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ON THE GABRIEL'S QUIVER OF SOME EQUIPPED POSETS

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Abstract

A complete description of the indecomposable representations and irreducible morphisms of some equipped posets of finite growth representation type is provided.

1. Introduction

Poset Representation Theory was introduced in the 1970's by Nazarova and Roiter with the purpose of giving a proof of the second Brauer-Thrall conjecture on the classification of algebras [10, 11]. The main problem in this theory is to give a complete description of all indecomposable representations of the additive category $\operatorname{rep}\mathcal{P}$ of k-linear representations of

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a given poset $\mathcal{P}$. In this case, a representation $U \in \operatorname{rep} \mathcal{P}$ is a system of k-vector spaces attached to the poset in such a way that

$$U = (U_0; U_x \mid x \in \mathcal{P}),$$

where $U_x \subseteq U_y$ provided $x \leq y$ [15].

Researches on the subject grew up so fast in the 1980's. For instance, soon afterwards the discovery by Nazarova and Kleiner of criteria to classify posets of finite and tame representation type, Zavadskij and Nazarova gave a criterion to classify posets of finite growth representation type via the algorithm of differentiation with respect to a suitable pair of points introduced by Zavadskij [9, 12, 13]. Furthermore, Zavadskij defined an apparatus of differentiation consisting of five algorithms which allowed to Bondarenko, Nazarova, Roiter and Zavadskij to classify posets endowed with an involution [1, 2, 19]. Posets of this kind belong to a broadest class of posets with additional structures whose classification was investigated in the 1980's and 1990's. Actually, in the earliest 2000, Zabarilo and Zavadskij defined equipped posets and gave criteria to classify equipped posets of one parameter. Soon afterwards, Zavadskij classified equipped posets of finite growth and tame representation type. To do that, he introduced some algorithms of differentiation named VII-XVII. In particular, algorithms of differentiations I, VII, VIII and IX were used to classify equipped posets of finite growth representation type [16, 20, 21].

As we can see, most of the researches regarding poset representation theory have been oriented to the classification of objects without paying much attention to the behavior of morphisms of the categories involved in the procedures. We quote that some of the few results regarding morphisms of categories of posets with additional structures have been obtained by the first author et al. in [3-5, 7]. In these works, categorical properties of algorithms of differentiations VII, VIII and IX for equipped posets were obtained. Such properties allow deducing that algorithms of differentiation induce a categorical equivalence between the corresponding categories. We recall that Gabriel in [8] proved that the algorithm of differentiation with respect to a maximal point induces a categorical equivalence between some quotient categories and that Zavadskij in [18] used the algorithm of differentiation with respect to suitable pair of points in order to describe the Auslander-Reiten quiver of ordinary posets of finite growth representation type [17, 22]. Actually, the current main goal of the poset representation theory is to give a complete description of the Auslander-Reiten quiver of the category of representations $R = \operatorname{rep} \mathcal{P}$ of a given poset $\mathcal{P}$. Such quiver is obtained by defining suitable translations to the Gabriel's quiver $\Gamma(R)$ of the category which has classes of indecomposable representations as vertices and irreducible morphisms as edges. In this paper, we describe the Gabriel's quiver of the category of representations of some equipped posets of finite representation type.

2. Preliminaries

In this section, we introduce main notation and definitions to be used throughout the paper. Authors refer the reader to [3-5, 7, 14, 15] for precise definitions.

A poset $(\mathcal{P}, \leq)$ is called *equipped* if all the order relations between its points $x \leq y$ are separated into strong (denoted $x \trianglelefteq y$) and weak (denoted $x \preceq y$) in such a way that

$$x \leq y \trianglelefteq z \text{ or } x \trianglelefteq y \leq z \text{ implies } x \trianglelefteq z, \tag{1}$$

i.e., a composition of a strong relation with any other relation is strong.

We let $x \leq y$ denote an arbitrary relation in an equipped poset $(\mathcal{P}, \leq)$. The order $\leq$ on an equipped poset $\mathcal{P}$ gives raise to the relations $\prec$ and $\lhd$ of *strict inequality*: $x \prec y$ (respectively, $x \lhd y$) in $\mathcal{P}$ if and only if $x \preceq y$ (respectively, $x \trianglelefteq y$) and $x \neq y$.

A point $x \in \mathcal{P}$ is called *strong* (*weak*) if $x \trianglelefteq x$ (respectively, $x \preceq x$). These points are denoted $\circ$ (respectively, $\otimes$) in diagrams. We also denote

$\mathcal{P}^{\circ} \subseteq \mathcal{P}$ (respectively, $\mathcal{P}^{\otimes} \subseteq \mathcal{P}$) the subset of strong points (respectively, weak points) of $\mathcal{P}$. If $\mathcal{P}^{\otimes} = \varnothing$, then the equipment is *trivial* and the poset $\mathcal{P}$ is ordinary.

If $\mathcal{P}$ is an equipped poset and $a \in \mathcal{P}$, then the subsets of $\mathcal{P}$ denoted $a^{\vee}$, $a_{\wedge}$, a^{∇}, a_{Δ}, $a^{\blacktriangledown}$, $a_{\blacktriangle}$, $a^{\curlyvee}$ and $a_{\curlywedge}$ are defined in such a way that

$$a^{\vee} = \{x \in \mathcal{P} \mid a \leq x\}, \quad a_{\wedge} = \{x \in \mathcal{P} \mid x \leq a\},$$

$$a^{\nabla} = \{x \in \mathcal{P} \mid a \trianglelefteq x\}, \quad a_{\Delta} = \{x \in \mathcal{P} \mid x \trianglelefteq a\},$$

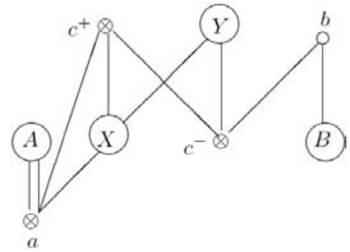
$$a^{\blacktriangledown} = a^{\vee} \setminus a, \quad a_{\blacktriangle} = a_{\wedge} \setminus a,$$

$$a^{\curlyvee} = \{x \in \mathcal{P} \mid a \preceq x\}, \quad a_{\curlywedge} = \{x \in \mathcal{P} \mid x \preceq a\}.$$

Subset $a^{\vee}$ $(a_{\wedge})$ is called the *ordinary upper* (*lower*) *cone*, associated to the point $a \in \mathcal{P}$ and subset a^{∇} (a_{Δ}) is called the *strong upper* (*lower*) *cone* associated to the point $a \in \mathcal{P}$, whereas subsets $a^{\blacktriangledown}$ and $a_{\blacktriangle}$ are called *truncated cones* (upper and lower) associated to the point $a \in \mathcal{P}$.

In general, subsets $a^{\curlyvee}$ and $a_{\curlywedge}$ are not cones. Note that, if $x \in \mathcal{P}^{\circ}$, then $x^{\curlyvee} = x_{\curlywedge} = \varnothing$.

