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COUNTABLE LOWER SEMILATTICES WHOSE SET OF JOIN-REDUCIBLE ELEMENTS IS WELL-ORDERED

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Abstract

In our previous papers [2] and [3], we have introduced the notion of special semigroups. They are commutative Boolean semigroups that can be built from a trivial semigroup in finitely or infinitely many steps, where in each step an idempotent is adjoined in one of the three allowed ways. In our paper [4], we characterized finite special semigroups. In this paper, we show that some countable special semigroups correspond to the lower semilattices whose set of join-reducible elements is well-ordered. In that way, we characterize these lattices.

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1. Introduction and Preliminaries

In Section 1, we briefly discuss the content of the paper. Then we introduce some notations and preliminary notions.

In Section 2, we discuss the notion of special semigroups. (This notion was introduced in our previous papers [2] and [3].) They are commutative Boolean semigroups that can be built from a trivial semigroup in finitely or infinitely many steps, where in each step an idempotent is adjoined in one of the three allowed ways. The ith step of the building process is completely described by an element a_i of the set $\{0, 1, ..., i-1, \text{‘id’}, \text{‘ze’}\}$ of $i+2$ elements. The sequence $\mathbf{a} = a_0, a_1, a_2, ...,$ obtained in that way, is called a *defining sequence* of the semigroup. It follows that the special semigroups form a large class of semigroups each of which can be easily defined. They can be arbitrarily big, nevertheless, checking of associativity is completely avoided. Because of that, they can be useful for various purposes.

In Section 2, we also discuss the notion of join-reducible elements of a lower semilattice.

In Section 3, in our main theorem, we show that some countable special semigroups correspond to the lower semilattices whose set of join-reducible elements is well-ordered (either finite, or of type ω). In that way, we characterize these lattices.

In Section 4, we give some examples which illustrate the conditions appearing in our main theorem. We also raise a question.

Now we introduce some notations and preliminaries.

All the notions or facts related to semigroups and partially ordered sets (posets), that we use but not define or state in this paper, can be found in [1].

The symbol ω will denote the smallest infinite ordinal number.

A nonempty set S with an associative binary operation is called a *semigroup*. The operation is multiplication and is denoted by $\cdot$, which is usually omitted. The *trivial semigroup* S_0 consists of a single element s_0, with the only possible operation $s_0 s_0 = s_0$. An element *e* of the semigroup *S* is called the *identity* if $se = es = s$ for all $s \in S$. An element *z* of *S* is called the *zero* if $sz = zs = z$ for all $s \in S$. It is usually denoted by 0. A semigroup *S* is called *Boolean* if $s^2 = s$ for every $s \in S$.

If $x \leq y$ are two elements of a poset $(P, \leq)$, then the *interval* $[x, y]$ is the set of all elements $z \in P$ such that $x \leq z \leq y$. The poset *P* is said to be *locally finite* if every interval of it is finite.

Let $(P, \leq)$ be a poset. For any two elements $x, y \in P$, we say that *x covers y* if $x < y$ and there is no element $z \in P$ such that $x < z < y$. We associate to $(P, \leq)$ the *covering relation* $\mathcal{C}(P)$, defined in the following way: $(x, y) \in \mathcal{C}(P)$ if and only if *y* covers *x*.

Remark 1.1. The *Hasse diagram* of a finite poset $(P, \leq)$ is a directed graph $\mathcal{H}(P)$ with the property that there is a directed edge from *x* to *y* if and only if *y* covers *x*. Furthermore, when drawing the directed graph on a Euclidean plane, instead of drawing a directed segment from *x* to *y*, we draw a segment without an arrow but putting *x* lower than *y*. We will use the notation $x - y$ to denote an edge in the Hasse diagram. This assumes $x < y$. Also, we will use the notation $x_1 - x_2 - \cdots - x_n$ to denote a path in the Hasse diagram. If *C* is a subgraph of the Hasse diagram with the greatest element *x*, then $C - y$ denotes the graph obtained by adding to *C* the directed edge $x - y$, assuming that *y* covers *x*. Similarly, we use the notation $y - C$ for the graph obtained by adding the directed edge $y - x$ to the subgraph *C* of the Hasse diagram, assuming that *C* has the least element *x* and that *x* covers *y*.

