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SURFACE AREAS OF CRYSTALS GENERATED BY TETRAHEDRONS

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Abstract

The main purpose of this paper is to evaluate surface areas of crystals generated from quadrilaterals in tetrahedrons. Explicit summation formulas for finding the surface areas of crystals are derived and some applications of these formulas are given.

1. Introduction

Calculating surface areas of some of the crystals generated by tetrahedrons can take considerable time. However, surface areas can be easily obtained if explicit formulas are available for the crystals. In 1999, Yetter [1] studied the area of a medial parallelogram in a tetrahedron with sides of length a, b, c, d, e and f (see Figure 1) and showed that the area was $\frac{1}{8}\sqrt{4e^2 f^2 - (a^2 - b^2 + c^2 - d^2)^2}$.

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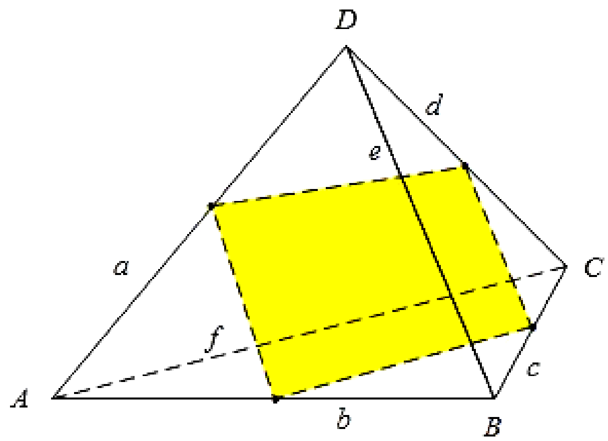

Figure 1. The medial parallelogram.

This result stimulated us to undertake the present study. The outline of the present paper is as follows. In Section 2, basic definitions and results related to tetrahedrons are summarized. In Section 3, an explicit formula is derived for calculating surface areas of some crystals generated from quadrilaterals in tetrahedrons. In Section 4, the results are applied to a practical problem and conclusions are made.

2. Preliminaries and Notation

In this section, we summarize some general mathematical background required in this work. Throughout this paper, we let $\mathbb{N}_4$ denote the set $\{1, 2, 3, 4\}$ and as shown in Figure 2, we assume that $\mathcal{T}$ denotes a tetrahedron with four non-coplanar vertices labeled by P_1, P_2, P_3 and P_4. Any set of indices $i, j \in \mathbb{N}_4$ or $i, j, k \in \mathbb{N}_4$ or $i, j, k, l \in \mathbb{N}_4$ specifies a set of distinct vertices of the tetrahedron $\mathcal{T}$.

Definition 2.1. Let $\mathcal{T}$ be a tetrahedron with vertices P_1, P_2, P_3 and P_4 as shown in Figure 2.

(1) For any $i, j \in \mathbb{N}_4$, let $x_{ij} = |P_iP_j| = x_{ji}$ be the *length of side* P_iP_j (or P_jP_i) which joins vertices P_i and P_j.

(2) For any $i, j \in \mathbb{N}_4,$ let P_{ij} be a *point on side* P_iP_j with distances from P_i to P_{ij} specified by the ratio $|P_iP_{ij}| : |P_iP_j| = r_{ij},$ where $0 < r_{ij} < 1.$

(3) For any $i, j, k, l \in \mathbb{N}_4,$ we say that sides P_iP_j and P_kP_l are *non-incident sides* of $\mathcal{T}$ if sides P_iP_j and P_kP_l have no common vertices in $\mathcal{T}.$

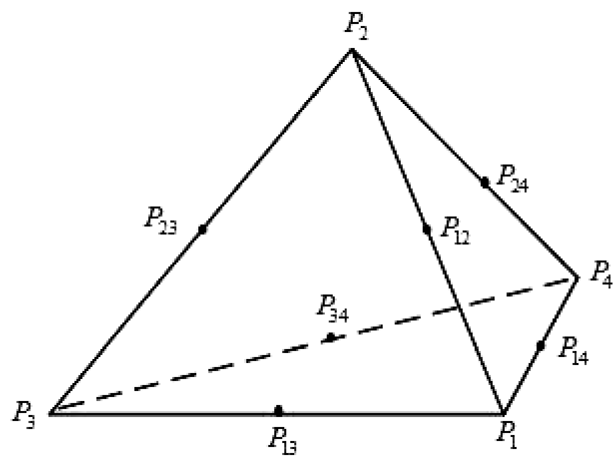

Figure 2. The tetrahedron $\mathcal{T}.$

Remarks 2.2. (1) We assume that P_iP_j and P_jP_i denote the same side. If the direction of a side is required, we use the usual vector notation $\overrightarrow{P_iP_j}.$