The diagram of an equipped poset $(\mathcal{P}, \leq)$ may be obtained via its Hasse diagram (with strong $(\circ)$ and weak points $(\otimes)$). In this case, a new line is added to the line connecting two points $x, y \in \mathcal{P}$ with $x \lhd y$ if and only if such relation cannot be deduced of any other relations in $\mathcal{P}$. In Figure 1, we show an example of this kind of diagram:



$$a^{\nabla} = A;\ a^{\curlyvee} = \{a, c^+\} + X + Y;\ (c^-)^{\curlyvee} = \{c^+\} + Y;\ b_{\Delta} = \{c^-\} + B$$

Figure 1

For an equipped poset $(\mathcal{P}, \leq)$ and $A \subset \mathcal{P}$, we define the subsets A^{∇}, $A^{\curlyvee}$ and $A^{\vee}$ in such a way that

$$A^{\nabla} = \bigcup_{a \in A} a^{\nabla}, \quad A^{\curlyvee} = \bigcup_{a \in A} a^{\curlyvee}, \quad A^{\vee} = \bigcup_{a \in A} a^{\vee}.$$

Subsets A_{Δ}, $A_{\curlywedge}$ and $A_{\wedge}$ are defined in the same way.

If $\mathcal{P}$ is an equipped poset, then a chain $C = \{c_i \in \mathcal{P} \,|\, 1 \leq i \leq n,\ c_{i-1} < c_i \text{ if } i \geq 2\} \subseteq \mathcal{P}$ is a *weak chain* if and only if $c_{i-1} \prec c_i$ for each $i \geq 2$. If $c_1 \prec c_n$, then we say that C is a *completely weak chain*. Moreover, a subset $X \subset \mathcal{P}$ is *completely weak* if $X = X^{\otimes}$ and weak relations are the only relations between points of X. Often, we let $\{c_1 \prec c_2 \prec \cdots \prec c_n\}$ denote a weak chain which consists of points $c_1, c_2, ..., c_n$. An ordinary chain C is denoted in the same way (by using the corresponding symbol <).

For an equipped poset $\mathcal{P}$ and $A, B \subset \mathcal{P}$, we write $A < B$ if $a < b$ for each $a \in A$ and $b \in B$. Notations $A \prec B$ and $A \lhd B$ are assumed in the same way.

The *complexification* of a real vector space can be generalized to the case (F, G), where $G = F(\mathbf{u})$ is a quadratic extension of F. In this case, we assume that $\mathbf{u}$ is a root of the minimal polynomial $t^2 + \mu t + \lambda$, $\lambda \neq 0$ $(\lambda, \mu \in F)$. In particular, if U_0 is an F-space, then the corresponding complexification is the G-vector space also denoted $U_0^2 = \widetilde{U}_0$. As in the case $(\mathbb{R}, \mathbb{C})$, we write $U_0^2 = U_0 + \mathbf{u}U_0 = \widetilde{U}_0$.

To each G-subspace W of $\widetilde{U}_0$, it is possible to associate the following F-subspaces of U_0,

$W^+ = \operatorname{Re} W_F = \operatorname{Im} W_F$ and $W^- = gen\{\alpha \in U_0 \mid (\alpha, 0)^t \in W\} \subset W^+$
and for a G-space Z, we have the following property:

$$\widetilde{Z}^+ = F(Z) \text{ is called the } F\text{-}hull \text{ of } Z \text{ such that } Z \subset F(Z).$$

The *category of representations of an equipped poset* over a pair of fields (F, G) is defined as a system of the form

$$U = (U_0; U_x \mid x \in \mathcal{P}), \tag{2}$$

where U_0 is a finite dimensional F-space and for each $x \in \mathcal{P}$, U_x is a G-subspace of $\widetilde{U}_0$ such that

$$x \leq y \Rightarrow U_x \subset U_y,$$

$$x \trianglelefteq y \Rightarrow F(U_x) \subset U_y.$$

For each $x \in \mathcal{P}$, we let $\underline{U_x}$ denote the *radical subspace* of U_x such that $\underline{U_x} = \sum_{z \lhd x} F(U_z) + \sum_{z \prec x} U_z$.

We let $\operatorname{rep} \mathcal{P}$ denote the category whose objects are the representations of an equipped poset $\mathcal{P}$ over a pair of fields (F, G). In this case, a morphism

$$\varphi : (U_0; U_x \mid x \in \mathcal{P}) \to (V_0; V_x \mid x \in \mathcal{P})$$

between two representations U and V is an F-linear map $\varphi : U_0 \to V_0$ such that

$$\widetilde{\varphi}(U_x) \subset V_x, \text{ for each } x \in \mathcal{P},$$

where $\widetilde{\varphi} : \widetilde{U}_0 \to \widetilde{V}_0$ is the complexification of φ, i.e., the application

G-linear induced by φ and defined in such a way that if $z = x + \mathbf{u}y \in \widetilde{U}_0$, then $\widetilde{\varphi}(z) = \varphi^2(z) = \varphi(x) + \mathbf{u}\varphi(y)$. The composition between morphisms of $\operatorname{rep} \mathcal{P}$ is defined in a natural way.

If $\mathcal{P}$ is an equipped poset and $U, V \in \operatorname{rep} \mathcal{P}$, then U is a *sub-representation* of V if and only if spaces U_0, V_0, U_x and V_x satisfy the inclusions $U_0 \subset V_0$ and $U_x \subset V_x$ for each $x \in \mathcal{P}$.

Two representations $U, V \in \operatorname{rep} \mathcal{P}$ are said to be *isomorphic* if and only if there exists an F-isomorphism $\varphi : U_0 \to V_0$ such that $\widetilde{\varphi}(U_x) = V_x$, for each $x \in \mathcal{P}$.

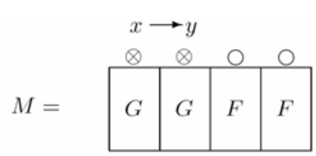
The *main problem* dealing with equipped posets consists of classifying its indecomposable representations up to isomorphisms.

The *dimension* of a representation $U \in \operatorname{rep} \mathcal{P}$ is a vector d such that $d = \underline{\dim}\, U = (d_0; d_x \,|\, x \in \mathcal{P})$, where $d_0 = \dim_F U_0$ and $d_x = \dim_G U_x / \underline{U_x}$. A representation $U \in \operatorname{rep} \mathcal{P}$ is *sincere* if $d_0 \neq 0$ and $d_x \neq 0$ for each $x \in \mathcal{P}$. In other words, the vector d of a sincere representation U has not null coordinates.

2.1. The matrix problem

In this subsection, we recall the *matrix problem* induced by an equipped poset $\mathcal{P}$, according to the new version described by Rodriguez and Zavadskij in [23].

Each equipped poset $\mathcal{P}$ naturally defines a matrix problem of mixed type over the pair (F, G). Consider a rectangular matrix M separated into vertical stripes M_x, $x \in \mathcal{P}$, with M_x being over F (over G) if the point x is strong (weak):



Such partitioned matrices M are called *matrix representation* of $\mathcal{P}$ over (F, G). Two matrix representations are said to be *equivalent* or *isomorphic* if they can be turned into each other with the help of the following admissible transformations:

Their *admissible transformations* are as follows:

(a) F-elementary row transformations of the whole matrix M;

(b) F-elementary (G-elementary) column transformations of a stripe M_x if the point x is strong (weak);

(c) in the case of a weak relation $x \prec y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in G;

(d) in the case of a strong relation $x \lhd y$, independent additions both real and imaginary parts of columns of the stripe M_x to real and imaginary parts (in any combinations) of columns of the stripe M_y with coefficients in F (assuming that, for y strong, there are no additions to the zero imaginary part of M_y).

The corresponding *matrix problem* of mixed type over the pair (F, G) consists of classifying indecomposables in the natural sense matrices M, up to equivalence.