Let *a*, *b* be two elements of *P*. An element $d \in P$ is called the *meet* of *a* and *b* (notation $a \wedge b$) if $d \leq a$, $d \leq b$ and $d \geq e$ for any element $e \in P$ such that $e \leq a$ and $e \leq b$. If for any elements $a, b \in P$ the meet $a \wedge b$ exists, then *P* is called a *lower semilattice*.

Lemma 1.2 [1, Proposition 1.3.2]**.** *Let* $(E, \cdot)$ *be a Boolean commutative semigroup. Then the relation* $\leq$ *on E defined by*

$$a \leq b \text{ if } ab = a$$

is a partial order on E with respect to which $(E, \leq)$ *is a lower semilattice. In* $(E, \leq)$, *the meet of a and b is their product ab.*

Conversely, let $(E, \leq)$ *be a lower semilattice. Then* $(E, \wedge)$ *is a Boolean commutative semigroup and*

$$(\forall a, b \in E)\ a \leq b \text{ if and only if } a \wedge b = a.$$

Thus, to quote [1, page 15], the notions of lower semilattice and Boolean commutative semigroup are equivalent and interchangeable.

Given a Boolean commutative semigroup $(E, \cdot)$, we say that the partial order $\leq$ on E, defined in the above lemma, is the *natural partial order* on E and that $(E, \leq)$ is the *natural semilattice* associated to the semigroup E. Conversely, if $(E, \leq)$ is a lower semilattice, then we say that the semigroup $(E, \cdot = \wedge)$ is the natural semigroup associated to $(E, \leq)$. Thus, we have the *natural correspondence* $(E, \cdot) \leftrightarrow (E, \leq)$ between commutative Boolean semigroups and lower semilattices.

2. Construction and Basic Properties of Special Semigroups; Join-reducible Elements

To any semigroup S (which might even be a monoid), we can adjoin a new element x and define the operation on $S \cup \{x\}$ by adding the rules $xx = x$ and $sx = xs = s$ for all $s \in S$ to already existing multiplication on S. Then the element x is the identity element of the new semigroup $S \cup \{x\}$ and this process is called the *adjoining of an identity*. If the operation on $S \cup \{x\}$ is defined by adding the rules $xx = x$ and $sx = xs = x$ for all $s \in S$ to already existing multiplication on S, then the element x is the zero element of the new semigroup $S \cup \{x\}$ and this process is called the *adjoining of a zero*.

We will consider one more type of adjoining. Namely, let S be a semigroup and let t be an element of S. We adjoin to S a new element x (which "behaves like t") and define the operation on $S \cup \{x\}$ by adding the rules $xx = x$ and $xs = ts$, $sx = st$ for all $s \in S$ to already existing multiplication on S. Then $S \cup \{x\}$ is a semigroup and x is an idempotent in it.

We start with the trivial semigroup $S_0 = \{s_0\}$. We will adjoin a new element s_1 to S_0 and obtain a new semigroup $S_1 = \{s_0, s_1\}$, we will then adjoin a new element s_2 to S_1 and obtain a new semigroup $S_2 = \{s_0, s_1, s_2\}$, etc. Which of the three described types of adjoining we have each time is described by a sequence $\mathbf{a} = a_0, a_1, a_2, \ldots,$ which is defined in the following way:

$a_0 =$ 'ze' or 'id', meaning that s_0 is adjoined as zero or as identity to the empty set; both values result in the same semigroup S_0;

