda_{ij} = \min_{i_2, i_3, ..., i_n}\{b_{i_1 i_2} + b_{i_2 i_3} + \cdots + b_{i_n j}\}.$$

If Λ is a reduced tiled order and $k \neq l$ with $k, l \geq 1$, then we let $\Lambda^{-}_{(k,l)}$ denote the ring generated by $B = (b_{ij})$ with $b_{kl} = \lambda_{kl} - 1$ and $b_{ij} = \lambda_{ij}$ for $(i, j) \neq (k, l)$ entries of this *generation matrix* satisfies formula (2).

A pair of points (k, l), $k \neq l$ is said to be *suitable* for differentiation, if it satisfies one of the conditions (a) or (b) in Lemma 4. The *derived ring* $\Lambda'_{(k,l)}$ is defined in such a way that:

(1) $\Lambda'_{(k,l)} = \Lambda^{-}_{(k,l)}$ if the pair (k, l) satisfies condition (a) in Lemma 4.

(2) If the pair (k, l) satisfies condition (b) in Lemma 4, then $\Lambda'_{(k,l)}$ is generated by matrix $B = (b_{ij})_{1 \leq i, j \leq n}$ with $m' = n + 1$ and

$$b_{kl} = \lambda_{kl} - 1; \quad b_{ml} = \lambda_{ml} - 1; \quad b_{km'} = \lambda_{km} - 1,$$

$$b_{m'm} = 1; \quad b_{m'm'} = 0,$$

$$b_{m'j} = \lambda_{mj}; \text{ for } j \neq m, m'; \, b_{im'} = \lambda_{im} \text{ for } i \neq k, m',$$

$$b_{ij} = \lambda_{ij} \text{ if } i, j \neq m'; \text{ and } (i, j) \neq (k, l), (m, l). \tag{3}$$

The following result describes the derivative of some reduced tiled orders.

Theorem 5. *The generation matrix is the derivative of a reduced tiled order* $\Lambda = (\lambda_{i,j})_{1 \leq i, j \leq 3}$ *with* $\lambda_{ki} + \lambda_{il} = \lambda_{kl}$ *for all* i.

Proof. It suffices to prove that the matrix obtained by subtract 1 to the entry (k, l) of Λ is a tiled order.

It is clear that $\lambda_{ii} = 0$, inequality (2) must to be verified if the entry $\lambda_{kl} - 1$ is included in the generation matrix. Thus, we have three cases:

(i) Since the pair (k, l) is a suitable pair of points, $\lambda_{ki} + \lambda_{il} = \lambda_{kl}$, therefore, $\lambda_{ki} + \lambda_{il} \geq \lambda_{kl} - 1$.

(ii) Since Λ is reduced $\lambda_{li} + \lambda_{il} \geq 1$, $\lambda_{kl} + \lambda_{li} + \lambda_{il} \geq 1 + \lambda_{kl}$ and $\lambda_{kl} + \lambda_{li} + \lambda_{kl} - \lambda_{ki} \geq 1 + \lambda_{kl}$, thus $\lambda_{li} + \lambda_{kl} - 1 \geq \lambda_{ki}$.

(iii) Inequality $\lambda_{ik} + \lambda_{kl} - 1 \geq \lambda_{il}$ is obtained by using arguments from (ii). □

For $(0, 1)$-tiled orders, we have the following result:

Theorem 6. *A* $(0, 1)$*-tiled order is of finite* (*tame*, *finite growth*, *one parameter*, *etc.*) *representation type if the poset* $\Re(\Lambda)$ *is.*

Remark 7. Up to for the finite representation type case, Theorem 6 is not true for arbitrary $(0, n)$-tiled orders, therefore one of the main problems regarding $(0, n)$-tiled orders consists of establishing for $n > 1$ which orders satisfy conditions described in such theorem. To give some advances to this problem, we prove that some $(0, n)$-tiled orders can be reduced to $(0, 1)$-tiled orders via differentiation.

2.1. The matrix problem

If A is a Λ-not null right admissible module, then the submodule $A_i = Ae_{ii}$ is said to be a $\mathbb{O}$-*net*. Therefore, it is possible to associate to the module A a system of the form $\mathbb{S}_A = (V; A_1, ..., A_n)$, where V is a finite dimensional T-vector space $(\dim_T V = d)$, further $A_i \subset V$ is a complete $\mathbb{O}$-net for each i, i.e., the rank of A_i as a $\mathcal{O}$-free module equals d. Furthermore,