Remark 1. The matrix problem for representations (a)-(d) occurs naturally in the classification of the objects $U \in \operatorname{rep} \mathcal{P}$ up to isomorphisms. In this case, it is associated to the representation U its matrix presentation $M_U = (M_x;\, x \in \mathcal{P})$ defined as follows:

If a point $x \in \mathcal{P}^0(\mathcal{P}^{\otimes})$, then the columns of the stripe M_x consist of coordinates (with respect to a fixed ordered basis $\mathcal{B}$ of U_0) of a system of generators $\mathcal{G}$ of U_x^+ (respectively, G-subspace U_x) modulo its radical subspace $\underline{U_x^+}$ (respectively, $\underline{U_x}$). Problem (a)-(d) may be obtained by changing basis $\mathcal{B}$ and the system of generators $\mathcal{G}$.

2.2. Some indecomposable objects

In this subsection, we give some examples of indecomposable objects in the category $\operatorname{rep}\mathcal{P}$, where $\mathcal{P}$ is an equipped poset.

If $\mathcal{P}$ is an equipped poset and $A \subset \mathcal{P}$, then $P(A) = P(\min A) = (F; P_x \mid x \in \mathcal{P})$, $P_x = G$ if $x \in A^{\vee}$ and $P_x = 0$ otherwise. In particular, $P(\varnothing) = (F; 0, \ldots, 0)$.

If $a, b \in \mathcal{P}^{\otimes}$, then $T(a)$ and $G_1(a, b)$ denote indecomposable objects with matrix representation of the following form:

$$T(a) = \begin{array}{c} a \\ \boxed{\begin{array}{c} 1 \\ \mathbf{u} \end{array}} \end{array}, \; a \in \mathcal{P}^{\otimes}, \; T(a, b) = \begin{array}{c|c} a & b \\ \hline 1 & 0 \\ \mathbf{u} & 1 \end{array} \text{ with } a \prec b.$$

If we consider the notation (2) for objects in $\operatorname{rep}\mathcal{P}$, then the object $T(a)$ may be described in such a way that $T(a) = (T_0; T_x \mid x \in \mathcal{P})$, where $T_0 = F^2$ and

$$T_x = \begin{cases} \widetilde{T}_0 = G^2, & \text{if } x \in a^{\nabla}, \\ G\{(1, \mathbf{u})^t\}, & \text{if } x \in a^{\curlyvee}, \\ 0, & \text{otherwise,} \end{cases}$$

where $(1, \mathbf{u})^t$ is the column of coordinates with respect to an ordered basis of T_0.

On the other hand, representation $T(a, b)$ may be described in such a way that $T(a, b) = (T_0; T_x \mid x \in \mathcal{P})$, where $T_0 = F^2$ and

$$T_x = \begin{cases} G\{(1, \mathbf{u})^t\}, & \text{if } a \preceq x \prec b, \\ \widetilde{T}_0 = G^2, & \text{if } x \in a^{\nabla} \cup b^{\vee}, \\ 0, & \text{otherwise.} \end{cases}$$

If $a \in \mathcal{P}^{\otimes}$ and $B \subset \mathcal{P}$ is a subset completely weak such that $a \prec B$, then we let $T(a, B)$ denote the representation of $\mathcal{P}$ which satisfies the following conditions with $T_0 = F^2$:

$$T_x = \begin{cases} G\{(1, \mathbf{u})^t\}, & \text{if } x \in a^{\curlyvee} \setminus B, \\ \widetilde{T}_0 = G^2, & \text{if } x \in a^{\nabla} + B^{\vee}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $T(a, \varnothing) = T(a)$.

Remark 2. In [20], it is proved that $P(\varnothing), P(c_i), T(c_i)$ and $T(c_i, c_j)$, for $1 \leq i < j \leq n$ are the only indecomposable representations (up to isomorphisms) over the pair $(\mathbb{R}, \mathbb{C})$ of a completely weak chain $C = \{c_1 \prec \cdots \prec c_n\}$. In fact, if $U = (U_0; U_{c_i} \mid 1 \leq i \leq n)$ is a representation of C over $(\mathbb{R}, \mathbb{C})$, then in the corresponding matrix representation each block U_{c_i}, $1 \leq i \leq n$, may be reduced via admissible transformations to the following standard form:

$$U_{c_i} = \begin{array}{|c|c|} \hline I & \\ \hline & I \\ \hline & \mathbf{i}I \\ \hline & \\ \hline \end{array},$$

where the columns consist of generators of U_{c_i} modulo its radical subspace $\underline{U_{c_i}} = U_{c_{i-1}}$ with respect to a fixed basis of U_0 (in this case, empty cells indicate null coordinates). This result can be generalized in a natural way to the case (F, G) by using a suitable scalar $\mathbf{u} \in G$ instead of the constant $\mathbf{i} \in \mathbb{C}$ in the matrix presentation of U_{c_i} showed above.

3. The Algorithm of Differentiation VII

The differentiation VII is one of the seventeen differentiations developed by Zavadskij to classify (in particular) equipped posets of tame type and of finite growth type [20, 21].

First of all, in this section, we recall the original construction of the algorithm of differentiation VII [20].

A pair of incomparable points (a, b) in an equipped poset $\mathcal{P}$ is said to be VII-*suitable*, if $a \in \mathcal{P}^{\otimes}$, $b \in \mathcal{P}^{\circ}$ and

$$\mathcal{P} = a^{\nabla} + b_{\Delta} + \{a \prec c_1 \prec \cdots \prec c_n\},$$

where $\{a \prec c_1 \prec \cdots \prec c_n\}$ is a completely weak chain incomparable with b, we can assume $n \geq 0$ by setting $c_0 = a$. We let C denote the set $\{c_1 \prec \cdots \prec c_n\}$.

If $\mathcal{P}$ is an equipped poset with a pair of points (a, b) VII-suitable, then the *derived poset*, $\mathcal{P}'$ is an equipped poset with the following shape:

$$\mathcal{P}'_{(a,b)} = \mathcal{P} \setminus \{a + C\} + \{a^- < a^+\} + C^- + C^+,$$

where $a \in (\mathcal{P}')^{\otimes}$, $a^+ \in (\mathcal{P}')^{\circ}$, C and C^+ are completely weak chains with $C^- = \{c_1^- \prec \cdots \prec c_n^-\}$ and $C^+ = \{c_1^+ \prec \cdots \prec c_n^+\}$, $c_i^- \prec c_i^+$ for each i. Furthermore, $a^- \prec c_1^-$, $a^+ < c_1^+$, $c_n^- < b$. Points in the derived poset $\mathcal{P}'$ satisfy the following conditions:

(1) Points a^-, a^+, c_i^- and c_i^+ inherit the relations that points a and c_i had in the original poset $\mathcal{P} \setminus \{a + C\}$.

(2) Order relations in $\mathcal{P} \setminus \{a + C\}$, together with the relations described above induce the order relations in $\mathcal{P}'_{(a,b)}$ (in particular, $a^- \prec c_n^-$).

Figure 2 shows the diagrams of $\mathcal{P}$ and the corresponding $\mathcal{P}'_{(a,b)}$.