$a_n \in \{$'id', 'ze', $0, 1, 2, \ldots, n-1\}$, $n = 1, 2, 3, \ldots;$ here $a_n =$ 'ze' means that s_n is *adjoined to* S_{n-1} as *zero*, $a_n =$ 'id' means that s_n is *adjoined to* S_{n-1} *as identity*, and $a_n = i \in \{0, 1, 2, \ldots, n-1\}$ means that s_n is *adjoined to* S_{n-1} *as an idempotent* "*behaving like* s_i". (This means that the new rules on $S_n = S_{n-1} \cup \{s_n\}$, added to the already existing multiplication on S_{n-1}, are: $s_n s_n = s_n$, $s_n s = s_i s$, $s s_n = s s_i$, for all $s \in S_{n-1}$.)

In this way to any sequence $\mathbf{a} = a_0, a_1, a_2, \ldots,$ we uniquely associate a semigroup S_ω, whose underlying set is $\{s_0, s_1, s_2, \ldots\}$. We say that a sequence $\mathbf{a} = a_0, a_1, a_2, \ldots$ is a *defining sequence* of the semigroup associated to it. We call any semigroup S_ω a *special semigroup of type* ω.

Observe that all the special semigroups S_N $(N \geq 0)$ and S_ω are commutative Boolean semigroups. However, not all commutative Boolean semigroups are special. So a natural question that arises is to characterize all commutative Boolean semigroups that are special. That is of interest because of the specific way in which special semigroups are defined (built up). Finite special commutative Boolean semigroups are characterized in our paper [3]. We give a characterization of the infinite ones of type ω in this paper in Section 3.

Remark 2.1. In order to construct an S_ω, one has to construct all the intermediate S_N, $N = 0, 1, 2, \ldots$. So this remark about the Hasse diagrams, and the comment after it, can be very useful for understanding the main theorem of the paper.

Let $S = S_N$ be a special semigroup and $i \leq N$. Then the Hasse diagram $\mathcal{H}(S_i)$ of the special semigroup $S_i = \{s_0, s_1, \ldots, s_i\}$ is the same as the induced subgraph of $\mathcal{H}(S_N)$ on $\{s_0, s_1, \ldots, s_i\}$.

Based on the previous remark, we describe how to construct the Hasse diagram of a special semigroup $S = S_N$. For $\{s_0\}$, we put a point. Then for $i = 1, 2, \ldots, N$, we repeat the following procedure. If s_i is added to $\{s_0, s_1, \ldots, s_{i-1}\}$ with $a_i = k \in \{0, 1, \ldots, n-1\}$, then, to get the Hasse diagram of $\{s_0, s_1, \ldots, s_i\}$, we add the segment $a_k - a_i$ to the Hasse diagram of $\{s_0, s_1, \ldots, s_{i-1}\}$. If s_i is added to $\{s_0, s_1, \ldots, s_{i-1}\}$ with $a_i =$ 'id', then, to get the Hasse diagram of $\{s_0, s_1, \ldots, s_i\}$, we add all the segments $b - s_i$ to the Hasse diagram of $\{s_0, s_1, \ldots, s_{i-1}\}$, where b is any maximal element of $\{s_0, s_1, \ldots, s_{i_1}\}$, i.e., where b is any uncovered element of the Hasse diagram of $\{s_0, s_1, \ldots, s_{i-1}\}$. Finally, if s_i is added to $\{s_0, s_1, \ldots, s_{i-1}\}$ with $a_i =$ 'ze', then, to get the Hasse diagram of $\{s_0, s_1, \ldots, s_i\}$, we add the segment $s_i - a$ to the Hasse diagram of $\{s_0, s_1, \ldots, s_{i-1}\}$, where a is the least element of $\{s_0, s_1, \ldots, s_{i-1}\}$.

Definition 2.2. Let $(P, \leq)$ be a poset. An element $x \in P$ is said to be *join-reducible* if the following condition holds:

$$(\exists Y \subset P)\ x = \sup(Y) \text{ and } x \notin Y.$$

Otherwise, x is called *join-irreducible*.

