Far East Journal of Theoretical Statistics
Published Online: March 2015
http://dx.doi.org/10.17654/FJTSJan2015_065_076
Volume 50, Number 1, 2015, Pages 65-76 **ISSN: 0972-0863**

UNIFIED CUMULATIVE SUM CHART FOR MONITORING SHIFTS IN THE PARAMETERS OF THE ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION

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Abstract

Cumulative sum (CUSUM) control chart has been proposed for detecting simultaneous shifts in the parameters of the Erlang-truncated exponential distribution. It was observed that the parameters of the CUSUM chart, that is, the lead distance and the mask angle change considerably for a slight shift in the parameters of the distribution. The average run length (ARL) of the control chart also changes considerably for a slight shift in the parameters of the distribution.

1. Introduction

The quality of a product is one fundamental thing that majority of consumers look for when purchasing a product. This makes producers keep researching into how to improve the quality of their product in order to

Received: October 15, 2014; Revised: December 27, 2014; Accepted: February 3, 2015
2010 Mathematics Subject Classification: 62K05, 62P30.
Keywords and phrases: cumulative sum chart, sequential probability ratio test, Erlang-truncated exponential distribution, lead distance, mask angle, average run length.
Communicated by Hideo Hirose

maintain old and woo consumers. Following the famous work of Shewart [10, 11], researchers in recent times are developing different quality control schemes for monitoring the quality of a product.

One of the quality control techniques that has found a niche in statistical process control, as a parallel process control technique, to the well known Shewart control charts is the cumulative sum (CUSUM) chart developed by Page [7, 8]. The CUSUM charts have the advantage of detecting small to moderate size shift from a simple acceptable quality level than the Shewart control charts.

Umpteen of researchers have studied and developed control charts for monitoring shifts in the parameters of life time distributions. Johnson and Leone [3] made use of simultaneous applications of sequential probability ratio test (SPRT) to test a simple null hypothesis against two separate simple alternative hypotheses on either sides of the null hypothesis. Edgeman [1] studied inverse Gaussian control charts. Nabar and Bilgi [6] developed a CUSUM chart for the inverse Gaussian distribution. Kantam and Rao [5] investigated the CUSUM control chart for the log-logistic distribution. Rao [9] developed a one-sided CUSUM control chart for monitoring a shift in one of the parameters of the Erlang-truncated exponential distribution (ETED) on the assumption that the other parameter is fixed and known. Parallel to the work of Rao [9], this study seeks to develop a CUSUM chart for monitoring simultaneous shifts in the parameters of the ETED.

2. Erlang-truncated Exponential Distribution

The ETED developed by El-Alosey [2] has the probability density function

$$f(x, v, \lambda) = v(1 - e^{-\lambda})\exp\{-vx(1 - e^{-\lambda})\} \tag{1}$$

for $x > 0$ and $v > 0, \lambda > 0.$

The distribution function is given by

$$F(x, v, \lambda) = 1 - \exp\{-vx(1 - e^{-\lambda})\}, \tag{2}$$

v is the shape parameter and λ is the scale parameter. The mean and variance of the ETED are $\mu = (v(1 - e^{-\lambda}))^{-1}$ and $\sigma^2 = (v(1 - e^{-\lambda}))^{-2}$, respectively.

3. Sequential Probability Ratio Test (SPRT)

Wald's [12] SPRT is a joint, subject by subject, likelihood ratio test (LRT). In this approach, each subject constitutes a stage. Let $f(x, v, \lambda)$ denote the distribution of a random variable x following and ETED. Let H_0 be the hypothesis that there is no shift in the parameters of the distribution of x and H_1 be the hypothesis that there is a shift in the parameters of the distribution of x. Thus, the distribution of x is given by $f(x, v_0, \lambda_0)$ when H_0 is true, and by $f(x, v_1, \lambda_1)$ when H_1 is true. For any positive integral value m, the probability that a sample $x_1, x_2, ..., x_m$ is obtained is given by

$$P_{1m} = \prod_{i=1}^{m} f(x_i, v_1, \lambda_1) \tag{3}$$

when H_1 is true, and by

$$P_{0m} = \prod_{i=1}^{m} f(x_i, v_0, \lambda_0) \tag{4}$$

when H_0 is true.