$$A_i \pi^{\lambda_{ij}} \subset A_j \text{ for all } i, j.$$

Thus, two admissible Λ-modules $A \to \mathbb{S}_A = (V; A_1, ..., A_n)$ and $A' \to \mathbb{S}'_A = (V'; A'_1, ..., A'_n)$ are isomorphic if and only if there exists a T-isomorphism, $\varphi : V \to V'$ such that $\varphi(A_i) = A'_i$ for all i. Thus, the problem of classifying right admissible Λ-modules is equivalent to the problem of classifying system of the type $(V; A_1, ..., A_n)$. The corresponding matrix problem is defined in such a way that if an admissible right Λ-module A which has associated a system of the form $\mathbb{S}_A = (V; A_1, ..., A_n)$, where A_i is a $\mathbb{O}$-net, then a matrix M_A is assigned to $\mathbb{S}_A$:

$$M_A = \begin{array}{ccc} & i & \\ \hline \multicolumn{1}{|c|}{\cdots} & \multicolumn{1}{c|}{M_i} & \multicolumn{1}{c|}{\cdots} \\ \hline \end{array}$$

where column of each stripe M_i consists of coordinates of $\mathbb{O}$-generators of the net A_i with respect to fixed basis of V modulo the subnet $\underline{A_i} = \sum_{j \neq i} A_j \pi^{\lambda_{ji}}$. The following are the admissible transformations which define equivalent matrices:

(1) T-elementary transformations of rows of the whole matrix M_A.

(2) $\mathbb{O}$-elementary transformations of columns within each stripe M_i.

(3) Additions of columns of type $M_j + M_i \pi^{\lambda_{ij}} \mathbb{O} \to M_j$.

3. Main Results

In this section, we use the algorithm of differentiation with respect to a suitable pair of points for tiled orders in order to reduce some $(0, 2)$-tiled orders to $(0, 1)$-orders.

For a given exponent matrix Λ, we let $i(\Lambda)$ denote the index of Λ in such a way that:

$$i(\Lambda) = \sum_{1 \leq i, j \leq n} \lambda_{ij}.$$

Note that if Λ is a $(0, 1)$-tiled order, then $\frac{n(n-1)}{2} \leq i(\Lambda) \leq n(n-1)$. Furthermore, $\Lambda \simeq \Lambda'$ implies $i(\Lambda) = i(\Lambda')$ as a consequence of Theorem 1.

The following result concerns $(0, 1)$-tiled orders.

Theorem 8. *The following is a complete list of representatives of isomorphic classes of* $(0, 1)$*-reduced tiled orders* $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$:

$$\Lambda_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The associated poset $\mathcal{P}(\Lambda)$ of the $(0, 1)$-tiled order Λ is an infinite ordinal sum of copies of $\mathfrak{R}(\Lambda)$, conversely the corresponding $(0, 1)$-tiled order $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ associated to $\mathfrak{R}$ is such that

$$\lambda_{ij} = \begin{cases} 0, & \text{if } i \le j, \\ 1, & \text{if } j < i. \end{cases}$$

Up to isomorphism and antiisomorphism, the only possibilities for poset $\Re(\Lambda)$ are

Figure 3

which have associated orders $\Lambda_1, \ldots, \Lambda_4$. □

Theorem 9. *The following is a complete list of representatives of isomorphic classes of* $(0, 1, 2)$*-reduced tiled orders* $\Lambda = (\lambda_{ij})_{1 \le i, j \le 3}$ *with* $\lambda_{ij} = 0$ *if* $j \le i$:

$$\Lambda_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let $\Lambda = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ 0 & 0 & \lambda_{23} \\ 0 & 0 & 0 \end{pmatrix}$ be a $(0, 1, 2)$-reduced tiled order. Then $\lambda_{ij} \ne 0$ for all $i < j$, thus:

(1) If $\lambda_{13} = 1$, then $\lambda_{12} = \lambda_{23} = 1$.

(2) If $\lambda_{13} = 2$, then pair of numbers

$$(\lambda_{12}, \lambda_{23}) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\},$$

thus the corresponding reduced exponent matrices are:

$$\Lambda_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_4 = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_5 = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\Lambda_3 \simeq \Lambda_4$, we are done. □

Theorem 10. *Any* $(0, 1, 2)$*-reduced tiled order* $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ *with* $i(\lambda) \leq 5$ *is isomorphic to a* $(0, 1)$*-tiled order.*

Proof. Let Λ be an order which satisfies hypothesis of the theorem and $r = | \{\lambda_{ij} \in \Lambda | \lambda_{ij} = 2\} |$ therefore if $r = 0$, then Λ is a $(0, 1)$-tiled order. Actually, if $i(\Lambda) = 3$, then $r = 0$ and Λ is a $(0, 1)$-tiled order.