(2) We assume that P_{ij} is a point on the side P_iP_j and that P_{ij} and P_{ji} are the same point. Then, from part (2) of Definition 2.1, the ratio $|P_jP_{ji}| = r_{ji}|P_iP_j|$ implies that $r_{ji} = 1 - r_{ij},$ since $|P_iP_{ij}| + |P_jP_{ji}| = |P_iP_j|.$

(3) In vector notation, we can write

$$x_{ij} = x_{ji} = |\overrightarrow{P_iP_j}| = |\overrightarrow{P_iP_{ij}}| + |\overrightarrow{P_{ji}P_j}| = (r_{ij} + r_{ji})|\overrightarrow{P_iP_j}|,$$

and

$$\overrightarrow{P_iP_{ij}} = r_{ij}\overrightarrow{P_iP_j}, \quad \overrightarrow{P_{ij}P_j} = r_{ji}\overrightarrow{P_iP_j} = (1 - r_{ij})\overrightarrow{P_iP_j}.$$

(4) Clearly, sides P_iP_j and P_kP_l are non-incident sides if and only if all integers i, j, k and l are different. From Figure 2, $\mathcal{T}$ has only three pairs of non-incident sides as follows:

(a) P_1P_2 is non-incident with P_3P_4,

(b) P_1P_3 is non-incident with P_2P_4, and

(c) P_1P_4 is non-incident with P_2P_3.

Definition 2.3. Let $\mathcal{T}$ be a tetrahedron with vertices P_1, P_2, P_3 and P_4.

(1) For any pairs of sides P_iP_j and P_kP_l which are non-incident and $i < j$, $k < l$, $i < k$, let $\mathcal{R}_{kl}^{ij}$ be a *quadrilateral* with four vertices given by points P_{qr} on four sides other than sides P_iP_j and P_kP_l. For example, the quadrilateral $\mathcal{R}_{34}^{12}$ shown in Figure 3 has vertices P_{13}, P_{23}, P_{14} and P_{24} but not P_{12} or P_{34}.

(2) For any quadrilateral $\mathcal{R}_{kl}^{ij}$ with $i < j$, $k < l$, $i < k$, let a *crystal* $\mathcal{G}_{kl}^{ij}$ be the octahedron generated from $\mathcal{R}_{kl}^{ij}$ by joining two points P_{ij} and P_{kl} to the four vertices of $\mathcal{R}_{kl}^{ij}$. For example, a crystal $\mathcal{G}_{34}^{12}$ is shown in Figure 3.

Remarks 2.4. There are exactly three distinct choices for the indices i, j, k, l that specify a quadrilateral and an associated crystal given by

(1) $\mathcal{R}_{34}^{12}$ and $\mathcal{G}_{34}^{12}$ generated from non-incident sides P_1P_2 and P_3P_4, as shown in Figure 3,

(2) $\mathcal{R}_{24}^{13}$ and $\mathcal{G}_{24}^{13}$ generated from non-incident sides P_1P_3 and P_2P_4,

(3) $\mathcal{R}_{23}^{14}$ and $\mathcal{G}_{23}^{14}$ generated from non-incident sides P_1P_4 and P_2P_3.

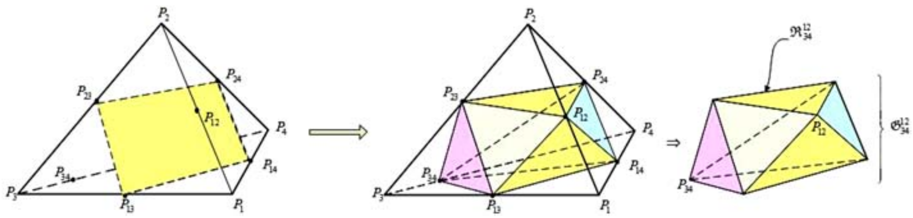

Figure 3. A quadrilateral $\mathcal{R}_{34}^{12}$ and a crystal $\mathcal{G}_{34}^{12}$.