Note that if P has the least element, say a, then the element a is join-reducible since $a = \sup(\varnothing)$ and $a \notin \varnothing$.

Remark 2.3. An element s_i is join-reducible in $S = S_N$ if and only if it is join-reducible in some/every S_k with $k \geq i$.

Indeed, it is enough to observe that if an element a is join-reducible in $\{s_0, ..., s_{i-1}\}$, then it is join-reducible in $\{s_0, ..., s_i\}$. That follows from the fact that $\mathcal{C}(\{s_0, ..., s_{i-1}\})$ is equal to the covering relation $\mathcal{C}(\{s_0, ..., s_i\})$ plus one or more elements added. So if for some element $a \in \{s_0, ..., s_{i-1}\}$, there are elements $b, c \in \{s_0, ..., s_{i-1}\}$ with $(a, b), (a, c) \in \mathcal{C}(\{s_0, ..., s_{i-1}\})$, then $(a, b), (a, c) \in \mathcal{C}(\{s_0, ..., s_i\})$.

3. Main Theorem

Definition 3.1. Let $(S, \leq)$ be a partially ordered set. A *path* in S is a finite sequence $x_1 > x_2 > \cdots > x_n$ such that x_i covers x_{i+1} for $i = 1, 2, ..., n-1$.

Lemma 3.2. *Let $(S, \leq)$ be a locally finite partially ordered set, $x, y, z \in S$. Let $y < x$ and $z < x$ and suppose that y and z are incomparable. Then there is a join-reducible element u in $[y, x] \cup [z, x]$ and $y < u,\ z < u$.*

Note that this lemma holds in greater generality, but for the purpose of this paper, this version is enough.

Proof. We may assume that S is the union $[y, x] \cup [z, x]$. Then we prove by induction on the total number of elements of S. If x is join-reducible, then we are done. Suppose x is not join-reducible. Let x' be the element covered by x (it exists since x is the greatest element in S). Then $x' > y$ and $x' > z$, so we may apply the inductive hypothesis to $S' = [y, x'] \cup [z, x']$. □

Theorem 3.3. *Every special semigroup* $S = S_\omega$ (*considered as a lower semilattice in the natural way*) *has the following four properties*:

(i) *the set of join-reducible elements of S is well-ordered*, *either finite or of type* ω;

(ii) *if there are infinitely many join-reducible elements*, *every element of S is smaller than some join-reducible element*;

(iii) *every element of S covers finitely many elements*;

(iv) *S is locally finite*.

Conversely, *if* $(S, \leq)$ *is a countable lower semi-lattice which satisfies the above four conditions*, *then S can be realized as a special semigroup* S_ω.

Proof. $(\Rightarrow)$ Since $S = S_\omega = \{s_0, s_1, s_2, \ldots\}$, the conditions (i) and (ii) are obviously satisfied.

Let us show that the condition (iii) holds. When an element s_i of S_ω is added as a zero, it does not cover any element at the moment when it is adjoined. At the moment when for the first time (if at all) another element s_j is added as a zero, then s_i covers s_j and in the final semigroup S_ω, s_j is the only element covered by s_i. When an element s_i is added as an element behaving like another element s_t, then s_i covers s_t and in the final semigroup S_ω, s_t is the only element covered by s_i. When an element s_i is added as an identity, it covers finitely many elements (some of $s_0, s_1, \ldots, s_{i-1}$) and they are the only elements covered by s_i in the final semigroup S_ω. This proves that the condition (iii) holds.

Let us show that the condition (iv) holds. Let s_t and s_k be two elements of S_ω such that $s_t < s_k$. (Note that t can be $\geq$ or $\leq k$.) The only elements s_l of S_ω with $l > k$, that are $< s_k$ are those that are added as zeros, but they are $< s_t$, so they are not in the interval $[s_t, s_k]$. So in the interval $[s_t, s_k]$ we can have only elements s_i such that $i \leq \max\{t, k\}$. This proves that the condition (iv) holds.