The SPRT for testing H_0 against H_1 is defined as follows: two positive constants A and B $(B < A)$ are chosen. At each stage of the experiment (at the mth trial for any integral value m), the probability ratio P_{1m}/P_{0m} is computed. If $B < \frac{P_{1m}}{P_{0m}} < A$, then the experiment is continued by taking additional observation. If $\frac{P_{1m}}{P_{0m}} \geq A$, then the process is terminated with the rejection of H_0. If $\frac{P_{1m}}{P_{0m}} \geq B$, then the process is terminated with the acceptance of H_0. The constants A and B are to be determined so that the test will have the prescribed strength (α, β), where α and β are the Type I and Type II errors, respectively. The constants $A = \frac{1-\beta}{\alpha}$ and $B = \frac{\beta}{1-\alpha}$.

For the purpose of practical computation, it is much more convenient to compute the logarithm of the ratio $\frac{P_{1m}}{P_{0m}}$ than the ratio itself. Thus, the continuation region can be given as $\ln B < \ln \frac{P_{1m}}{P_{0m}} < \ln A$.

4. CUSUM Chart for Controlling Parameters v and λ

The likelihood ratio test to test the null hypothesis that there is no shift in the parameters of the ETED against the alternative that there is a shift in the parameters of the ETED is given by

$$\frac{P_{1m}}{P_{0m}} = \frac{\prod_{i=1}^{m} f(x_i, v_1, \lambda_1)}{\prod_{i=1}^{m} f(x_i, v_0, \lambda_0)}, \tag{5}$$

$$\frac{P_{1m}}{P_{0m}} = \left(\left(\frac{v_1}{v_0}\right)\left(\frac{(1-e^{-\lambda_1})}{(1-e^{-\lambda_0})}\right)\right)^m \exp\left\{(v_0(1-e^{-\lambda_0}) - v_1(1-e^{-\lambda_1}))\sum_{i=1}^{m} x_i\right\}. \tag{6}$$

The continuation region of the SPRT discriminating between the two hypotheses is given by

$$\ln\left(\frac{\beta}{1-\alpha}\right) < m \ln\left\{\frac{v_1(1-e^{-\lambda_1})}{v_0(1-e^{-\lambda_0})}\right\} + \{v_0(1-e^{-\lambda_0}) - v_1(1-e^{-\lambda_1})\}\sum_{i=1}^{m} x_i$$

$$< \ln\left(\frac{1-\beta}{\alpha}\right). \tag{7}$$

Considering the mean and variance of the ETED, any shift in the parameters affects these two moments. Let v_0 and λ_0 be the target values, let $v = v_1$ $(v_1 > v_0)$ and $\lambda = \lambda_1$ $(\lambda_1 > \lambda_0)$ be the changed values due to shift in the parameters. The SPRT will stop by rejecting or accepting, or continue to sample, as $\ln \frac{P_{1m}}{P_{0m}}$ is outside or in between $\ln A$ and $\ln B$. The process stops by rejecting H_0 if $\ln \frac{P_{1m}}{P_{0m}} \geq \ln A$; this gives a rejection line $v_1 > v_0$ and $\lambda_1 > \lambda_0$. Similarly, if we employ SPRT with the same strength to the cases $v_1 < v_0$ and $\lambda_1 < \lambda_0$, then another rejection line is obtained. These two rejection lines give a geometrical nature of masking. The observations in the sample enter the mask in a sequential way. Thus, the mask for the CUSUM chart was developed. For $\ln \frac{P_{1m}}{P_{0m}} \geq \ln A$ and considering equation (7),

$$\sum_{i=1}^{m} x_i \geq \frac{\ln \alpha + m \ln \left\{ \frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})} \right\}}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}$$

$$\Rightarrow \sum_{i=1}^{m} x_i \geq C + mD, \tag{8}$$

where

$$C = \frac{\ln \alpha}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}$$

and

$$D = \frac{\ln \left\{ \frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})} \right\}}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}.$$