However, for each distinct choice of indices i, j, k, l there are an infinite number of quadrilaterals $\mathcal{R}_{kl}^{ij}$ and an infinite number of crystals $\mathcal{G}_{kl}^{ij}$ because there are an infinite number of points P_{ij} on the line P_iP_j defined by ratios $0 < r_{ij} < 1$.

The following mathematical background relates to areas of triangles. In this paper, we use the notation T_{BC}^{A} for the area of a triangle with vertices A, B and C because the triangles we consider usually have one vertex defined differently from the other two. Of course, $T_{BC}^{A} = T_{AC}^{B} = T_{AB}^{C}$. Some standard theorems on areas of triangles are as follows. A proof of Theorem 2.5 is given in [2], and Theorem 2.6, Theorem 2.7 are given in [4] and [5].

Theorem 2.5. *If a triangle* $\triangle ABC$ *has sides of lengths* a, b, c *opposite the vertices* A, B *and* C, *respectively*, *then*

(1) $T_{BC}^{A} = \sqrt{s(s-a)(s-b)(s-c)}$, *where* $s = \dfrac{a+b+c}{2}$,

(2) $T_{BC}^{A} = \dfrac{1}{4}\sqrt{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}$.

Proof. Formula (1) is the well-known Heron's formula. A proof is given in [2]. The proof of formula (2) follows immediately from Heron's formula.

$\square$

Theorem 2.6 (Magnitude of cross product). *Let* A, B *and* C *be any points in Euclidean space* $\mathbb{R}^3$. *If* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are nonzero vectors in* $\mathbb{R}^3$ *with the* *angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *is* θ *and* $0 \leq \theta \leq \pi$, *then*

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta.$$

Proof. See the proof in [4] and [5]. □

Theorem 2.7 (Areas of triangles)**.** *Let A, B and C be any points in Euclidean space. Then the area of triangle* $\triangle ABC$ *is equal to*

$$T_{BC}^{A} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin\theta,$$

where $\theta\,(0 \leq \theta \leq \pi)$ *is the angle between* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$.

Proof. Use Theorem 2.6 and theorems on areas of triangles in [4] and [5]. □

Remarks 2.8. Since $T_{BC}^{A} = T_{AC}^{B} = T_{AB}^{C}$ are different expressions for the area of triangle $\triangle ABC$, we also have from Theorem 2.7 that

$$T_{BC}^{A} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}| = \frac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|.$$

Theorem 2.9. *For the tetrahedron* $\mathcal{T}$ *and point* P_{ij} *defined by* $|P_iP_{ij}| = r_{ij}|P_iP_j|$, *we have for any distinct set* $i, j, k \in \mathbb{N}_4$ *that* $T_{P_{ij}P_{ik}}^{P_i} = r_{ij}r_{ik}T_{P_jP_k}^{P_i}$.

Proof. For any distinct set of indices $i, j, k \in \mathbb{N}_4$, we have from Theorem 2.7 that

$$T_{P_{ij}P_{ik}}^{P_i} = \frac{1}{2}|\overrightarrow{P_iP_{ij}} \times \overrightarrow{P_iP_{ik}}| = \frac{1}{2}r_{ij}r_{ik}|\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_k}| = r_{ij}r_{ik}T_{P_jP_k}^{P_i}.$$ □

3. Main Results

In this section, we prove the main Theorem 3.6 for explicitly calculating surface areas of the crystals generated by tetrahedrons. We begin with some preliminary lemmas and theorems.

