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THE HERMITIAN REFLEXIVE SOLUTION FOR A CLASS OF MATRIX EQUATIONS

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Abstract

In this paper, we investigate the solvability of the Hermitian reflexive solution for a class of matrix equations. The findings of this paper extend some known results in the literature.

1. Introduction

Throughout, we denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m \times n}$. For a matrix $A \in \mathbb{C}^{m \times n}$, the symbols A^*, $A^\dagger$, $r(A)$ stand for the conjugate transpose, the Moore-Penrose inverse and the rank of A, respectively. If $A = A^*$, then A is called *Hermitian*. The set of all $n \times n$ Hermitian matrices

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is denoted by $\mathbb{CH}^{n\times n}$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^{\dagger}A$, $R_A = I - AA^{\dagger}$ induced by A, where I denotes identity matrix with the appropriate size.

As a generalization of centrosymmetric matrices, reflexive matrices were defined in [1, 2]. If a matrix A is reflexive and Hermitian, then A is called *Hermitian reflexive matrix*. It is noted that the Hermitian reflexive matrices have practical applications in information theory, linear system theory, linear estimate theory and numerical analysis (see, e.g., [1-4]). Therefore, there are lots of papers in which the aim is to find necessary and sufficient conditions for the existence of a solution X to some matrix equations such that X belongs to Hermitian reflexive matrix (see, e.g., [5, 6, 8, 11]). In [8] and [11], Xu et al. considered the Hermitian reflexive solutions to matrix equations $AX = B$ and matrix equations $AX = C$, $XB = D$, respectively.

As a nontrivial generalization of above matrix equations, the matrix equations

$$A_aX = C_a, \quad XB_a = D_a, \quad A_bXA_b^* = C_b \tag{1.1}$$

is also studied by many authors. However, to our knowledge, there has been little information about the Hermitian reflexive solution of (1.1) at present. Motivated by the work mentioned above, we, in this paper, investigate the Hermitian reflexive solution to matrix equations (1.1).

2. Hermitian Reflexive Solution to Matrix Equations (1.1)

A matrix A is called *reflexive* with respect to the matrix P if $A = PAP$, where P is the nontrivial generalized reflection matrix, i.e., $P = P^* \neq I$ and $P^2 = I$. If A is reflexive and $A = A^*$, A is called a *Hermitian reflexive matrix* (associated with P). Put

$$\mathbb{C}_r^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} \mid A = PAP\},$$

$$\mathbb{CH}_r^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} \mid A = PAP,\ A = A^*\}.$$

In this section, we consider Hermitian reflexive solution to matrix equations (1.1). We begin with the following lemmas.

Lemma 2.1 [9]. *Let* $A_2, C_2 \in \mathbb{C}^{s\times n}$, $B_2, D_2 \in \mathbb{C}^{n\times t}$, $A_3, C_3 \in \mathbb{C}^{q\times p}$, $B_3, D_3 \in \mathbb{C}^{p\times l}$, $C_4 \in \mathbb{C}^{m\times n}$, $D_4 \in \mathbb{C}^{m\times p}$, $A_4 \in \mathbb{CH}^{m\times m}$ *be known and* $X_1 \in \mathbb{C}^{n\times n}$, $X_2 \in \mathbb{C}^{p\times p}$ *be unknown. Set*

$$E = \begin{bmatrix} A_2 \\ B_2^* \end{bmatrix}, \quad F = \begin{bmatrix} C_2 \\ D_2^* \end{bmatrix}, \quad G = \begin{bmatrix} A_3 \\ B_3^* \end{bmatrix}, \quad H = \begin{bmatrix} C_3 \\ D_3^* \end{bmatrix} \tag{2.1}$$

and

$$\phi = E^{\dagger}F + F^*(E^{\dagger})^* - E^{\dagger}EF^*(E^{\dagger})^*, \quad \varphi = G^{\dagger}H + H^*(G^{\dagger})^* - G^{\dagger}GH^*(G^{\dagger})^*,$$

$$C = C_4L_E, \quad D = D_4L_G, \quad M = R_CD, \quad S = DL_M,$$

$$A = A_4 - C_4\phi C_4^* - D_4\varphi D_4^*.$$

Then the following statements are equivalent:

(a) *The matrix equations*

$$A_2X_1 = C_2, \quad X_1B_2 = D_2, \quad A_3X_2 = C_3, \quad X_2B_3 = D_3,$$

$$C_4X_1C_4^* + D_4X_2D_4^* = A_4 \tag{2.2}$$

have a pair of Hermitian solution $X_1 \in \mathbb{CH}^{n\times n}$, $X_2 \in \mathbb{CH}^{p\times p}$.