Similarly, the rejection line, when $\ln \frac{P_{1m}}{P_{0m}} \leq \ln A,$ is given by

$$\sum_{i=1}^{m} x_i \leq \frac{\ln \alpha + m \ln\left\{\frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})}\right\}}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}$$

$$\Rightarrow \sum_{i=1}^{m} x_i \leq C^* + mD^*, \tag{9}$$

where

$$C^* = \frac{\ln \alpha}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}$$

and

$$D^* = \frac{\ln\left\{\frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})}\right\}}{v_1(1 - e^{-\lambda_1}) - v_0(1 - e^{-\lambda_0})}.$$

Equations (8) and (9) form the regions above and below the plane $\left(m, \sum_{i=1}^{m} x_i\right)$. If m is allowed sequentially, at some stage $\sum_{i=1}^{m} x_i$ satisfies either equation (8) or equation (9). Till then, the procedure continues. Using the slopes and intercepts of the two lines (equations (8) and (9)), the parameters of the CUSUM chart, called the *angle* and *lead distance*, are obtained. From Figure 1, $\tan \theta_1 =$ slope of the line $P_1Q_1 = D,$ and the lead distance OP_1 is $d_1,$ where

$$d_1 = \frac{-\ln \alpha}{\ln\left\{\frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})}\right\}}$$

when $v_1 > v_0$ and $\lambda_1 > \lambda_0$.

$\tan \theta_{-1}$ = slope of the line $P_{-1}Q_{-1} = D^*$, and the lead distance OP_{-1} is d_{-1}, where

$$d_{-1} = \frac{-\ln \alpha}{\ln\left\{\frac{v_1(1 - e^{-\lambda_1})}{v_0(1 - e^{-\lambda_0})}\right\}}$$

when $v_1 < v_0$ and $\lambda_1 < \lambda_0$.

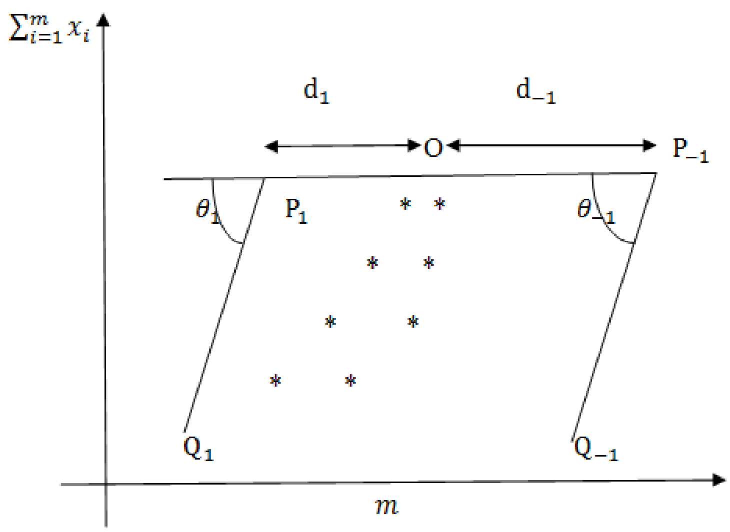

Figure 1. CUSUM control chart.

Let $x_1, x_2, ..., x_m$ be a sample from ETED. If the points $\left(m, \sum_{i=1}^m x_i\right)$ are plotted with a suitable scale, then the ordinates of the points represent the cumulative sum of the data. Equations (8) and (9) are the consequences of a shift in the population parameters v and λ. Figure 1 indicates a considerable shift in the population parameters if $\sum_{i=1}^m x_i$ falls outside the lines P_1Q_1 and $P_{-1}Q_{-1}$. The chart is interpreted by placing the mask over the chart as shown in Figure 1 with point O over the last plotted point on the chart with line

P_1P_{-1} parallel to the axis m. If any of the points lies below $P_{-1}Q_{-1}$, then it is an indication of a decrease in v and λ and if any of the points lies above P_1Q_1, then it indicates an increase in v and λ. The graph of the points $\left(m, \sum_{i=1}^{m} x_i\right)$ is superimposed with the mask as shown in Figure 1 such that the point O of the mask and each point of the CUSUM chart coincide with the horizontal line parallel to the m axis.