Lemma 3.1. *For the tetrahedron* $\mathcal{T}$ *and a set of distinct vertices specified by* $i, j, k \in \mathbb{N}_4$, *we have*

$$T^{P_i}_{P_jP_k} = \frac{1}{2}|\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_k}|$$

$$= \frac{1}{4}\sqrt{2(x_{ij}^2x_{ik}^2 + x_{ij}^2x_{jk}^2 + x_{ik}^2x_{jk}^2) - (x_{ij}^4 + x_{ik}^4 + x_{jk}^4)}.$$

Proof. Using Theorem 2.7, we obtain $T^{P_i}_{P_jP_k} = \frac{1}{2}|\overrightarrow{P_iP_j} \times \overrightarrow{P_iP_k}|$. Then, from the results of Theorem 2.5, we have

$$T^{P_i}_{P_jP_k} = \frac{1}{4}\sqrt{2(x_{ij}^2x_{ik}^2 + x_{ij}^2x_{jk}^2 + x_{ik}^2x_{jk}^2) - (x_{ij}^4 + x_{ik}^4 + x_{jk}^4)}. \qquad \square$$

Lemma 3.2. *For the tetrahedron* $\mathcal{T}$ *and a set of distinct vertices specified by* $i, j, k \in \mathbb{N}_4$, *we have*

$$T^{P_{ij}}_{P_{ik}P_{jk}} = (1 - r_{ij}r_{ik} - r_{ji}r_{jk} - r_{ki}r_{kj})T^{P_i}_{P_jP_k}.$$

Proof. From the geometry of the triangle $\triangle P_iP_jP_k$, the definitions of the points $P_{ij} = P_{ji}$, $P_{jk} = P_{kj}$ and $P_{ki} = P_{ik}$, and the result in Theorem 2.9, we have

$$T^{P_{ij}}_{P_{ik}P_{jk}} = T^{P_i}_{P_jP_k} - T^{P_i}_{P_{ij}P_{ik}} - T^{P_j}_{P_{ji}P_{jk}} - T^{P_k}_{P_{ki}P_{kj}}$$

$$= (1 - r_{ij}r_{ik} - r_{ji}r_{jk} - r_{ki}r_{kj})T^{P_i}_{P_jP_k}. \qquad \square$$

Theorem 3.3. *The areas of the triangular surfaces of each crystal* $\mathcal{G}^{ij}_{kl}$ *generated in the tetrahedron* $\mathcal{T}$ *by the quadrilateral* $\mathcal{R}^{ij}_{kl}$ *have the following properties*:

(1) $T^{P_{ij}}_{P_{ik}P_{jk}} = (1 - r_{ij}r_{ik} - r_{ji}r_{jk} - r_{ki}r_{kj})T^{P_i}_{P_jP_k}.$

(2) $T_{P_{il}P_{jl}}^{P_{ij}} = (1 - r_{ij}r_{il} - r_{ji}r_{jl} - r_{li}r_{lj})T_{P_jP_l}^{P_i}$.

(3) $T_{P_{ki}P_{li}}^{P_{kl}} = (1 - r_{kl}r_{kj} - r_{ki}r_{kl} - r_{li}r_{lk})T_{P_lP_i}^{P_k}$.

(4) $T_{P_{kj}P_{lj}}^{P_{kl}} = (1 - r_{kl}r_{kj} - r_{lk}r_{lj} - r_{jk}r_{jl})T_{P_lP_j}^{P_k}$.

Proof. The proof of each of the 4 formulas follows from Lemma 3.2. □

We now introduce the following notation. For the tetrahedron $\mathcal{T}$ with vertices P_1, P_2, P_3 and P_4 and for a distinct set of vertices specified by $i, j, k \in \mathbb{N}_4$, we let

$$\mathcal{A}_{jk}^{i} = \mathcal{A}_{kj}^{i} = (r_{ij}x_{ij})^2 + (r_{ik}x_{ik})^2,$$

$$\mathcal{B}_{jk}^{i} = \mathcal{B}_{kj}^{i} = 2r_{ij}r_{ik}\sqrt{(x_{ij}x_{ik})^2 - 4(T_{P_jP_k}^{P_i})^2}.$$

Lemma 3.4. *For the tetrahedron $\mathcal{T}$ with vertices P_1, P_2, P_3 and P_4 and a distinct set of vertices specified by $i, j, k \in \mathbb{N}_4$, we obtain*

$$|P_{ij}P_{ik}| = \sqrt{\mathcal{A}_{jk}^{i} - \mathcal{B}_{jk}^{i}}.$$

Proof. We will prove this lemma using the results of Theorem 2.6, Theorem 2.7 and Theorem 2.9. From the cosine law [4] applied to $\triangle P_{ij}P_iP_{ik}$, we obtain

$$\cos(\widehat{P_{ij}P_iP_{ik}}) = \frac{|P_iP_{ij}|^2 + |P_iP_{ik}|^2 - |P_{ij}P_{ik}|^2}{2|P_iP_{ij}||P_iP_{ik}|}. \tag{3.1}$$