If $i(\Lambda) = 4$, then $r \in \{0, 1\}$. If $i, j, k \in \{1, 2, 3\}$ and $r = 1$, thus if $\lambda_{kl} = 2$, then the remaining entries belong to $\{0, 1\}$. Since $\lambda_{li} + \lambda_{ik} \geq \lambda_{lk}$, $\lambda_{li} = \lambda_{ik} = 1$ the other entries are null, a $(0, 1)$-tiled order can be received if we subtract 1 to the row l and simultaneously add 1 to the column l of the matrix Λ.

If $i(\Lambda) = 5$, then $r \leq 2$, therefore, if $r = 1$, then it is possible to assume $r = 1$ and $\lambda_{lk} = 2$, thus $\lambda_{li} = \lambda_{ik} = 1$. Since $\lambda_{kl} = 0$, either $\lambda_{ki} = 1$ or $\lambda_{il} = 1$, if $\lambda_{ki} = 1$, then Λ can be transformed to a $(0, 1)$-tiled order by subtracting 1 to the row l and adding 1 to the column l. If $\lambda_{il} = 1$, then a $(0, 1)$-tiled order can be obtained from Λ by subtracting 1 to the column k and adding 1 to row k.

Finally, if $r = 2$, then different options arise, for instance, a $(0, 1)$-tiled order isomorphic to Λ can be obtained if all entries with value 2 belong to the same row or column. On the other hand, if entries with value 2 belong to different rows or columns, then a $(0, 1)$-tiled order can be received from Λ via t-admissible transformations. For instance, if $\lambda_{il} = \lambda_{ki} = 2,$ then either $\lambda_{kl} = 1$ or $\lambda_{lk} = 1.$ Note that, if $\lambda_{kl} = 1,$ then Λ is not a tiled order, on the other hand, if $\lambda_{lk} = 1,$ then Λ is isomorphic to a triangular tiled order which is a $(0, 1)$-tiled order as a consequence of Theorem 9. Since same arguments can be used to all the other cases, we are done. □

The following theorem concerns tiled orders with $i(\Lambda) = 6.$

Theorem 11. *Every* $(0, 1, 2)$*-tiled order* $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ *with* $i(\Lambda) = 6$ *has a suitable pair of points* (k, l) *such that* $\Lambda'_{(k, l)}$ *is isomorphic to a* $(0, 1)$*-tiled order.*

Proof. Let $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ be an order which satisfies the proposed hypothesis and $r = |\{\lambda_{ij} \in \Lambda \,|\, \lambda_{ij} = 0,\, i \neq j\}|.$ It is easy to see that $r \leq 3,$ so we must to prove that in any case it is possible to find a suitable pair of points with the property (a) mentioned in Lemma 4.

Let us suppose that $\lambda_{ml} \neq \lambda_{mn} + \lambda_{nl}$ for any $m, n, l \in \{1, 2, 3\}.$

If $r = 1,$ then we receive a contradiction. If $r = 2,$ then two entries of Λ are 2's and the other two are 1's. If $\lambda_{lm} = 0,$ then $\lambda_{mn} \neq 0.$ If $\lambda_{mn} = 1,$ then $\Lambda \simeq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$ If $\lambda_{mn} = 2,$ then $\lambda_{ln} \in \{0, 1\}$ which is a contradiction. The case $r = 3$ defines a tiled order with three entries equal to 2 which contradicts the definition of $\Lambda.$ □

The following results concern $(0, 1, 2)$-tiled orders with $i(\Lambda) \geq 6.$

Theorem 12. *Any* $(0, 1, 2)$*-tiled order* $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ *with* $8 \leq i(\Lambda) \leq 11$ *has a suitable pair of points* (k, l) *such that* $\Lambda'_{(k, l)}$ *is not a* $(0, 1)$*-tiled order.*

Proof. If Λ is a $(0, 1, 2)$-tiled order with $i(\Lambda) < 12$ and width $w(\Lambda) = 3$, then the properties of the corresponding poset of projective modules ensure the existence of a suitable pair of points (k, l) for differentiation.

If (k, l) satisfies condition (a) of Lemma 4, then $i(\Lambda'_{(k,l)}) \geq 7$ by Theorem 5, therefore $\Lambda'_{(k,l)}$ cannot be a $(0, 1)$-order.

Now let us suppose that Λ has a pair of points (k, l) which satisfies condition (b) of Lemma 4 but does not satisfy condition (a). Therefore, the following arguments prove that the matrix B defined as in (3) is a tiled order such that $B = \Lambda'_{(k,l)}$.