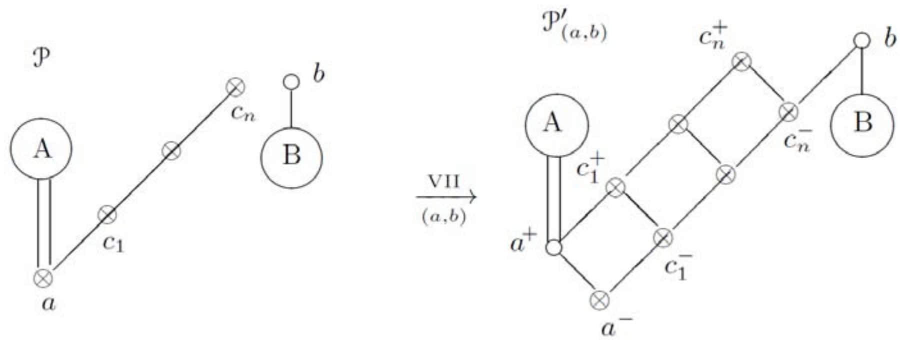


Figure 2

The *functor of differentiation* $D_{(a,\,b)} : \operatorname{rep}\mathcal{P} \to \operatorname{rep}\mathcal{P}'_{(a,b)}$, also denoted $' : \operatorname{rep}\mathcal{P} \to \operatorname{rep}\mathcal{P}'_{(a,b)}$, it is defined in such a way that for $U' = D_{(a,\,b)}(U) \in \operatorname{rep}\mathcal{P}'_{(a,b)}$, if $1 \le i \le n$, then

$$U'_0 = U_0,$$

$$U'_a = U_a \cap U_b,$$

$$U'_{a^+} = F(U_a),$$

$$U'_{c_i^-} = U_{c_i} \cap U_b,$$

$$U'_{c_i^+} = U_{c_i} + F(U_a),$$

$$U'_x = U_x \text{ for the remaining points } x \in \mathcal{P}'_{(a,b)},$$

$$\varphi' = \varphi, \text{ for a linear map-morphism } \varphi : U_0 \to V_0. \tag{3}$$

For example: $P'(a) = P(a^+),\ T'(a) = T'(a,\,c_i) = P^2(a^+)$.

The following result is known as the integration process of the algorithm of differentiation VII [20].

Lemma 3. *For each object* $W \in \mathcal{R}'$, *there exists an object* $U = W^{\uparrow} \in \mathcal{R}$ *such that* $U' \simeq W \oplus P^m(a^+)$, *for some* $m \geq 0$.

In [20], Zavadskij proved that $(W^{\uparrow})^{\downarrow} \simeq W$, $(U^{\downarrow})^{\uparrow} \simeq U$, where $W \simeq U^{\downarrow}$. Zavadskij also proved the following results.

Theorem 4. *In the case of the differentiation VII*, *there exist mutually inverse bijections between classes of indecomposables of the form*:

$$\operatorname{Ind}\mathcal{P}\backslash[T(a),\, T(a, c_i),\, P(a) \mid 1 \leq i \leq n] \rightleftarrows \operatorname{Ind}\mathcal{P}'_{(a,b)}\backslash[P(a^+)]$$

realized by the operations $\downarrow$ *and* $\uparrow$.

Corollary 5. *If* $\Gamma(\mathcal{R})$ *and* $\Gamma(\mathcal{R}')$ *are Gabriel's quiver of the categories* $\mathcal{R}$ *and* $\mathcal{R}'$, *then*

$$\Gamma(\mathcal{R})\backslash[P(a),\, T(a),\, T(a, c_i) \mid 1 \leq i \leq n] \simeq \Gamma(\mathcal{R}')\backslash[P(a^+)].$$

4. On the Auslander-Reiten Quiver of Some Equipped Posets

In this section, we describe the Gabriel's quiver and the Auslander-Reiten quiver of some posets of finite representation type.

Henceforth, for a given equipped poset $\mathcal{P}$ and representations $U, V \in \operatorname{rep}\mathcal{P}$, we let $\operatorname{rad}\mathcal{P}$ denote the radical ideal of the category $\operatorname{rep}\mathcal{P}$, $\operatorname{Hom}(U, V) = \mathcal{R}_{\mathcal{P}}(U, V)$ denote the F-vector space of morphisms between U and V, sometimes we write $\mathcal{R}(U, V)$ instead of $\mathcal{R}_{\mathcal{P}}(U, V)$ if it is clear that U and V are objects of the category of representations of a given poset $\mathcal{P}$. Further, if $X \subset \operatorname{Ind}\mathcal{P}$, then we say that a representation $U \in \operatorname{rep}\mathcal{P}$ is *X-free* whenever no direct summand of U is isomorphic to an element of X.

The following result concerns X-free representations for a given subset $X \subset \operatorname{Ind}\mathcal{P}$.

Proposition 6. *For a given equipped poset* $\mathcal{P}$, *the following results hold*:

(1) $U \in \operatorname{rep}\mathcal{P}$ *is* $P(\varnothing)$*-free if and only if* $\mathcal{R}_{\mathcal{P}}(U, P(\varnothing)) = 0$.

(2) *If* $U \in \operatorname{Ind}\mathcal{P}\backslash P(\varnothing)$, *then* $\operatorname{rad}(P(\varnothing), U) \neq 0$. *Furthermore, any morphism* $f \in \operatorname{rad}(P(\varnothing), U)$ *is a non-split monomorphism*.

Proof. (1) $\Rightarrow$ If $f \in \mathcal{R}(U, P(\varnothing))$ and $U = (U_0; U_x \mid x \in \mathcal{P})$ is $P(\varnothing)$-free, then $U_0 = \sum_{x\in\mathcal{P}} U_x$ with $f(U_x) = 0$ for all $x \in \mathcal{P}$ thus $f = 0$.

$\Leftarrow$ Let $\varphi \neq 0$ be a morphism $f \in \mathcal{R}_{\mathcal{P}}(U, P(\varnothing))$. Since $\widetilde{\varphi}(U_x) = 0$ for all $x \in \mathcal{P}$, $W_0 = \sum_{x\in\mathcal{P}} U_x^+ \subset \operatorname{Ker}\varphi$ ($\operatorname{Ker}\varphi$ denotes the *kernel* of φ). Moreover, since $\varphi \neq 0$, $W_0 \neq U_0$ and $U_0 = E_0 \oplus W_0$. Therefore, U can be written as a sum of representations of the form $W \oplus E$, where W and E are subrepresentations of U induced by W_0 and E_0, respectively, in this case, $E \cong P^m(\varnothing)$ ($m = \dim E_0$).

(2) Clearly, $\mathcal{R}(P(\varnothing), U) = \operatorname{Hom}_F(\mathbb{F}, U_0) \cong U_0 \neq 0$ and any morphism $0 \neq f \in \operatorname{Hom}_F(\mathbb{F}, U_0)$ is a non-split monomorphism as a consequence of the first part of this proof. $\square$

As a direct consequence of Proposition 6, we have that the set of vertices of $\Gamma(R)$ denoted $\Gamma(R)_0$ coincides with $P(\varnothing)$ for any equipped poset $\mathcal{P}$.

For a representation $U \in \operatorname{rep}\mathcal{P}$ of an equipped poset $\mathcal{P}$, we consider the following subsets:

$$\operatorname{supp}_d U = \{x \in \mathcal{P} \mid U_x^- = 0 \text{ and } U_x \neq 0\},$$

$$\operatorname{supp}_f U = \{x \in \mathcal{P} \mid U_x = U_x^+ \neq 0\},$$

$$\operatorname{supp} U = \{x \in \mathcal{P} \mid U_x \neq 0\} \tag{4}$$

which are called the *weak support*, *strong support* and the *support* of the representation U, respectively. Furthermore, we let $\operatorname{nul} U$ denote the complementary subspace of $\operatorname{supp} U$ in $\mathcal{P}$. Note that $\operatorname{supp}_d U \subset \mathcal{P}^{\otimes}$ and $\operatorname{supp}_d U \cup \operatorname{supp}_f U \subset \operatorname{supp} U$.