Then, from Theorem 2.6, Theorem 2.7, equation (3.1) and a basic identity of trigonometry, we have

$$T_{P_{ij}P_{ik}}^{P_i} = \frac{1}{2}|P_iP_{ij}||P_iP_{ik}|\sqrt{1 - \left(\frac{|P_iP_{ij}|^2 + |P_iP_{ik}|^2 - |P_{ij}P_{ik}|^2}{2|P_iP_{ij}||P_iP_{ik}|}\right)^2}.$$

Then, it can easily be shown that

$$\left(\frac{|P_iP_{ij}|^2 + |P_iP_{ik}|^2 - |P_{ij}P_{ik}|^2}{2|P_iP_{ij}||P_iP_{ik}|}\right)^2 = 1 - \left(\frac{2}{|P_iP_{ij}||P_iP_{ik}|} T^{P_i}_{P_{ij}P_{ik}}\right)^2$$

and that

$$|P_{ij}P_{ik}|^2 = |P_iP_{ij}|^2 + |P_iP_{ik}|^2 - \sqrt{(2|P_iP_{ij}||P_iP_{ik}|)^2 - (4T^{P_i}_{P_{ij}P_{ik}})^2}. \quad (3.2)$$

Clearly, we have seen that

$$|P_iP_{ij}|^2 + |P_iP_{ik}|^2 = (r_{ij}x_{ij})^2 + (r_{ik}x_{ik})^2 = \mathcal{A}^i{}_{jk}. \quad (3.3)$$

Obviously, from Theorem 2.9, we obtain

$$(2|P_iP_{ij}||P_iP_{ik}|)^2 - (4T^{P_i}_{P_{ij}P_{ik}})^2 = \left(2r_{ij}r_{ik}\sqrt{(x_{ij}x_{ik})^2 - 4(T^{P_i}_{P_jP_k})^2}\right)^2$$

$$= (\mathcal{B}^i{}_{jk})^2. \quad (3.4)$$

We substitute equations (3.3) and (3.4) in (3.2) and we obtain

$$|P_{ij}P_{ik}| = \sqrt{\mathcal{A}^i{}_{jk} - \mathcal{B}^i{}_{jk}}. \qquad \square$$

We now introduce the following notation. For the triangular surface of a crystal $\mathcal{G}^{ij}_{kl}$ generated in the tetrahedron $\mathcal{T}$ by the quadrilateral $\mathcal{R}^{ij}_{kl}$, we define

1. $$\mathcal{C}^{P_{ij}}_{P_{ik}P_{il}} = 2 \sum_{\{(q,r): q=k,l,\, r=j,k\} - \{(l,j)\}} (\mathcal{A}^i_{jq} - \mathcal{B}^i_{jq})(\mathcal{A}^i_{rl} - \mathcal{B}^i_{rl}),$$

$$\mathcal{D}^{P_{ij}}_{P_{ik}P_{il}} = \sum_{\{(q,r): q=j,k,\, r=k,l\} - \{(k,k)\}} (\mathcal{A}^i_{qr} - \mathcal{B}^i_{qr})^2.$$

2. $\mathcal{C}_{P_{jk}P_{jl}}^{P_{ij}} = 2 \sum_{\{(q, r): q=k, l, r=i, k\}-\{(l, i)\}} (\mathcal{A}_{iq}^{j} - \mathcal{B}_{jq}^{j})(\mathcal{A}_{rl}^{j} - \mathcal{B}_{rl}^{j}),$

$\mathcal{D}_{P_{jk}P_{jl}}^{P_{ij}} = \sum_{\{(q, r): q=i, k, r=k, l\}-\{(k, k)\}} (\mathcal{A}_{qr}^{j} - \mathcal{B}_{qr}^{j})^2.$