($\Leftarrow$) (1) Let r_0 be the smallest join-reducible element. If r_0 has no predecessors, then put $p_0 = r_0$. Otherwise, the set of predecessors of r_0 is finite (by (iii)). Let p_0 be their infimum. By (iv), the interval $[p_0, r_0]$ is finite, hence $[p_0, r_0)$ is a finite tree.

(2) Label $s_0 = p_0$ (as if we are starting creating S_ω by adding s_0 as a zero). Then label all the elements of the tree $[p_0, r_0)$ as if we are continuing with creating S_ω by adding new elements behaving like already existing elements: $s_1, s_2, \ldots, s_{n_0-1}$. Then label r_0 (if $r_0 \neq p_0$) by s_{n_0} as if we are adding a new element to S_ω as an identity.

(3) We claim that all the elements of S, smaller than r_0, are either in $[p_0, r_0)$, or on a unique decreasing path $DP(p_0)$, starting at p_0. (Note that $[p_0, r_0)$ might be just p_0 and that the unique decreasing path might be finite or infinite, including the possibility that it just consists of the vertex p_0.)

(4) It is enough to show that for every element $x < r_0$, x is comparable with p_0. (Then if $x \geq p_0$, $x \in [p_0, r_0]$; if $x < p_0$, $x \in DP(p_0)$.) Suppose $x < r_0$ is incomparable with p_0. Let y be a predecessor of r_0 which is $\geq x$. Then, by Lemma 3.2, there is a join-reducible element in $[p_0, y] \cup [x, y]$, contradicting the assumption that r_0 is the smallest join-reducible element.

(5) If $DP(p_0)$ does not consist just of p_0, then we label by s_{n_0+1} the next element (after p_0) on that path as if we are adding a new element to S_ω as a zero.

(6) If s_{n_0+1} is added to S_ω, then we denote $p_0' = s_{n_0+1}$, otherwise, $p_0' = p_0$.

(7) Now all the elements of S smaller than r_0 are either in $[p_0', r_0]$ or on a unique decreasing path $DP(p_0')$ starting at p_0'. (This follows from (3).)

(8) If there are more join-reducible elements, then let r_1 be the next one. Let p_1 be the infimum of the predecessors of r_1. (The set of predecessors is finite by (ii).)

(9) We claim that p_1 is comparable with r_0. (Suppose to the contrary. Let π be a predecessor of r_1 which is between r_0 and r_1. Then $\pi \neq r_0$, otherwise p_1 would be comparable with r_0. Hence, by Lemma 3.2, there is a join-reducible element in $[p_1, \pi] \cup [r_0, \pi]$, bigger than p_1 and r_0. This contradicts the assumption that r_1 is the next join-reducible element after r_0.)

(10) It follows from (7) and (9) that there are three possibilities for p_1: (a) $p_1 > r_0$; (b) $p_1 \in [p_0', r_0]$; and (c) $p_1 < p_0'$.

(11) (a) If $p_1 > r_0$, then there is a unique decreasing path from p_1 to r_0. The set of leaves of the tree $[r_0, r_1)$ (finite by (iv)) coincides with the set of the predecessors of r_1. We label the non-root vertices of that tree (a finite set by (iii)) by $s_{n_0+1}, \ldots, s_{n_1-1}$ or by $s_{n_0+2}, \ldots, s_{n_1-1}$ (depending on whether $p_0' = p_0$ or $p_0' < p_0$) as if we are adding new elements to S_ω behaving as already existing elements.

(12) (b) If $p_1 \in [p_0', r_0]$, then the vertices from $[p_1, r_1] \cap [p_0', r_0]$ are roots of the trees (some of which are possibly just roots) of a forest whose set of leaves coincides with the set of predecessors of r_1. We label the non-root vertices of this forest (a finite set by (iii)) by $s_{n_0+1}, \ldots, s_{n_1-1}$ or by $s_{n_0+2}, \ldots, s_{n_1-1}$ (depending on whether $p_0' = p_0$ or $p_0' < p_0$) as if we are adding new elements to S_ω behaving as already existing elements.

