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LIPSCHITZ STABILITY FOR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

Sang Il Choi and Yoon Hoe Goo*

Department of Mathematics
Hanseo University
Seosan Chungnam 356-706
Republic of Korea
e-mail: schoi@hanseo.ac.kr
yhgoo@hanseo.ac.kr

Abstract

In this paper, we investigate Lipschitz and asymptotic stability for perturbed functional differential systems.

1. Introduction

Dannan and Elaydi introduced a new notion of uniformly Lipschitz stability (ULS) [9]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer [4] and uniformly stability in variation of Brauer and Strauss [3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite

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*Corresponding author

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distinct. Furthermore, uniform Lipschitz stability neither implies asymptotic stability nor is it implied by it. Also, Elaydi and Farran [10] introduced the notion of exponential asymptotic stability (EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Gonzalez and Pinto [11] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al. [7, 8] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo et al. [5, 12] investigated Lipschitz and asymptotic stability for perturbed differential systems.

The purpose of this paper is to employ the theory of integral inequalities to study Lipschitz and asymptotic stability for solutions of the nonlinear differential systems. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear systems of differential equations. In the present situation, the method of integral inequalities is as efficient as the direct Lyapunov's method.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed differential system of (2.1),

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)), \quad y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel [9].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called

(S) *stable* if for any $\varepsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for all $t \ge t_0 \ge 0$,

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$,

($ULSV$) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \le M$ for $|x_0| \le \delta$ and $t \ge t_0 \ge 0$, (*EAS*) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K|x_0|e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,

(*EASV*) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq Ke^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

Remark 2.2 [11]**.** The last definition implies that for $|x_0| \leq \delta$,

$$|x(t)| \leq K|x_0|e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t.$$

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.3. *Let x and y be solutions of* (2.1) *and* (2.5), *respectively*. *If* $y_0 \in \mathbb{R}^n$, *then for all t such that* $x(t, t_0, y_0) \in \mathbb{R}^n$,

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s))ds.$$

Lemma 2.4 (Bihari-type inequality)**.** *Let* $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$

and $w(u)$ *be nondecreasing in u. Suppose that, for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \qquad t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda(s) ds\right], \qquad t_0 \le t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \operatorname{dom} W^{-1}\right\}.$$

Lemma 2.5 [15]**.** *Let* $u, p, q, v, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, $w(u)$ *be nondecreasing in u, and* $u \le w(u)$. *Suppose that for some* $c \ge 0$,

$$u(t) \le c + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) u(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) w(u(a)) da \right) d\tau \right) ds, \quad t \ge t_0.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \right],$$

$$t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma* 2.4 *and*

$$b_1 = \sup\Bigg\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \in \operatorname{dom} W^{-1} \Bigg\}.$$

Lemma 2.6 [13]**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$

be nondecreasing in u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \left(\int_{t_0}^{s} \lambda_3(\tau) u(\tau) d\tau \right) ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \right) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma* 2.4, *and*

$$b_1 = \sup\left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \right) ds \in \operatorname{dom} W^{-1} \right\}.$$

Lemma 2.7 [14]. *Let* $u, p, q, r, v \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c \geq 0$,

$$u(t) \leq c + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) w(u(\tau)) + v(t) \int_{t_0}^{\tau} r(a) w(u(a)) da \right) d\tau \right) ds, \quad t \geq t_0.$$

Then

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma* 2.4, *and*

$$b_1 = \sup\left\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \in \operatorname{dom} W^{-1} \right\}.$$

Lemma 2.8 [6]. *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds$$
$$+ \int_{t_0}^{t} \lambda_2(s) \left(\int_{t_0}^{s} \lambda_3(\tau) w(u(\tau)) d\tau \right) ds, \quad 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \right) ds \right], \quad t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma* 2.4, *and*

$$b_1 = \sup\left\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) d\tau \right) ds \in \operatorname{dom} W^{-1} \right\}.$$

3. Main Results

In this section, we investigate Lipschitz and asymptotic stability for solutions of the perturbed functional differential systems.

We need the lemma to prove the following theorem.