3. $\mathcal{C}_{P_{ki}P_{kj}}^{P_{kl}} = 2 \sum_{\{(q, r): q=i, j, r=l, i\}-\{(l, j)\}} (\mathcal{A}_{lq}^{k} - \mathcal{B}_{lq}^{k})(\mathcal{A}_{rj}^{k} - \mathcal{B}_{rj}^{k}),$

$\mathcal{D}_{P_{ki}P_{kj}}^{P_{kl}} = \sum_{\{(q, r): q=l, i, r=i, j\}-\{(i, i)\}} (\mathcal{A}_{qr}^{k} - \mathcal{B}_{qr}^{k})^2.$

4. $\mathcal{C}_{P_{li}P_{lj}}^{P_{kl}} = 2 \sum_{\{(q, r): q=i, j, r=k, i\}-\{(k, j)\}} (\mathcal{A}_{kq}^{l} - \mathcal{B}_{kq}^{l})(\mathcal{A}_{rj}^{l} - \mathcal{B}_{rj}^{l}),$

$\mathcal{D}_{P_{li}P_{lj}}^{P_{kl}} = \sum_{\{(q, r): q=k, i, r=i, j\}-\{(i, i)\}} (\mathcal{A}_{qr}^{l} - \mathcal{B}_{qr}^{l})^2.$

Theorem 3.5. *The areas of the triangular surfaces of the crystal* $\mathcal{G}_{kl}^{ij}$ *generated in tetrahedron* $\mathcal{T}$ *by the quadrilateral* $\mathcal{R}_{kl}^{ij}$ *are given by the following equations*:

$$T_{P_{ik}P_{il}}^{P_{ij}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{ik}P_{il}}^{P_{ij}} - \mathcal{D}_{P_{ik}P_{il}}^{P_{ij}}}, \quad T_{P_{jk}P_{jl}}^{P_{ij}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{jk}P_{jl}}^{P_{ij}} - \mathcal{D}_{P_{jk}P_{jl}}^{P_{ij}}},$$

$$T_{P_{ki}P_{kj}}^{P_{kl}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{ki}P_{kj}}^{P_{kl}} - \mathcal{D}_{P_{ki}P_{kj}}^{P_{kl}}}, \quad T_{P_{li}P_{lj}}^{P_{kl}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{li}P_{lj}}^{P_{kl}} - \mathcal{D}_{P_{li}P_{lj}}^{P_{kl}}}.$$

Proof. The results of this theorem follow directly from Heron’s formula, Theorem 2.5 and Lemma 3.4. □

We now introduce the following notation. For a crystal $\mathcal{G}_{kl}^{ij}$ generated in tetrahedron $\mathcal{T}$ by the quadrilateral $\mathcal{R}_{kl}^{ij}$, the areas of the triangular surfaces of the crystal, $T_{P_{ik}P_{jk}}^{P_{ij}}$, $T_{P_{il}P_{jl}}^{P_{ij}}$, $T_{P_{ki}P_{li}}^{P_{kl}}$, and $T_{P_{kj}P_{lj}}^{P_{kl}}$ are defined in Theorem

3.3 and on the other hand $T_{P_{ik}P_{il}}^{P_{ij}}$, $T_{P_{jk}P_{jl}}^{P_{ij}}$, $T_{P_{ki}P_{kj}}^{P_{kl}}$, and $T_{P_{li}P_{lj}}^{P_{kl}}$ are defined in Theorem 3.5, we define:

1. The sum $\mathcal{S}_{ij}$ of surface areas of triangles with vertex at P_{ij}. The sum is given by $\mathcal{S}_{ij} = T_{P_{ik}P_{jk}}^{P_{ij}} + T_{P_{il}P_{jl}}^{P_{ij}} + T_{P_{ik}P_{il}}^{P_{ij}} + T_{P_{jk}P_{jl}}^{P_{ij}}$.

2. The sum $\mathcal{S}_{kl}$ of surface areas of triangles with vertex at P_{kl}. The sum is given by $\mathcal{S}_{kl} = T_{P_{ki}P_{li}}^{P_{kl}} + T_{P_{kj}P_{lj}}^{P_{kl}} + T_{P_{ki}P_{kj}}^{P_{kl}} + T_{P_{li}P_{lj}}^{P_{kl}}$.