Given representations $U, V \in \operatorname{rep}\mathcal{P}$ of an equipped poset $\mathcal{P}$ and $\varphi: U_0 \to V_0 \in \operatorname{rep}\mathcal{P}$, then the following identities hold:

$$\sum_{x\in \operatorname{supp}_d U} U_x^- + \sum_{x\in \operatorname{nul} V} U_x^+ \subset Ker\,\varphi \quad \text{and} \quad \sum_{x\in \operatorname{supp} U} U_x^+ \subset \operatorname{Im}\varphi. \tag{5}$$

Furthermore, if $V \in \operatorname{rep}\mathcal{P}$ is $P(\varnothing)$-free, then:

$$\sum_{x\in \operatorname{supp} U} \widetilde{\varphi}(U_x)^+ \subset \operatorname{Im}\varphi \subset \sum_{x\in \operatorname{supp} V} V_x^+. \tag{6}$$

As a consequence of identities (5), we have the following result:

Proposition 7. *If* $U, V \in \operatorname{rep}\mathcal{P}$ *and*

$$U_0 = \sum_{x\in \operatorname{supp}_d V} U_x^- + \sum_{x\in \operatorname{nul} V} U_x^+,$$

then $\mathcal{R}(U, V) = 0$.

Proof. See identities (5).

Proposition 8. *If* $U \in \operatorname{rep}\mathcal{P}$ *such that for some* $X \subset \operatorname{supp}_f U$, $U_0 = \sum_{x\in X} U_x^+$ *and* $V \in \operatorname{rep}\mathcal{P}$ *is a representation such that* $X \subset \operatorname{supp}_d V$, *then* $\mathcal{R}_\mathcal{P}(U, V) = 0$.

Proof. If $\varphi \in \mathcal{R}_\mathcal{P}(U, V)$, then

$$\begin{aligned} \varphi(U_0) &= \varphi\left(\sum_{x\in X} U_x^+\right) = \varphi\left(\sum_{x\in X} U_x\right) \\ &= \sum_{x\in X} \varphi(U_x^-) \subset \sum_{x\in X} \widetilde{\varphi}(U_x)^- \\ &\subset \sum_{x\in X} V_x^- = 0. \end{aligned}$$

Thus, $\varphi = 0$. □

Remark 9. Note that, if $\mathcal{C}$ is a weak chain and $U, V \in \operatorname{Ind} \mathcal{C}$, then a morphism $f \in \mathcal{R}_{\mathcal{C}}(U, V)$ is neither split mono (for split monomorphism) nor split epi (for split epimorphism).

4.1. The Auslander-Reiten quiver of a completely weak chain

In this subsection, we let $\mathcal{C} = \{a_1 \prec a_2 \prec \cdots \prec a_n\}$ denote a weak chain of size *n*.

Proposition 10. *If* $\mathcal{P}$ *is an equipped poset such that* $a \prec B \subset \mathcal{P}$ *with* $a \in \mathcal{P}^{\otimes}$, *then the following conditions hold*:

(1) $\operatorname{End}(P(x)) \cong F$.

(2) *The ring of endomorphisms* $\operatorname{End}(T(x))$ *is a division ring isomorphic to the ring*

$$\Lambda = \left\{\begin{pmatrix} a & b \\ -\lambda b & a - \mu b \end{pmatrix} \middle| a, b \in F\right\}.$$

(3) *The ring of endomorphisms* $\operatorname{End}(T(a, B))$ *is a division ring isomorphic to the ring* Λ.

Proof. (1) First assertion is easy to see provided that $\operatorname{End}(P(x)) = \operatorname{End}(F) \cong F$.

(2) If $\varphi \in \operatorname{End}(T(x))$, then the corresponding matrix representation has the following form:

$$A := \begin{pmatrix} a & b \\ -\lambda b & a - \mu b \end{pmatrix}. \tag{7}$$

Thus, φ is invertible if and only if $a^2 - \mu ab + \lambda b^2 \neq 0$. We note that

$$a^2 - \mu ab + \lambda b^2 \tag{8}$$

is null whenever $a = 0$ and $b = 0$.

If $b \neq 0,$ then (8) can be rewritten as:

$$b^2\left(\left(-\frac{a}{b}\right)^2 + \mu\left(-\frac{a}{b}\right) + \lambda\right),$$

which is not null. On the other hand, if $b = 0,$ then equation (8) can be reduced to a^2 which is null if $a = 0.$ Therefore, φ is invertible as an F-operator if and only if $a \neq 0$ or $b \neq 0.$

Now, we denote $\nu = a^2 - \mu ab + \lambda b^2$ and suppose that A is invertible, thus

$$A^{-1}\begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \frac{1}{\nu}\begin{pmatrix} a - \mu b & -b \\ \lambda b & a \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \frac{1}{\nu}\begin{pmatrix} a - \mu b - \mathbf{u}b \\ \lambda b + \mathbf{u}a \end{pmatrix}$$

$$= \frac{a - \mu b - \mathbf{u}b}{\nu}\begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \in \mathrm{gen}_G\left\{\begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}\right\}.$$

This fact proves that if φ^{-1} exists, then it defines an endomorphism of $\mathrm{End}(T(x)).$ Thus, if an endomorphism $\varphi \in \mathrm{End}(T(x))$ is not null, then it is invertible, therefore, $\mathrm{End}(T(x))$ is a division ring. Moreover, the correspondence $\varphi \to M_\varphi$ which applies to each endomorphism its corresponding matrix representation M_φ defines an isomorphism between $\mathrm{End}(T(x))$ and $\Lambda.$

(3) Arguments described above prove this item. □

The following result is a consequence of Proposition 10.

Corollary 11. *If* $U \in \mathrm{Ind}\,\mathcal{C},$ *then* $rad\,U = 0.$

The following results are consequences of Propositions 6 and 7.

Proposition 12. *If* $\mathcal{C}$ *is a completely weak chain*, *then the following identities hold*:

(1) *If* $i < j$, *then* $\mathcal{R}_{\mathcal{C}}(P(a_i), P(a_j)) = 0$.

(2) $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j)) = 0$, *for each* $1 < i, j \leq n$.

(3) *If* $i < k$, *then* $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j, a_k)) = 0$.

(4) *If* $i < j$, *then* $\mathcal{R}_{\mathcal{C}}(T(a_i), P(a_j)) = 0$.

(5) *If* $i < j$, *then* $\mathcal{R}_{\mathcal{C}}(T(a_i), T(a_j)) = 0$.

(6) *If* $i < j$, *then* $\mathcal{R}_{\mathcal{C}}(T(a_i), T(a_j, a_k)) = 0$.

(7) *If* $i < j$, *then* $\mathcal{R}_{\mathcal{C}}(T(a_i, a_k), P(a_j)) = 0$.

(8) $\mathcal{R}_{\mathcal{C}}(T(a_i, a_s), T(a_j)) = 0$, *for each* $1 \leq i, j \leq n$.

(9) *If* $i < j$ *or* $s < r$, *then* $\mathcal{R}_{\mathcal{C}}(T(a_i, a_s), T(a_j, a_r)) = 0$.