Lemma 3.1. *Let* $k, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c \ge 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \left[\int_{t_0}^{s} \left[\lambda_2(\tau) w(u(\tau)) \right.\right.$$
$$\left.\left. + \lambda_3(\tau) \int_{t_0}^{\tau} k(r) w(u(r)) dr \right] d\tau + \lambda_4(s) w(u(s)) \right] ds, \tag{3.1}$$

for $t \ge t_0 \ge 0$ *and for some* $c \ge 0$. *Then*

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda_1(s)\left(\int_{t_0}^{s}\left(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau} k(r)dr\right)d\tau + \lambda_4(s)\right)ds\right], \tag{3.2}$$

for $t_0 \leq t < b_1$, *where* W, W^{-1} *are the same functions as in Lemma* 2.4, *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_1(s) \right.$$

$$\left. \times \left(\int_{t_0}^{s}\left(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau} k(r)dr\right)d\tau + \lambda_4(s)\right)ds \in \operatorname{dom}W^{-1}\right\}.$$

Proof. Define a function $v(t)$ by the right member of (3.1). Then

$$v'(t) = \lambda_1(t)\left[\int_{t_0}^{t}\left(\lambda_2(s)w(u(s)) + \lambda_3(s)\int_{t_0}^{s} k(\tau)w(u(\tau))d\tau\right)ds + \lambda_4(t)w(u(t))\right],$$

which implies

$$v'(t) \leq \lambda_1(t)\left[\int_{t_0}^{t}\left(\lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} k(\tau)d\tau\right)ds + \lambda_4(t)\right]w(v(t)),$$

since v and w are nondecreasing and $u(t) \leq v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$v(t) \leq c + \int_{t_0}^{t} \lambda_1(s)\left[\int_{t_0}^{s}\left(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau} k(r)dr\right)d\tau + \lambda_4(s)\right]w(v(s))ds. \tag{3.3}$$

Then, by the well-known Bihari-type inequality, (3.3) yields the estimate (3.2). □

Theorem 3.2. *For the perturbed* (2.2), *we assume that*

$$|g(t, y)| \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{3.4}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)w(|y(t)|), \tag{3.5}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *is nondecreasing in u and* $\frac{1}{v}w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\left[W(M) + M\int_{t_0}^{\infty}\left(\int_{t_0}^{s}\left(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr\right)d\tau + c(s)\right)ds\right], \tag{3.6}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (2.2) *is ULS whenever the zero solution of* (2.1) *is ULSV*.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS by [9, Theorem 3.3]. Using the nonlinear variation of constants formula, (3.4) and (3.5), we have

$$\begin{aligned}
&|y(t)| \\
&\le |x(t)| + \int_{t_0}^{t}|\Phi(t, s, y(s))|\left(\int_{t_0}^{s}|g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|\right)ds \\
&\le M|y_0| + \int_{t_0}^{t}M|y_0| \\
&\quad\times\left[\int_{t_0}^{s}\left[a(\tau)w\left(\frac{|y(\tau)|}{|y_0|}\right) + b(\tau)\int_{t_0}^{\tau}k(r)w\left(\frac{|y(r)|}{|y_0|}\right)dr\right]d\tau + c(s)w\left(\frac{|y(s)|}{|y_0|}\right)\right]ds.
\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 3.1 yields

$$|y(t)| \le |y_0|W^{-1}\left[W(M) + M\int_{t_0}^{t}\left(\int_{t_0}^{s}\left(a(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr\right)d\tau + c(s)\right)ds\right].$$

Thus, by (3.6), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Remark 3.3. Letting $c(t) = 0$ in Theorem 3.2, we obtain the same result as that of Theorem 3.2 in [5].

Theorem 3.4. *For the perturbed* (2.2), *we assume that*

$$|g(t, y)| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{3.7}$$

and

$$|h(t, y(t), Ty(t))| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{3.8}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *is nondecreasing in u and* $\frac{1}{v}w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\left[W(M) + M\int_{t_0}^{\infty}\int_{t_0}^{s}\left(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr\right)d\tau ds\right], \tag{3.9}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (2.2) *is ULS whenever the zero solution of* (2.1) *is ULSV*.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Using the nonlinear variation of constants formula, (3.7) and (3.8), we have

$$\begin{aligned}
&|y(t)| \\
&\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\left(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|\right)ds \\
&\le M|y_0| + \int_{t_0}^{t} M|y_0|\int_{t_0}^{s}\left[(a(\tau) + c(\tau))w\left(\frac{|y(\tau)|}{|y_0|}\right)d\tau ds\right. \\
&\quad \left. + \int_{t_0}^{t} M|y_0|\int_{t_0}^{s} b(\tau)\int_{t_0}^{\tau} k(r)w\left(\frac{|y(r)|}{|y_0|}\right)dr\right]d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then it follows from Lemma 2.7 that

$$|y(t)| \le |y_0| W^{-1}\left[W(M) + M\int_{t_0}^{t}\int_{t_0}^{s}\left(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr\right)d\tau ds\right].$$

From (3.9), we get $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Remark 3.5. Letting $c(s) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.2 in [5].