3. The sum $\mathcal{S}_{\mathcal{G}_{kl}^{ij}}$ of surface area of a crystal $\mathcal{G}_{kl}^{ij}$.

Theorem 3.6. *Let $\mathcal{T}$ be a tetrahedron with vertices P_1, P_2, P_3 and P_4. For each crystal $\mathcal{G}_{kl}^{ij}$ which generated by tetrahedron $\mathcal{T}$ from $\mathcal{R}_{kl}^{ij}$, then we obtain its surface areas of the crystal $\mathcal{G}_{kl}^{ij}$ equals to $\mathcal{S}_{\mathcal{G}_{kl}^{ij}} = \mathcal{S}_{ij} + \mathcal{S}_{kl}$.*

Proof. The result of this theorem is followed immediately from Theorem 3.3 and Theorem 3.5. □

4. Conclusions and Applications

In general, the calculation of the crystal surface areas generated by tetrahedron can be a tedious task. However, the calculation can be carried out easily by using the closed form of the crystal surface areas. The important conclusion of this study is that the crystal surface areas generated by tetrahedrons can be carried out using the closed form as in Theorem 3.6.

Of course, the above results can be used to solve some practical problems need to find the crystal surface areas generated by tetrahedron. We now give an example of such a problem.

Example 4.1. Let $\mathcal{T}$ be a tetrahedron with vertices P_1, P_2, P_3 and P_4. Suppose that for any distinct $q, r \in \mathbb{N}_4$, the length of all sides P_qP_r which joint between vertices P_q and P_r are equal to x units of length and setting P_{qr} is any midpoint on side P_qP_r. Calculate the surface areas $\mathcal{S}_{\mathcal{G}_{kl}^{ij}}$ of the crystal $\mathcal{G}_{kl}^{ij}$ generated from the quadrilateral $\mathcal{R}_{kl}^{ij}$ in $\mathcal{T}$.

Solution. We can analyze and solve this problem as follows: From the hypotheses for any $q, r \in \mathbb{N}_4$, in tetrahedron $\mathcal{T}$, let x_{qr} or x_{rq} be the length of side P_qP_r which joint between vertices P_i and P_j, then $x_{qr} = x$ and since P_{qr} is the midpoint of the side P_qP_r, all ratios r_{qr} are equal to $\frac{1}{2}$. Obviously, by Theorem 2.5 we see that

$$T_{P_jP_k}^{P_i} = \frac{1}{4}\sqrt{2(x_{ij}^2x_{ik}^2 + x_{ij}^2x_{jk}^2 + x_{ik}^2x_{jk}^2) - (x_{ij}^4 + x_{ik}^4 + x_{jk}^4)} = \frac{\sqrt{3}}{4}x^2. \quad (4.1)$$

By Theorem 3.3 and equation (4.1), we obtain

$$T_{P_{ik}P_{jk}}^{P_{ij}} = \frac{1}{4}T_{P_jP_k}^{P_i} = \frac{\sqrt{3}}{16}x^2, \quad T_{P_{il}P_{jl}}^{P_{ij}} = \frac{1}{4}T_{P_jP_l}^{P_i} = \frac{\sqrt{3}}{16}x^2,$$

$$T_{P_{ki}P_{li}}^{P_{kl}} = \frac{1}{4}T_{P_lP_i}^{P_k} = \frac{\sqrt{3}}{16}x^2, \quad T_{P_{kj}P_{lj}}^{P_{kl}} = \frac{1}{4}T_{P_lP_j}^{P_k} = \frac{\sqrt{3}}{16}x^2.$$

Next, on the other hand, we will calculate

$$T_{P_{ik}P_{il}}^{P_{ij}}, \quad T_{P_{jk}P_{jl}}^{P_{ij}}, \quad T_{P_{ki}P_{kj}}^{P_{kl}}, \text{ and } T_{P_{li}P_{lj}}^{P_{kl}}.$$