Proof. (1) If $i < j$, then $a_i \in \operatorname{nul} P(a_j)$ and $P_0(a_i) = P_i(a_i)^+$, then $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j)) = 0$ by Proposition 6.

(2) If $k = \max\{i, j\}$, then $a_k \in \operatorname{supp}_f P(a_i)$, $P_0(a_i) = P_k(a_i)^+$ and $a_k \in \operatorname{supp}_d T(a_j)$, therefore, $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j)) = 0$ by Proposition 8.

(3) If $i < k$, then $a_i \in \operatorname{nul} T(a_j, a_k) \bigcup \operatorname{supp}_d T(a_j, a_k)$. If $a_i \in \operatorname{nul} T(a_j, a_k)$, then $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j, a_k)) = 0$ by Proposition 7. On the other hand, if $a_i \in \operatorname{supp}_d T(a_j, a_k)$, then $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j, a_k)) = 0$ by Proposition 8. Assertions (4), (5), (6) and (7) are the consequences of Proposition 7.

(8) If $k = \max\{j, s\}$, then $a_k \in \operatorname{supp}_f T(a_i, a_s)$, $T_0(a_i, a_s) = T_k(a_i, a_s)^+$ and $a_k \in \operatorname{supp}_d T(a_j)$. Proposition 8 allows to conclude $\mathcal{R}_{\mathcal{C}}(T(a_i, a_s), T(a_j)) = 0$.

(9) Suppose that $i < j$. Then $a_i \in \operatorname{supp}_d T(a_i, a_s)$, $T_0(a_i, a_s) = T_i(a_i, a_s)^+$

and $a_i \in \operatorname{nil} T(a_j, a_r)$. Proposition 7 allows to deduce $\mathcal{R}_{\mathcal{C}}(T(a_i, a_s), T(a_j, a_r)) = 0$. On the other hand, if $s < r$, then $a_s \in \operatorname{supp}_f T(a_i, a_s)$, $T_0(a_i, a_s) = T_s(a_i, a_s)^+$ and $a_s \in \operatorname{supp}_d T(a_j, a_r)$, then $\mathcal{R}_{\mathcal{C}}(T(a_i, a_s), T(a_j, a_r)) = 0$ by Proposition 8. □

Remark 13. The following relationships will be used to characterize irreducible morphisms of the category of representations of a completely weak chain:

(i) $\mathcal{R}_{\mathcal{C}}(P(a_i), P(a_j)) \subset \mathcal{R}_{\mathcal{C}}(P(a_i), P(a_k))$ for each $i \leq k \leq j$.

(ii) $\mathcal{R}_{\mathcal{C}}(T(a_i), P(a_j)) \subset \mathcal{R}_{\mathcal{C}}(T(a_i), P(a_k))$ for each $i \leq k \leq j$.

(iii) If $i = j \neq n$, then $\mathcal{R}_{\mathcal{C}}(T(a_i), P(a_i)) \subset \mathcal{R}_{\mathcal{C}}(T(a_i, a_{i-1}), P(a_i))$.

(iv) $\mathcal{R}_{\mathcal{C}}(P(a_i), T(a_j, a_k)) \subset \mathcal{R}_{\mathcal{C}}(P(a_i), T(a_r, a_s))$ for each $j \leq r < s \leq k$.

(v) $\mathcal{R}_{\mathcal{C}}(T(a_j, a_k), P(a_i)) \subset \mathcal{R}_{\mathcal{C}}(T(a_j, a_k), P(a_s))$ for each $i \leq s \leq j$.

(vi) $\mathcal{R}_{\mathcal{C}}(T(a_i), T(a_j)) \subset \mathcal{R}_{\mathcal{C}}(T(a_i), T(a_k))$ for each $j \leq k \leq i$.

(vii) If $j < r < i$, then $\mathcal{R}_{\mathcal{C}}(T(a_i, a_k), T(a_j, a_r)) \subset \mathcal{R}_{\mathcal{C}}(T(a_i, a_k), T(a_s, a_t))$ for each $j \leq s < r$ and $r \leq t \leq i$.

(viii) If $i < r < k$, then $\mathcal{R}_{\mathcal{C}}(T(a_i, a_k), T(a_j, a_r)) \subset \mathcal{R}_{\mathcal{C}}(T(a_i, a_k), T(a_s, a_t))$ for each $j \leq s < i$ and $r \leq t \leq k$.

The next result describes irreducible morphisms of a completely weak chain.

Proposition 14. *For an indecomposable representation* $P(a_i) \in \operatorname{rep} \mathcal{C}$, *the following relationships hold*:

(1) $\operatorname{Irr}(P(a_i), P(a_j)) = 0$ *for each* $1 \leq i, j \leq n$.

(2) $\operatorname{Irr}(T(a_i), P(a_j)) \neq 0$ *if and only if* $i = j = n$. *Moreover, any irreducible morphism* $\varphi \in \operatorname{Irr}_{\mathcal{C}}(T(a_n), P(a_n))$ *is an epimorphism and*

$$\dim_F \operatorname{Irr}(T(a_n), P(a_n)) = 2.$$

(3) $\operatorname{Irr}(P(a_i), T(a_j, a_k))$ *if and only if* $i = k$ *and* $i = j + 1$. *Furthermore, any irreducible morphism* $\varphi \in \operatorname{Irr}(P(a_i), T(a_{i-1}, a_i))$ *is a monomorphism and*

$$\dim_F \operatorname{Irr}(P(a_i), T(a_{i-1}, a_i)) = 2.$$

(4) $\operatorname{Irr}(T(a_j, a_k), P(a_i)) \neq 0$ *if and only if* $i = j$ *and* $k = i + 1$. *Further, any irreducible morphism* $\varphi \in \operatorname{Irr}(T(a_i, a_{i+1}), P(a_i))$ *is an epimorphism and*

$$\dim_F \operatorname{Irr}(T(a_i, a_{i+1}), P(a_i)) = 2.$$

(5) $\operatorname{Irr}(T(a_i), T(a_j)) \neq 0$ *if and only if* $j = i - 1$. *Further, any irreducible morphism* $\varphi \in \operatorname{Irr}(T(a_i), T(a_{i-1}))$ *is an isomorphism and*

$$\dim_F \operatorname{Irr}(T(a_i), T(a_{i-1})) = 2.$$

(6) $\operatorname{Irr}(T(a_i, a_k), T(a_j, a_r)) \neq 0$ *if and only if* $i = j$ *and* $k = r + 1$ $= i + 2$ *or* $k = r$ *and* $i = j - 1 = k - 2$. *Furthermore, any irreducible morphism* $\varphi \in \operatorname{Irr}(T(a_i, a_{i+1}), T(a_i, a_{i+2}))$ *and any irreducible morphism* $\varphi \in \operatorname{Irr}(T(a_{i-1}, a_i), T(a_{i-2}, a_i))$ *is an isomorphism and*

$$\dim_F \operatorname{Irr}(T(a_i, a_{i+1}), T(a_i, a_{i+2})) = \dim_F (T(a_{i-1}, a_i), T(a_{i-2}, a_i)) = 2.$$

Proof. (1) For the case $j \neq i - 1$, we can use Corollary 11, the first part of Proposition 12 and item (i) in Remark 13 to conclude that there are not irreducible morphisms in $\mathcal{R}_{\mathcal{C}}(P(a_i), P(a_j))$. On the other hand, consider that $j = i - 1$, then we can assume that there exists a morphism

$$\varphi \in \mathcal{R}_{\mathcal{C}}(P(a_i), P(a_{i-1}))$$

inducing a morphism $\varphi' = [\varphi \ \ 0]^t \in \mathcal{R}_{\mathcal{C}}(P(a_i), T(a_{i-1}, a_i))$ such that if $\tau = [1 \ \ 0] \in \mathcal{R}_{\mathcal{C}}(T(a_{i-1}, a_i), P(a_i))$, then $\varphi = \tau\varphi'$. Because of this, there are not irreducible morphisms in $\mathcal{R}_{\mathcal{C}}(P(a_i), P(a_{i-1}))$.