(13) (c) If $p_1 \leq p_0'$, then there is a unique decreasing path from p_0' to p_1. We label the elements of this path by $s_{n_0+1}, \ldots, s_{n_0+m_1}$, or by $s_{n_0+2}, \ldots, s_{n_0+m_1}$ (depending on whether $p_0' = p_0$ or $p_0' < p_0$) as if we are adding new elements to S_ω as zeros. (If $p_1 = p_0'$, then $m_1 = 1$ in the first case and $m_1 = 2$ in the second case.) The vertices from $[p_1, r_1)$ are roots of the trees (some of which are possibly just roots) of a forest whose set of leaves coincides with the set of predecessors of r_1. We label the non-root vertices of this forest by $s_{n_0+m_1+1}, \ldots, s_{n_1-1}$ as if we are adding new elements to S_ω behaving as already existing elements. Then label r_1 by s_{n_1} as if we are adding a new element to S_ω as an identity.

(14) We claim that all the elements of S, smaller than r_1, are either in $[p_1, r_1)$, or on a unique decreasing path $DP(p_1)$, starting at p_1.

(15) It is enough to show that for every element $x < r_1$, x is comparable with p_1. (Then if $x \geq p_1$, $x \in [p_1, r_1]$; if $x < p_1$, $x \in DP(p_1)$.) Suppose $x < r_1$ is incomparable with p_1. Let y be a predecessor of r_1 which is $\geq x$. Then, by Lemma 3.2, there is a join-reducible element in $[p_1, y] \cup [x, y]$. Hence $x \leq r_0$. However, all the elements comparable with r_0 are either in $[p_0, r_0]$ or on $DP(p_0)$. In either case, x is comparable with p_1, a contradiction.

(16) If $DP(p_1)$ does not consist just of p_1, then we label by s_{n_1+1} the next element (after p_1) on that path as if we are adding a new element to S_ω as a zero.

(17) If s_{n_1+1} is added to S_ω, then we denote $p_1' = s_{n_1+1}$, otherwise, $p_1' = p_1$.

(18) We now proceed by induction. Suppose that we have labeled the elements of the interval $[p_{k-1}', r_{k-1}]$ and that r_k is the next join-reducible element. Let p_k be the infimum of the predecessors of r_k. (The set of predecessors is finite by (ii).)

(19) We claim that p_k is comparable with r_{k-1}. (Suppose to the contrary. Let π be a predecessor of r_k which is between r_{k-1} and r_k. Then $\pi \neq r_{k-1}$, otherwise, p_k would be comparable with r_{k-1}. Hence, by Lemma 3.2, there is a join-reducible element in $[p_k, \pi] \cup [r_{k-1}, \pi]$, bigger than p_k and r_{k-1}. This contradicts the assumption that r_k is the next join-reducible element after r_{k-1}.)

(20) It follows from the analogue of (14) and from (19) that there are three possibilities for p_k: (a) $p_k > r_{k-1}$; (b) $p_k \in [p_{k-1}', r_{k-1}]$; and (c) $p_k < p_{k-1}'$.

(21) (a) If $p_k > r_{k-1}$, then there is a unique decreasing path from p_k to r_{k-1}. The set of leaves of the tree $[r_{k-1}, r_k)$ (finite by (iv)) coincides with the set of the predecessors of r_k. We label the non-root vertices of that tree (a finite set by (iii)) by $s_{n_{k-1}+1}, \ldots, s_{n_k-1}$ or by $s_{n_{k-1}+2}, \ldots, s_{n_k-1}$ (depending on whether $p_{k-1}' = p_{k-1}$ or $p_{k-1}' < p_{k-1}$) as if we are adding new elements to S_ω behaving as already existing elements.