(b)

$$EF^* = FE^*, \quad GH^* = HG^*, \quad R_EF = 0, \quad R_GH = 0,$$

$$R_MR_CA = 0, \; R_CAR_D = 0. \tag{2.3}$$

(c)

$$EF^* = FE^*, \quad GH^* = HG^*, \quad r[E,\, F] = r(E), \quad r[G,\, H] = r(G),$$

$$r\begin{bmatrix} A_4 & C_4 & D_4 \\ FC_4^* & E & 0 \\ HD_4^* & 0 & G \end{bmatrix} = r\begin{bmatrix} C_4 & D_4 \\ E & 0 \\ 0 & G \end{bmatrix},$$

$$r\begin{bmatrix} A_4 & C_4 & D_4H^* \\ D_4^* & 0 & G^* \\ FC_4^* & E & 0 \end{bmatrix} = r\begin{bmatrix} 0 & C_4 & 0 \\ D_4^* & 0 & G^* \\ 0 & E & 0 \end{bmatrix}. \tag{2.4}$$

In that case, the general Hermitian solution of the matrix equations (2.2) *can be expressed as*

$$X_1 = \phi + L_E V_1 L_E, \quad X_2 = \varphi + L_G V_2 L_G, \tag{2.5}$$

where

$$V_1 = V_1^* = C^\dagger A(C^\dagger)^* - \frac{1}{2}C^\dagger DM^\dagger A[I + (D^\dagger)^* S^*](C^\dagger)^*$$

$$- \frac{1}{2}C^\dagger(I + SD^\dagger)A(C^\dagger DM^\dagger)^* - C^\dagger SU_2(C^\dagger S)^* + L_C W_1 + W_1^* L_C, \tag{2.6}$$

$$V_2 = V_2^* = \frac{1}{2}M^\dagger A(D^\dagger)^*(I + S^\dagger S) + \frac{1}{2}(I + S^\dagger S)D^\dagger A(M^\dagger)^*$$

$$+ L_M U_2 L_M + L_M L_S U_1 + U_1^* L_S L_M + U_3 L_D + L_D U_3^*, \tag{2.7}$$

where $W_1,\ U_1,\ U_3$ *and* $U_2 = U_2^*$ *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes.*

Lemma 2.2 [7]**.** *Let* $P \in \mathbb{C}^{n\times n}$ *be a nontrivial generalized reflection matrix, i.e.,* $P = P^*,\ P^2 = I_n$*. Then there exists unitary matrix* $U \in \mathbb{C}^{n\times n}$ *such that*

$$P = U^*\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}U. \tag{2.8}$$

Lemma 2.3 [7]**.** *A matrix* $A \in \mathbb{C}_r^{n\times n}(P)$ *if and only if* A *can be expressed as*

$$A = U^* \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U, \tag{2.9}$$

where $A_1 \in \mathbb{C}^{r\times r}$, $A_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ *and U is defined as* (2.8).

Now we consider the Hermitian reflexive solution to matrix equations (1.1).

Theorem 2.4. *Let* $A_a, C_a \in \mathbb{C}^{m\times n}$, $B_a, D_a \in \mathbb{C}^{n\times t}$, $A_b \in \mathbb{C}^{k\times n}$, $C_b \in \mathbb{C}^{k\times k}$ *be known and* $X \in \mathbb{CH}_r^{n\times n}(P)$ *be unknown*; $A_2, A_3, C_2, C_3, B_2, B_3, D_2, D_3, C_4, D_4, A_4$ *be defined by* (2.12)-(2.14). *Then* (1.1) *has a solution* $X \in \mathbb{CH}_r^{n\times n}(P)$ *if and only if* $A_4 = A_4^*$ *and the equalities in* (2.3) *or* (2.4) *are all satisfied*.

In that case, *the Hermitian reflexive solution X to matrix equations* (1.1) *can be expressed as*

$$X = U^* \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U, \tag{2.10}$$

where X_1, X_2 *are a pair of Hermitian solutions to* (2.2), *i.e.*, *the expression of* (2.5), *where* V_1, V_2 *can be expressed as* (2.6) *and* (2.7), *respectively*.

Proof. It follows from Lemma 2.3 that $X \in \mathbb{CH}_r^{n\times n}$ can be expressed as

$$X = U^* \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U, \tag{2.11}$$

where $X_1 \in \mathbb{CH}^{r\times r}$, $X_2 \in \mathbb{CH}^{(n-r)\times(n-r)}$ and U is defined as in (2.8). Suppose that

$$A_a U^* = [A_2, A_3], \quad C_a U^* = [C_2, C_3], \tag{2.12}$$

$$UB_a = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \quad UD_a = \begin{bmatrix} D_2 \\ D_3 \end{bmatrix}, \tag{2.13}$$

$$A_b U^* = [C_4, D_4], \quad C_b = A_4, \tag{2.14}$$

where $A_2, C_2 \in \mathbb{C}^{m\times r}$, $B_2, D_2 \in \mathbb{C}^{r\times t}$, $A_3, C_3 \in \mathbb{C}^{m\times(n-r)}$, $B_3, D_3 \in \mathbb{C}^{(n-r)\times t}$, $C_4 \in \mathbb{C}^{k\times r}$, $D_4 \in \mathbb{C}^{k\times(n-r)}$, $A_4 = A_4^* \in \mathbb{C}^{k\times k}$. Then (1.1) has a solution $X \in \mathbb{CH}_r^{n\times n}$ if and only if (2.2) has a pair of solutions $X_1 \in \mathbb{CH}^{r\times r}$ and $X_2 \in \mathbb{CH}^{(n-r)\times(n-r)}$. In view of Lemma 2.1, we get Theorem 2.4. □

3. Conclusion

In this paper, we have derived the Hermitian reflexive solution to matrix equations (1.1). We point out that the results of this paper can be generalized to a general associative ring and a semiring under some assumptions. Moreover, in view of the definition and characterization of bisymmetric matrix given in [10], we can also consider bisymmetric solution to matrix equations (1.1) with similar approach. Motivated by the work in this paper, it would be of interest to investigate the least square solutions with Hermitian reflexive matrix and bisymmetric matrix to matrix equations (1.1).

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