The values of the parameters angle θ and the lead distance d, for various values of α and choices of v_0, v_1, λ_0, λ_1 and $\beta = 0$ have been tabulated in Table 1 and Table 2 for ready reference. From Table 1, it is obvious that as the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$ increase, the values of the lead distance d decrease for fixed value of α. Also, the smaller the value of α the higher the value of d. Similarly, the value of angle θ decreases as the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$ increase as shown in Table 2.

Table 1. Values of the lead distance d for controlling λ and v

				α		
λ_0	λ_1	v_0	v_1	0.1	0.05	0.01
0.60	0.65	0.70	0.75	18.18	23.66	36.37
0.60	0.70	0.70	0.80	9.47	12.33	18.95
0.60	0.75	0.70	0.85	6.57	8.54	13.13
0.60	0.8	0.70	0.90	5.11	6.65	10.22
0.60	0.85	0.70	0.95	4.24	5.51	8.47
0.60	0.90	0.70	1.00	3.65	4.75	7.30
0.60	0.95	0.70	1.05	3.23	4.21	6.46
0.60	1.00	0.70	1.10	2.92	3.80	5.84

Table 2. Values of angle θ for controlling λ and v

λ_0	λ_1	v_0	v_1	θ
0.60	0.65	0.70	0.75	71.39
0.60	0.70	0.70	0.80	70.33
0.60	0.75	0.70	0.85	69.28
0.60	0.80	0.70	0.90	68.25
0.60	0.85	0.70	0.95	67.24
0.60	0.90	0.70	1.00	66.24
0.60	0.95	0.70	1.05	65.27
0.60	1.00	0.70	1.10	64.32

5. Average Run Length (ARL)

The ARL is the average number of trials required to detect a shift in the process average for the first time. According to Johnson [4] and Johnson and Leone [3], if α is the producer risk (Type I error), then the approximate ARL for detecting a shift in the parameters from λ_0 to λ_1 and from v_0 to v_1 is given by

$$ARL = \frac{-\ln \alpha}{E(\ln Z)_{\lambda=\lambda_1,\, v=v_1}},$$

where $Z = \dfrac{f(x, v_1, \lambda_1)}{f(x, v_0, \lambda_0)}$.

Therefore, the ARL is given by

$$ARL = \frac{-\ln \alpha}{\ln\left\{\dfrac{v_1(1-e^{-\lambda_1})}{v_0(1-e^{-\lambda_0})}\right\} - \left\{1 - \left(\dfrac{v_0}{v_1}\right)\left(\dfrac{1-e^{-\lambda_0}}{1-e^{-\lambda_1}}\right)\right\}}. \tag{10}$$

Table 3 provides the ARL for various values of α and choices of v_0, v_1, λ_0 and λ_1. From Table 3, it is clear that as the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$ increase, the values of the ARL decrease for a fixed value of α. Also, the smaller the value of α the higher the ARL.

Table 3. Average run length (ARL) for controlling parameters λ and v

				α		
λ_0	λ_1	v_0	v_1	0.1	0.05	0.01
0.60	0.65	0.70	0.75	299.47	389.62	598.95
0.60	0.70	0.70	0.80	84.39	109.80	168.78
0.60	0.75	0.70	0.85	41.95	54.58	83.90
0.60	0.80	0.70	0.90	26.21	34.11	52.43
0.60	0.85	0.70	0.95	18.53	24.11	37.06
0.60	0.90	0.70	1.00	14.13	18.39	28.27
0.60	0.95	0.70	1.05	11.35	14.77	22.70
0.60	1.00	0.70	1.10	9.46	12.31	18.92