Note that $\lambda_{km} + \lambda_{ml} = \lambda_{kl} + 1$ with $(k, m, l) \in \{1, 2, 3\}$, $l \neq k \neq m \neq l$ since (k, l) is a suitable pair of points. Furthermore, $\lambda_{ip} + \lambda_{pj} > \lambda_{ij}$ for all $i, j, p \in \{1, 2, 3\}$ provided that there is not a suitable pair of points which satisfies condition (a) of Lemma 4.

Formulas (3) allow to define $B = (b_{ij})$ in such a way that:

$$
\begin{aligned}
&b_{ii} = 0 \text{ for all } 1 \leq i \leq 4,\\
&b_{km} = \lambda_{km},\\
&b_{kl} = \lambda_{kl} - 1,\\
&b_{k4} = \lambda_{km} - 1,\\
&b_{mk} = b_{4k} = \lambda_{mk},\\
&b_{ml} = \lambda_{ml} - 1,\\
&b_{m4} = 0,\\
&b_{lk} = \lambda_{lk},\\
&b_{lm} = b_{l4} = \lambda_{lm},\\
&b_{4m} = 1,\\
&b_{4l} = \lambda_{ml},
\end{aligned}
\tag{4}
$$

thus $b_{rn} + b_{nq} \geq b_{rq}$ for all $r, n, q \in \{k, m, l, 4\}$. Therefore, B is a tiled order and $B = \Lambda'_{(k,l)}$. In particular, we have that

$$i(\Lambda'_{(k,l)}) = 2i(\Lambda) - (2 + \lambda_{kl} + \lambda_{lk}).$$

If $i(\Lambda) = 8$, then Λ is isomorphic to the tiled order

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the derived tiled order is isomorphic to

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

whose associated poset of projective modules $\mathcal{P}(\Lambda')$ is not a cardinal sum of a finite poset, thus it is not a $(0, 1)$-tiled order.

If $i(\Lambda) = 9$, then we take into account that $\lambda_{kl} + 1 = \lambda_{km} + \lambda_{ml}$ to see that $\lambda_{kl} \neq 0$, otherwise we have that $1 = \lambda_{km} + \lambda_{ml}$ and $\lambda_{mk} + \lambda_{lk} + \lambda_{lm} = 8$ which cannot be possible.

If $\lambda_{kl} = 1$, then $2 = \lambda_{km} + \lambda_{ml}$ and $\lambda_{mk} + \lambda_{lk} + \lambda_{lm} = 6$, thus $\lambda_{mk} = \lambda_{lk} = \lambda_{lm} = 2$, therefore $i(\Lambda'_{(k,l)}) = 2(9) - (2 + 1 + 2) = 13$.

If $\lambda_{kl} = 2$, then $3 = \lambda_{km} + \lambda_{ml}$ and $\lambda_{mk} + \lambda_{lk} + \lambda_{lm} = 4$. Since $\lambda_{lk} < \lambda_{lm} + \lambda_{mk}$, $\lambda_{lk} < 4 - \lambda_{lk}$, that is, $\lambda_{lk} < 2$. Note that $\lambda_{lk} = 0$ implies $\lambda_{mk} = \lambda_{lm} = 2$ and $\lambda_{lk} + \lambda_{km} > \lambda_{lm}$ implies $\lambda_{km} > 2$, a contradiction. Therefore, $\lambda_{lk} = 1$ and again $i(\Lambda'_{(k,l)}) = 2(9) - (2 + 2 + 1) = 13$. We conclude that $i(\Lambda'_{(k,l)}) = 13$ and that $\Lambda'_{(k,l)}$ is not a $(0,1)$-order.

Finally, if $i(\Lambda) \geq 10$, then we have $\lambda_{kl} + \lambda_{lk} \leq 4$, thus $i(\Lambda'_{(k,l)}) \geq 2(10) - (6) = 14$ therefore $\Lambda'_{(k,l)}$ is not a $(0, 1)$-tiled order. $\square$

As an example, the following are $(0, 1, 2)$-tiled orders with $i(\Lambda) \in \{7, 8\}$ whose derivative is not a $(0, 1)$-tiled order:

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix},$$

$$\Lambda'_{(1,3)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix},$$

$$\Delta'_{(2,3)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix},$$

$$\Delta''_{(3,2)} = (\Delta'_{(2,3)})_{(3,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}. \tag{5}$$

In this case, $(1, 3)$ is a suitable pair of points of the tiled order Λ with $i(\Lambda) = 7$, and $\Lambda'_{(1,3)}$ is not a $(0, 1)$-tiled order. On the other hand, $(2, 3)$ and $(3, 2)$ are suitable pairs of points of the tiled order Δ with $i(\Delta) = 8$, note that, $\Delta'_{(2,3)}$ is not a $(0, 1)$-tiled order but $\Delta''_{(3,2)} \simeq \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a $(0, 1)$-tiled order.

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