Theorem 3.6. *For the perturbed* (2.2), *we assume that*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{3.10}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)w(|y(t)|), \tag{3.11}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *is nondecreasing in u and* $\frac{1}{v}w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\left[W(M) + M\int_{t_0}^{\infty}\left(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\right)ds\right], \tag{3.12}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (2.2) *is ULS whenever the zero solution of* (2.1) *is ULSV.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Applying Lemma 2.3, (3.10), and (3.11), we have

$$|y(t)|$$

$$\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left(\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|\right) ds$$

$$\leq M|y_0| + \int_{t_0}^{t} M|y_0|(a(s)+c(s))w\left(\frac{|y(s)|}{|y_0|}\right)ds$$

$$+ \int_{t_0}^{t} M|y_0|b(s)\int_{t_0}^{s} k(\tau)w\left(\frac{|y(\tau)|}{|y_0|}\right)d\tau ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.8 yields

$$|y(t)| \leq |y_0|W^{-1}\left[W(M) + M\int_{t_0}^{t}\left(a(s)+c(s)+b(s)\int_{t_0}^{s} k(\tau)d\tau\right)ds\right].$$

Hence, by (3.12), we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 3.7. Letting $c(t) = 0$ in Theorem 3.6, we obtain the same result as that of Theorem 3.3 in [5].

Lemma 3.8. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds$$

$$+ \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds$$

$$+ \int_{t_0}^{t} \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\Bigg[W(c) + \int_{t_0}^{t}\Bigg(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau$$

$$+ \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau\Bigg)ds\Bigg], \quad t_0 \leq t < b_1, \tag{3.13}$$

where W, W^{-1} *are the same functions as in Lemma* 2.4 *and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^{t}\left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau\right.\right.$$

$$\left.\left. + \int_{t_0}^{t}\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\right)ds \in \operatorname{dom}W^{-1}\right\}.$$

Proof. Setting

$$z(t) = c + \int_{t_0}^{t}\lambda_1(s)u(s)ds + \int_{t_0}^{t}\lambda_2(s)w(u(s))ds$$

$$+ \int_{t_0}^{t}\lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)u(\tau)d\tau ds$$

$$+ \int_{t_0}^{t}\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)w(u(\tau))d\tau ds,$$

we have $z(t_0) = c$ and

$$z'(t) = \lambda_1(t)u(t) + \lambda_2(t)w(u(t))$$

$$+ \lambda_3(t)\int_{t_0}^{t}\lambda_4(s)u(s)ds + \lambda_5(t)\int_{t_0}^{t}\lambda_6(s)w(u(s))ds$$

$$\le \left(\lambda_1(t) + \lambda_2(t) + \lambda_3(t)\int_{t_0}^{t}\lambda_4(s)ds + \lambda_5(t)\int_{t_0}^{t}\lambda_6(s)ds\right)w(z(t)),$$

$$t \ge t_0,$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \le w(u)$, and $u(t) \le z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$z(t) \le c + \int_{t_0}^{t}\left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau\right.$$

$$\left. + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\right)w(z(s))ds. \tag{3.14}$$

It follows from Lemma 2.4 that (3.14) yields the estimate (3.13). □

Theorem 3.9. *For the perturbed* (2.2), *we assume that*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le a(t) w(|y(t)|) + b(t) \int_{t_0}^{t} k(s) w(|y(s)|) ds \qquad (3.15)$$

and

$$|h(t, y(t), Ty(t))| \le c(t) \left(|y(t)| + \int_{t_0}^{t} q(s) |y(s)| ds \right), \qquad (3.16)$$

where $a, b, c, k, q \in C(\mathbb{R}^+)$, $a, b, c, k, q \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *is nondecreasing in* $u, u \le w(u)$, *and* $\frac{1}{v} w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \Big(a(s) + c(s) + b(s) \int_{t_0}^{s} k(\tau) d\tau$$

$$+ c(s) \int_{t_0}^{s} q(\tau) d\tau \Big) ds \Big], \qquad (3.17)$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of* (2.2) *is ULS whenever the zero solution of* (2.1) *is ULSV*.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Applying Lemma 2.3, (3.15) and (3.16), we have

$$|y(t)|$$

$$\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left(\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds$$

$$\le M|y_0| + \int_{t_0}^{t} M|y_0| c(s) \frac{|y(s)|}{|y_0|} ds + \int_{t_0}^{t} M|y_0| a(s) w\left(\frac{|y(s)|}{|y_0|}\right) ds$$

$$+ \int_{t_0}^{t} M|y_0| c(s) \int_{t_0}^{s} q(\tau) \frac{|y(\tau)|}{|y_0|} d\tau$$

$$+ \int_{t_0}^{t} M|y_0| b(s) \int_{t_0}^{s} k(\tau) w\left(\frac{|y(\tau)|}{|y_0|}\right) d\tau ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 3.8 yields

$$|y(t)| \leq |y_0| W^{-1}\left[W(M) + M\int_{t_0}^{t}\left(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\right.\right.$$

$$\left.\left. + c(s)\int_{t_0}^{s} q(\tau)d\tau\right)ds\right].$$

By (3.17), we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 3.10. Letting $c(t) = 0$ in Theorem 3.9, we obtain the same result as that of Theorem 3.3 in [5].