For $\{(q, r) : q = k, l, r = j, k\} - \{(l, j)\}$, we have

$$\mathcal{A}_{jq}^i = \left(\frac{x_{ij}}{2}\right)^2 + \left(\frac{x_{iq}}{2}\right)^2 = \frac{x^2}{2}, \quad \mathcal{B}_{jp}^i = \frac{1}{2}\sqrt{(x_{ij}x_{iq})^2 - 4(T_{P_jP_q}^{P_i})^2} = \frac{x^2}{4},$$

$$\mathcal{A}_{rl}^i = \left(\frac{x_{ir}}{2}\right)^2 + \left(\frac{x_{il}}{2}\right)^2 = \frac{x^2}{2}, \quad \mathcal{B}_{rl}^i = \frac{1}{2}\sqrt{(x_{ir}x_{il})^2 - 4(T_{P_rP_l}^{P_i})^2} = \frac{x^2}{4}.$$

For $\{(q, r): q = j, k, r = k, l\} - \{(k, k)\}$, we have

$$\mathcal{A}_{qr}^{i} = \left(\frac{x_{iq}}{2}\right)^2 + \left(\frac{x_{ir}}{2}\right)^2 = \frac{x^2}{2}, \quad \mathcal{B}_{qr}^{i} = \frac{1}{2}\sqrt{(x_{iq}x_{ir})^2 - 4(T_{P_qP_r}^{P_i})^2} = \frac{x^2}{4}.$$

Then, it infers that

$$\mathcal{C}_{P_{ik}P_{il}}^{P_{ij}} = 2\sum_{\{(q, r):q=k, l, r=j, k\}-\{(l, j)\}} (\mathcal{A}_{jq}^{i} - \mathcal{B}_{jq}^{i})(\mathcal{A}_{rl}^{i} - \mathcal{B}_{rl}^{i}) = \frac{3}{8}x^4,$$

and

$$\mathcal{D}_{P_{ik}P_{il}}^{P_{ij}} = \sum_{\{(q, r):q=j, k, r=k, l\}-\{(k, k)\}} (\mathcal{A}_{qr}^{i} - \mathcal{B}_{qr}^{i})^2 = \frac{3}{16}x^4.$$

Then, by Theorem 3.5, we deduce

$$T_{P_{ik}P_{il}}^{P_{ij}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{ik}P_{il}}^{P_{ij}} - \mathcal{D}_{P_{ik}P_{il}}^{P_{ij}}} = \frac{\sqrt{3}}{16}x^2.$$

The same calculation as above, we obtain

$$T_{P_{jk}P_{jl}}^{P_{ij}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{jk}P_{jl}}^{P_{ij}} - \mathcal{D}_{P_{jk}P_{jl}}^{P_{ij}}} = \frac{\sqrt{3}}{16}x^2,$$

$$T_{P_{ki}P_{kj}}^{P_{kl}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{ki}P_{kj}}^{P_{kl}} - \mathcal{D}_{P_{ki}P_{kj}}^{P_{kl}}} = \frac{\sqrt{3}}{16}x^2,$$

$$T_{P_{li}P_{lj}}^{P_{kl}} = \frac{1}{4}\sqrt{\mathcal{C}_{P_{li}P_{lj}}^{P_{kl}} - \mathcal{D}_{P_{li}P_{lj}}^{P_{kl}}} = \frac{\sqrt{3}}{16}x^2.$$

Then we have

$$\mathcal{S}_{ij} = T_{P_{ik}P_{jk}}^{P_{ij}} + T_{P_{il}P_{jl}}^{P_{ij}} + T_{P_{ik}P_{il}}^{P_{ij}} + T_{P_{jk}P_{jl}}^{P_{ij}} = \frac{\sqrt{3}}{4}x^2,$$

and

$$\mathcal{S}_{kl} = T_{P_{ki}P_{li}}^{P_{kl}} + T_{P_{kj}P_{lj}}^{P_{kl}} + T_{P_{ki}P_{kj}}^{P_{kl}} + T_{P_{li}P_{lj}}^{P_{kl}} = \frac{\sqrt{3}}{4}x^2.$$

Therefore, by Theorem 3.6, we conclude that the crystal surface areas generated from the quadrilateral $\mathcal{R}_{kl}^{ij}$ in $\mathcal{T}$ is $\mathcal{S}_{\mathcal{G}_{kl}^{ij}} = \mathcal{S}_{ij} + \mathcal{S}_{kl} = \frac{\sqrt{3}}{8}x^2$ square units.

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