(2) $\Rightarrow$) See (4) in Proposition 12 and items (ii), (iii) in Remark 13.

$\Leftarrow$) If $i = j = n$ and $\varphi \in \mathcal{R}_{\mathcal{C}}(T(a_n), P(a_n))$ can be written as a composition of the form

$$T(a_n) \xrightarrow{f} U \xrightarrow{g} P(a_n)$$

in $\operatorname{rep}\mathcal{C}$, then U is a representation $\{P(a_i), T(a_i), T(a_i, a_k)\}$-free, $1 \leq i < n-1$, therefore, U must have as direct summands only representations of types $T(a_n)$ and $P(a_n)$, therefore, f is split mono or g is split epi, furthermore,

$$\operatorname{Irr}(T(a_n), P(a_n)) = \mathcal{R}_{\mathcal{C}}(T(a_n), P(a_n)),$$

thus, any irreducible morphism is an epimorphism and

$$\dim_F \operatorname{Irr}(T(a_n), P(a_n)) = 2.$$

(3) $\Rightarrow$) See proof of Proposition 12 and (iv) in Remark 13.

$\Leftarrow$) Use the arguments as in part (2) of this proof. Parts (4)-(6) can be proved by using arguments of parts (1)-(3) and Propositions 10, 12, Corollary 11 and Remark 13.

The following figure shows the Auslander-Reiten quiver of the completely weak chain $\mathcal{C} = a_1 \prec a_2 \prec \cdots \prec a_{n-1} \prec a_n$.

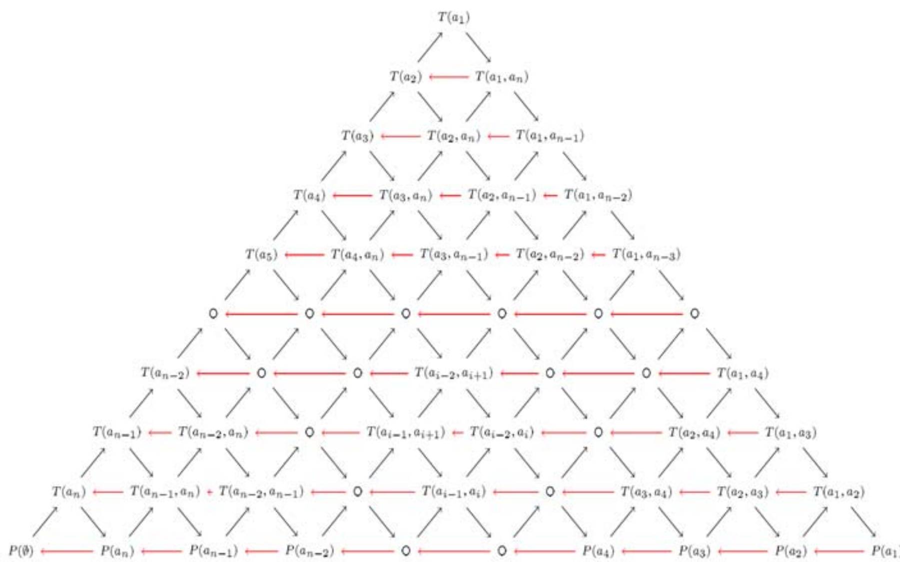


Figure 3. Auslander-Reiten quiver of a completely weak chain.

5. The Auslander-Reiten Quiver of F_{15}

In this section, we give a description of the Auslander-Reiten quiver of the equipped poset $F_{15} = \{\overset{a}{\otimes} \overset{b}{\circ}\}$. To do that, we begin giving the complete list of its indecomposable representations via Lemma 3 and algorithms of differentiation D-VII and completion. Same procedures can be used to describe the Auslander-Reiten quiver of posets F_{16} - F_{19}.

The following figures show the completion (with a strong relation $a < b$) $\overline{F'_{15}}$ of the derived poset F'_{15} and some of its irreducible representations:

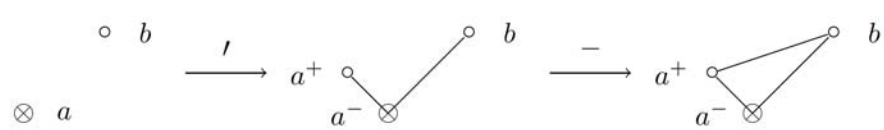
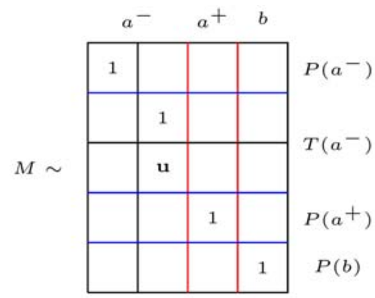


Figure 4



$P(a^-)^{\uparrow} = P(a, b)$	$P(a^+)^{\uparrow} = \begin{array}{\|c\|c\|} \hline 1 & 1 \\ \mathbf{u} & 0 \\ \hline \end{array}$
$T(a^-)^{\uparrow} = \begin{array}{\|c\|cc\|} \hline 1 & 1 & 0 \\ \mathbf{u} & 0 & 1 \\ \hline \end{array}$	$P(b^+)^{\uparrow} = P(b)$

Thus, the complete list of indecomposable representations of F_{15} is:

$$\operatorname{Ind} F_{15} = \{P(\varnothing), P(a), P(b), T(a), P(a, b), P(a^+)^{\uparrow}, T(a^-)^{\uparrow}\}.$$

For the sake of simplicity, let us assume notations $G_1(a, b)$ and $G_2(a, b)$ for representations $P(a^+)^{\uparrow}$ and $T(a^-)^{\uparrow}$, respectively, [6]. Now, we can make a description of irreducible morphisms of F_{15}.

As in the case for a completely weak chain, it is easy to see that End U is a division ring if $U \in \operatorname{Ind} F_{15}$. Actually, the following isomorphisms can be defined:

$$\operatorname{End}(P(\varnothing)) \cong \operatorname{End}(P(a)) \cong \operatorname{End}(P(b)) \cong \operatorname{End}(P(a, b)) \cong \operatorname{End}(G_1((a, b))) \cong \mathbb{F}$$

and

$$\operatorname{End}(T(a)) \cong \operatorname{End}(G_2(a, b)) \cong \Lambda.$$

The following results are consequences of Propositions 6 and 7:

(i) $\mathcal{R}_{F_{15}}(P(a, b), I) = 0 \Leftrightarrow I \in \operatorname{Ind} F_{15} \setminus \{P(a, b)\}$.

(ii) $\mathcal{R}_{F_{15}}(P(a), I) = 0 \Leftrightarrow I \in \operatorname{Ind} F_{15} \setminus \{P(a), P(a, b)\}$.