Theorem 3.11. *Let the solution* $x = 0$ *of* (2.1) *be EASV. Suppose that the perturbing term* $g(t, y)$ *satisfies*

$$|g(t, y(t))| \leq e^{-\alpha t}\left(a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds\right), \quad (3.18)$$

and

$$|h(t, y(t), Ty(t))| \leq \int_{t_0}^{t} e^{-\alpha s}c(s)|y(s)|ds, \quad (3.19)$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L_1(\mathbb{R}^+)$, $w(u)$ *is nondecreasing in* u, $u \leq w(u)$, *and* $\frac{1}{v}w(u) \leq w\left(\frac{u}{v}\right)$ *for some* $v > 0$. *If*

$$M(t_0) = W^{-1}\left[W(c) + \int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}\left[a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr\right]d\tau ds\right] < \infty,$$

$$t \geq t_0, \quad (3.20)$$

where $c = |y_0|Me^{\alpha t_0}$, *then all solutions of* (2.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is

EAS by Remark 2.2. Using Lemma 2.3, (3.18) and (3.19), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|$$

$$\times \left(\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} e^{-\alpha\tau} \Big((a(\tau) + c(\tau))|y(\tau)|$$

$$+ b(\tau) \int_{t_0}^{\tau} k(r) e^{-\alpha r} w(|y(r)|) dr \Big) d\tau ds$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)} \int_{t_0}^{s} \Big((a(\tau) + c(\tau))|y(\tau)|e^{\alpha\tau}$$

$$+ b(\tau) \int_{t_0}^{\tau} k(r) w(|y(r)|e^{\alpha r}) dr \Big) d\tau ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Lemma 2.5 and (3.20), we obtain

$$|y(t)| \le e^{-\alpha t} W^{-1} \left[W(c) + \int_{t_0}^{t} Me^{\alpha s} \int_{t_0}^{s} \left[a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr \right] d\tau ds \right]$$

$$\le e^{-\alpha t} M(t_0), \; t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. □

Remark 3.12. Letting $c(s) = 0$ in Theorem 3.11, we obtain the same result as that of Theorem 3.4 in [5].

Theorem 3.13. *Let the solution* $x = 0$ *of* (2.1) *be EASV. Suppose that the perturbed term* $g(t, y)$ *satisfies*

$$\int_{t_0}^{t} |g(s, y(s))| ds \le e^{-\alpha t} \left(a(t) w(|y(t)|) + b(t) \int_{t_0}^{t} k(s) |y(s)| ds \right) \quad (3.21)$$

and

$$|h(t, y(t), Ty(t))| \le e^{-\alpha t} c(t) w(|y(t)|), \tag{3.22}$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L_1(\mathbb{R}^+)$ *and* $w(u)$ *is nondecreasing in* u, $u \le w(u)$, *and* $\frac{1}{v} w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$. *If*

$$M(t_0) = W^{-1}\left[W(c) + M\int_{t_0}^{\infty}\left(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\right)ds\right] < \infty,$$

$$b_1 = \infty, \tag{3.23}$$

where $c = M|y_0|e^{\alpha t_0}$, *then all solutions of* (2.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is EAS. Using Lemma 2.3, (3.21) and (3.22), we have

$$\begin{aligned}
&|y(t)| \\
&\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\left(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|\right)ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big[e^{-\alpha s}a(s)w(|y(s)|) \\
&\quad + e^{-\alpha s}b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + e^{-\alpha s}c(s)w(|y(s)|)\Big]ds \\
&\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}(a(s) + c(s))w(|y(s)|e^{\alpha s})ds \\
&\quad + \int_{t_0}^{t} Me^{-\alpha t}b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|e^{\alpha\tau}d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, it follows from Lemma 2.6 and (3.23) that

$$|y(t)| \le e^{-\alpha t} W^{-1}\left[W(c) + M\int_{t_0}^{t}\left(a(s) + c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau\right)ds\right]$$

$$\le e^{-\alpha t} M(t_0),\ t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

Remark 3.14. Letting $c(t) = 0$ in Theorem 3.13, we obtain the same result as that of Theorem 3.5 in [5].

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