(22) (b) If $p_k \in [p_{k-1}', r_{k-1}]$, then the vertices from $[p_k, r_k] \cap [p_{k-1}', r_{k-1}]$ are roots of the trees (some of which are possibly just roots)

of a forest whose set of leaves coincides with the set of predecessors of r_k. We label the non-root vertices of this forest (a finite set by (iii)) by $s_{n_{k-1}+1}, \ldots, s_{n_k-1}$, or by $s_{n_{k-1}+2}, \ldots, s_{n_k-1}$ (depending on whether $p'_{k-1} = p_{k-1}$ or $p'_{k-1} < p_{k-1}$) as if we are adding new elements to S_ω behaving as already existing elements.

(23) (c) If $p_k \leq p'_{k-1}$, then there is a unique decreasing path from p'_{k-1} to p_k. We label the elements of this path by $s_{n_{k-1}+1}, \ldots, s_{n_{k-1}+m_1}$, or by $s_{n_{k-1}+2}, \ldots, s_{n_{k-1}+m_1}$ (depending on whether $p'_{k-1} = p_{k-1}$ or $p'_{k-1} < p_{k-1}$) as if we are adding new elements to S_ω as zeros. (If $p_k = p'_{k-1}$, then $m_k = 1$ in the first case and $m_k = 2$ in the second case.) The vertices from $[p_k, r_k)$ are roots of the trees (some of which are possibly just roots) of a forest whose set of leaves coincides with the set of predecessors of r_k. We label the non-root vertices of this forest by $s_{n_{k-1}+m_k+1}, \ldots, s_{n_k-1}$ as if we are adding new elements to S_ω behaving as already existing elements. Then label r_k by s_{n_k} as if we are adding a new element to S_ω as an identity.

(24) We claim that all the elements of *S*, smaller than r_k, are either in $[p_k, r_k)$ or on a unique decreasing path $DP(p_k)$, starting at p_k.

(25) It is enough to show that for every element $x < r_k$, x is comparable with p_k. (Then, if $x \geq p_k$, $x \in [p_k, r_k]$; if $x < p_k$, $x \in DP(p_k)$.) Suppose $x < r_k$ is incomparable with p_k. Let y be a predecessor of r_k which is $\geq x$. Then, by Lemma 3.2, there is a join-reducible element in $[p_k, y] \cup [x, y]$. Hence $x \leq r_{k-1}$. However, all the elements comparable with r_{k-1} are either in $[p_{k-1}, r_{k-1}]$ or on $DP(p_{k-1})$. In either case, x is comparable with p_k, a contradiction.

(26) If $DP(p_k)$ does not consist just of p_k, then we label by s_{n_k+1} the next element (after p_k) on that path as if we are adding a new element to S_ω as a zero.

(27) If s_{n_k+1} is added to S_ω, then we denote $p'_k = s_{n_k+1}$, otherwise, $p'_k = p_k$.

(28) Now we consider two possibilities: (A) there are finitely many join-reducible elements; and (B) there are infinitely many join-reducible elements.

(29) (A) Suppose there are finitely many join-reducible elements. We consider the situation after we label all of them in the above inductive procedure (1)-(27). Let r_t be the last join-reducible element. The remaining (non-labeled) elements of S form a forest whose set of roots is $[p'_t, r_t] \cup DP(p'_t)$. (Some of the trees of this forest might be just roots.) Let us list these trees (before that let us denote $N = n_t$ or $N = n_t + 1$, depending on whether $p'_t < p_t$ or $p'_t = p_t$): $T_0, T_1, \ldots, T_N, T_{N+1}, T_{N+2}, \ldots$. Here $T_{N+1}, T_{N+2}, \ldots$ are trees with roots $v_1, v_2, \ldots$, where $v_1, v_2, \ldots$ are vertices on $DP(p'_t)$ after p'_t in decreasing order (from some point on these vertices and the trees might not exist). We now label the remaining vertices in the following order:

- label one leaf on each of $T_0, \ldots, T_N$ (if they have unlabeled leaves);
- label v_1;
- label one leaf on each of $T_0, \ldots, T_{N+1}$, (if they have unlabeled leaves);
- label v_2;
- label one leaf on each of $T_0, \ldots, T_{N+2}$ (if they have unlabeled leaves);
- label v_3;
- label one leaf on each of $T_0, \ldots, T_{N+3}$ (if they have unlabeled leaves);
- ...