(iii) $\mathcal{R}_{F_{15}}(P(b), I) = 0 \Leftrightarrow I \in \{P(\varnothing), P(a), T(a)\}$.

(iv) $\mathcal{R}_{F_{15}}(T(a), I) = 0 \Leftrightarrow I \in \{P(\varnothing), P(b)\}$.

Note that the matrix representation of a morphism $\varphi \in \mathcal{R}_{F_{15}}(G_2(a, b), G_1(a, b))$ has the form shown in Figure 4. Furthermore, the following curbing holds $\widetilde{\varphi}(\mathbb{G}^2) \subset \mathbb{G}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$. In particular, $\varphi\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$ and $\varphi\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$ are elements of the subspace $\mathbb{G}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, therefore, $\varphi = 0$. Thus,

(v) $\mathcal{R}_{F_{15}}(G_2(a, b), I) = 0 \Leftrightarrow I \in \operatorname{Ind} F_{15} \setminus \{G_2(a, b), P(a, b)\}$.

On the other hand, if $\varphi \in \mathcal{R}_{F_{15}}(T(a, b), T(a))$, then the matrix representation of φ has the shape given in Figure 4. In fact, $\widetilde{\varphi}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = 0$, therefore, $\varphi = 0$. Thus, as a consequence of Propositions 7 and 8, we have:

(vi) $\mathcal{R}_{F_{15}}(T(a, b), I) = 0 \Leftrightarrow I \in \{P(\varnothing), P(b), T(a)\}$.

The following results regarding morphisms of category $\operatorname{rep} F_{15}$ are consequences of items (i)-(vi) and Proposition 6:

(I) $\operatorname{Irr}(P(\varnothing), P(b)) = \mathcal{R}_{F_{15}}(P(\varnothing), P(b))$ and $\dim \operatorname{Irr}(P(\varnothing), P(b)) = 1$.

(II) $\operatorname{Irr}(P(\varnothing), T(a)) = \mathcal{R}_{F_{15}}(P(\varnothing), P(a))$ and $\dim \operatorname{Irr}(P(\varnothing), T(a)) = 2$.

(III) $\operatorname{Irr}(G_2(a, b), P(a, b)) = \mathcal{R}_{F_{15}}(G_2(a, b), P(a, b))$ and $\dim \operatorname{Irr}(G_2(a, b), P(a, b)) = 2$.

(IV) $\operatorname{Irr}(P(a), P(a, b)) = \mathcal{R}_{F_{15}}(P(a), P(a, b))$ and $\dim \operatorname{Irr}(P(a), P(a, b)) = 1$.

Note that

$$\mathcal{R}_{F_{15}}(T(a),\, G_1(a,\, b)) \cong \mathcal{R}_{F_{15}}(T(a),\, G_2(a,\, b))$$

$$\cong \mathcal{R}_{F_{15}}(G_1(a,\, b),\, G_2(a,\, b)) \cong \Lambda, \tag{9}$$

$$\mathcal{R}_{F_{15}}(P(b),\, G_1(a,\, b)) \cong \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle|\, x \in \mathbb{F} \right\}, \tag{10}$$

$$\mathcal{R}_{F_{15}}(G_1(a,\, b),\, P(a)) \cong \{(0\ x) \mid x \in \mathbb{F}\}. \tag{11}$$

Isomorphisms described in formulas (9) allow defining the following results for a morphism $\varphi := (\alpha\ \ \beta) : T(a) \to P(a)$:

$$\varphi = (\alpha\ \ \beta) = \underbrace{(0\ \ 1)}_{\tau} \underbrace{\begin{pmatrix} \beta - \frac{\mu\alpha}{\lambda} & -\frac{\alpha}{\lambda} \\ \alpha & \beta \end{pmatrix}}_{\gamma},$$

where $\gamma \in \mathcal{R}_{F_{15}}(T(a),\, G_1(a,\, b))$ and $\tau \in \mathcal{R}_{F_{15}}(G_1(a,\, b),\, T(a))$, therefore, $\mathrm{Irr}(T(a),\, P(a)) = 0$. On the other hand, any morphism in $\mathcal{R}_{F_{15}}(T(a),$ $G_1(a,\, b))$ (in $\mathcal{R}_{F_{15}}(G_1(a,\, b),\, P(a))$) passes through direct sums of $T(a)$ and $G_1(a,\, b)$ ($P(a)$ and $T(a,\, b)$, respectively), then:

$$\mathcal{R}_{F_{15}}(T(a),\, G_1(a,\, b)) = \mathrm{Irr}(T(a),\, G_1(a,\, b)), \quad \dim \mathrm{Irr}(T(a),\, G_1(a,\, b)) = 2$$

and

$$\mathcal{R}_{F_{15}}(G_1(a,\, b),\, P(a)) = \mathrm{Irr}(G_1(a,\, b),\, P(a)) \text{ and } \dim \mathrm{Irr}(G_1(a,\, b),\, P(a)) = 1.$$

On the other hand, if $\varphi := \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{R}_{F_{15}}(P(b),\, G_2(a,\, b))$, then

$$\varphi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & -\frac{\beta}{\lambda} \\ \beta & \alpha + \frac{\mu}{\lambda}\beta \end{pmatrix}}_{\tau} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\gamma},$$

where $\gamma \in \mathcal{R}_{F_{15}}(P(b), G_1(a, b))$ and $\tau \in \mathcal{R}_{F_{15}}(G_1(a, b), G_2(a, b))$, then $\mathrm{Irr}(P(b), G_2(a, b)) = 0$. Furthermore,

$\mathcal{R}_{F_{15}}(P(b), G_1(a, b)) = \mathrm{Irr}(P(b), G_1(a, b))$, $\dim \mathrm{Irr}(P(b), G_1(a, b)) = 1$,

$\mathcal{R}_{F_{15}}(G_1(a, b), G_2(a, b)) = \mathrm{Irr}(G_1(a, b), G_2(a, b))$ and

$\dim \mathrm{Irr}(G_1(a, b), G_2(a, b)) = 2$.

Results described above allow defining the Gabriel's quiver of F_{15} in the following way:

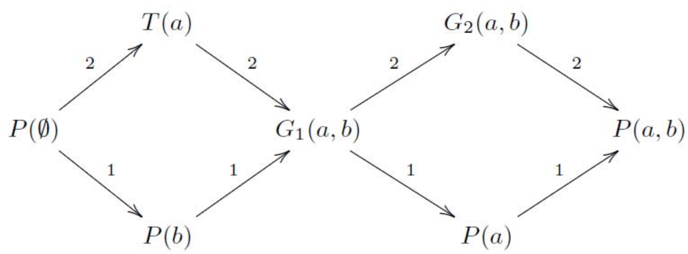


Figure 5

The following Figure 6 shows the Auslander-Reiten quiver of F_{15}.

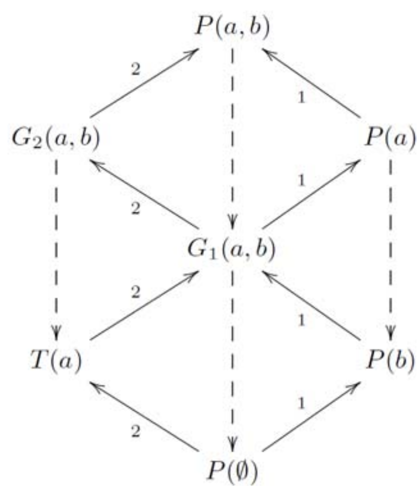


Figure 6

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