6. Hypothetical Example

In this section, the application of the proposed CUSUM chart has been demonstrated using a hypothetical data simulated from the ETED. The first ten observations were simulated with $v_0 = 2.0$ and $\lambda_0 = 0.5$. The last five observation were simulated with $v_1 = 3.5$ and $\lambda_1 = 1.5$. Table 4 displays the simulated data and their cumulative sum. The parameters of the V-mask were calculated using $v_0 = 2.0$, $v_1 = 3.5$, $\lambda_0 = 0.5$ and $\lambda_1 = 1.5$. The lead distance and the angle of the mask were 2.4 and 33°, respectively. The sample number (m) was plotted against the cumulative sum of the data. The point O of the V-mask was then placed at the last plotted point to monitor whether the process is in control as shown in Figure 2. Clearly, from

Figure 2, the process was out of control as observations 1 to 13 fell below line $P_{-1}Q_{-1}$ indicating a decrease in ν and λ. This calls for an action to be taken in order to bring the process under control.

Table 4. Simulated hypothetical data

Sample number (m)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Data	2.7	2.9	0.8	0.8	0.2	0.5	0.8	0.3	1.8	1.9	3.6	0.9	1.7	1.8	1.2
Cumulative sum	2.7	5.6	6.3	7.1	7.3	7.8	8.6	8.9	9.7	11.6	15.1	16	17.7	19.5	20.7

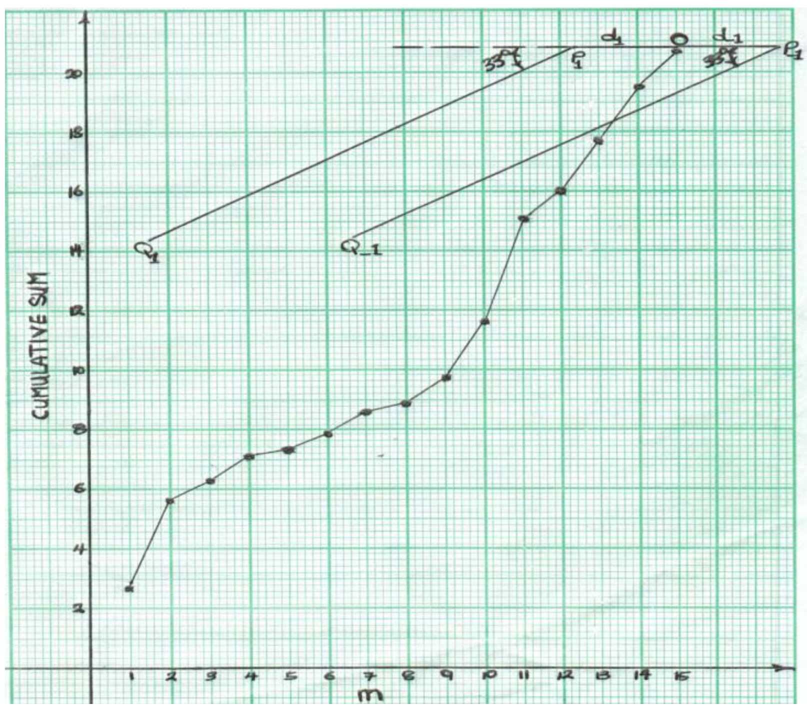

Figure 2. CUSUM plot of simulated data.

7. Conclusion

In this study, a CUSUM chart for monitoring simultaneous shift in the parameters of an ETED has been proposed and various parameters of the CUSUM chart estimated. The results revealed that for an increase in the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$, the values of the lead distance d decrease for fixed value of α. Also, the smaller the value of α the higher the value of d. Similarly, the value of angle θ decreases as the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$ increase. In addition, as the ratios $\frac{\lambda_1}{\lambda_0}$ and $\frac{v_1}{v_0}$ increase, the values of the ARL decrease for a fixed value of α.

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