In this way, all the elements of S will be labeled and we will finish the creation of S_ω.

(30) (B) Suppose there are finitely many join-reducible elements. In each step of the above inductive procedure (1)-(27), we finished with the labeling of the subsemigroup $[p'_k, r_k]$ of S. The semigroup $[p'_{k-1}, r_{k-1}]$ is a subsemigroup of $[p'_k, r_k]$ for every k, so $\bigcup_{k\in\mathbb{N}} [p'_k, r_k]$ is a subsemigroup of the intended semigroup S_ω. We claim that, in fact, $S_\omega = \bigcup_{k\in\mathbb{N}}[p'_k, r_k]$. This follows from the assumed condition (ii). Thus, all the elements of S will be labeled and we will finish the creation of S_ω in this case as well.

The proof of the theorem is finished. □

4. Some Examples

The following examples show that the four conditions in our main theorem are independent.

Example 4.1. Let $S = \{a_1, a_2, a_3, ...\} \cup \{a_\omega\}$ be a lower semilattice in which $a_i < a_j$ for every $i < j$ and $a_i < a_\omega$ for every i. This semigroup satisfies the conditions (i), (ii) and (iii), but not (iv). It cannot be realized as an S_ω (but it can as an $S_{\omega+1}$).

Example 4.2. Let $S = \{a\} \cup \{b_1, b_2, b_3, ...\} \cup \{c_1, c_2, c_3, ...\}$ and let the covering relation of S be $\mathcal{C}(S) = \{(a, b_i) | i = 1, 2, 3, ...\} \cup \{(b_i, c_{i-1}) | i = 2, 3, 4, ...\} \cup \{(b_1, c_1)\} \cup \{(c_i, c_{i+1}) | i = 1, 2, 3, ...\}$. This semigroup satisfies the conditions (i), (ii) and (iv), but not (iii). It cannot be realized as an S_ω (but it can as an $S_{\omega+1}$).

Example 4.3. Let $S = \bigcup_{i=1}^{\infty}\{a_{i,1}, a_{i,2}, a_{i,3}\} \cup \{b\}$ and let the covering relation of S be $\mathcal{C}(S) = \bigcup_{i=1}^{\infty}\{(a_{i,1}, a_{i,2}), (a_{i,1}, a_{i,3})\} \cup \bigcup_{i=1}^{\infty}\{(a_{i,2}, a_{i+1,1}),$

$(a_{i,3}, a_{i+1,1})\} \cup \{(a_{1,1}, b)\}$. This semigroup satisfies the conditions (i), (iii) and (iv), but not (ii). It cannot be realized as an S_ω (but it can as an $S_{\omega+1}$).

Example 4.4. Let $S = \{a_1, a_2, a_3, ...\} \cup \{b_1, b_2, b_3\} \cup \{c_1, c_2, c_3\}$ and let the covering relation of S be $\mathcal{C}(S) = \{(a_i, a_{i+1}) \mid i = 1, 2, 3, ...\} \cup \{(a_1, b_1), (a_1, b_2), (b_1, b_3), (b_2, b_3)\} \cup \{(a_2, c_1), (a_2, c_2), (c_1, c_3), (c_2, c_3)\}$. This semigroup satisfies the conditions (ii), (iii) and (iv), but not (i). It cannot be realized as an S_ω, in fact, it cannot be realized as any S_τ, τ an ordinal.

The above examples inspire the following.

Question 4.5. Characterize all the lower semilattices that can be realized as an S_τ, τ an ordinal. More concretely, is the well-orderedness of the set of join-reducible elements a necessary and sufficient condition?

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