JP Journal of Algebra, Number Theory and Applications

Published Online: March 2015
http://dx.doi.org/10.17654/JPANTAFeb2015_001_015
Volume 36, Number 1, 2015, Pages 1-15 **ISSN: 0972-5555**

EXPRESSING INTEGERS AS SUMS OF MANY DISTINCT PRIMES

Abdullah N. Arslan

Computer Science and Information Systems
Texas A&M University - Commerce
TX 75428, U. S. A.
e-mail: abdullah.arslan@tamuc.edu

Abstract

Let $L(n) = \max\{r \mid n$ can be partitioned into $r > 1$ distinct primes, i.e. n can be written as the sum of $r > 1$ distinct primes$\}$ for all sufficiently large positive integers n. We show that $L(n)$ is defined for all $n \geq 12$, i.e. every integer $n \geq 12$ can be partitioned into two or more distinct primes, and we find a nontrivial initial lower bound $\ell(n)$ approaching $\log_2 n$, and an integer n_0 such that $L(n) \geq \ell(n)$ for all $n \geq n_0$. We pose finding an optimal lower bound $\ell(n)$ for $L(n)$ as a challenge to the research community.

1. Introduction

The literature is very rich with works on expressing integers as sums of a small number of primes. The following are conjectured by Goldbach: *Every even integer greater than* 2 *can be expressed as the sum of two primes* (Goldbach's strong conjecture); *every odd number greater than* 7 *can be expressed as the sum of three odd primes* (Goldbach's weak conjecture).

Received: July 27, 2014; Revised: September 29, 2014; Accepted: September 30, 2014
2010 Mathematics Subject Classification: 11P32, 11P81, 11P84.
Keywords and phrases: prime numbers, partitions.
Communicated by Hasan Coskun

Schnirelmann proved [12, 13] that any natural number greater than 1 can be written as the sum of not more than a constant C prime numbers. The lowest such constant C is called *Schnirelmann's constant* which Schnirelmann calculated to be less than 800000. There have been many improvements on decreasing this constant. Ramaré and Saouter showed that every even number $n \geq 4$ is in fact the sum of at most six primes [10]. In 2012, Tao [16] proved that every integer larger than 1 can be written as the sum of at most five primes. Recently, Helfgott [5, 6] claimed to have fully proved Goldbach's weak conjecture for all odd integers greater than 7. Significant progress has been achieved for proving that integers larger than 1 can be expressed as sums of a small number of primes.

Definition 1. Let f and g be two functions defined over the set of positive integers. We say that f is a *lower bound* for g if there exists a positive integer n_0 such that for all $n \geq n_0$, $f(n) \leq g(n)$.

In the current work, if f is a function defined on reals, we still use f as a lower bound for g, if for all integers $n \geq n_0$, $f(n) \leq g(n)$. One can always define a new function f', only on integers, using f such that $f'(n) = f(n)$ for all integers $n \geq n_0$.

In expressing integers as sums of primes, we change the direction, and set the following objective: partition a given positive integer n into maximum number of distinct primes. Naturally, one wonders first if every sufficiently large positive integer can be partitioned into two or more distinct primes (please see Question 1 below). More formally, let $L(n) = \max\{r \mid n$ can be partitioned into $r > 1$ distinct primes$\}$ for all sufficiently large positive integers n. We ask the following questions:

Q.1. Is $L(n)$ defined for all $n \geq n_0$ for some integer $n_0 > 1$?

Q.2. What are some lower bounds $\ell(n)$, integers n_0 such that $L(n) \geq \ell(n) > 1$ for all $n \geq n_0 > 1$?

Q.3. What is an optimal lower bound $\ell(n)$ such that $L(n) \geq \ell(n)$ for all $n \geq n_0$ for some integer $n_0 > 1$?

We analyze Questions 1 and 2 and answer them together. We show that $L(n)$ is defined for all integers $n \geq n_0 \geq 12,$ and find several lower bounds $\ell(n) > 1$ for $L(n).$ We pose Question 3 as a challenge for the research community.

For all $n \geq 17,$ the number of primes smaller than n is $\pi(n) > n/\ln n$ [11]. We note that in our questions, the distinctness requirement for primes in partitioning n makes finding a lower bound $\ell(n)$ for $L(n)$ challenging.

The contributions of the current work are the following: we show that $L(n)$ is actually a function defined for all integers $n \geq 12.$ We prove that, for every real $K > 5,$

$$L(n) \geq \left\lceil \log_{\frac{2K}{K-1}} n + \left(\log_{5/2}\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil \right) + 3 - \log_{5/2} 50 - \log_{\frac{2K}{K-1}} (2n_0(K)) \right) \right\rceil$$

for all integers $n \geq \dfrac{K-1}{K} n_0(K),$ where $n_0(K) \geq 25.$ Parameters K and $n_0(K)$ are as described in the following implication of the Prime Number Theorem: for every real $K > 1,$ there exists a positive integer $n_0(K)$ such that there is always a prime in $(n, (1 + 1/K)n)$ for all $n \geq n_0(K).$ Two particular corollaries from our results are the following: (1) Every integer $n \geq 14$ can be partitioned into at least 3 distinct prime numbers (without repeating primes); (2) Every integer $n \geq 4021520$ can be expressed as the sum of at least $\lceil \log_{2.000120511} n - 6.9375 \rceil$ distinct primes (no prime is used more than once in the sum).

The outline of this paper is the following: we summarize the literature on existence of primes in precalculated intervals in Section 2. We give nontrivial lower bounds $\ell(n)$ for function $L(n)$ in Section 3. We have concluding remarks in Section 4.

2. Existence of Primes in Precalculated Intervals

Calculating small intervals which are guaranteed to contain prime numbers has been a topic of research in the literature. For example, Ramanujan [9] shows that there exists a prime number in $(n, 2n)$ for every integer n larger than 1. The Bertrand-Chebyshev Theorem states that for every integer $n > 3$, there always exists at least one prime number p in $(n, 2n - 2)$. Existence of primes in different intervals has been shown by many researchers (e.g. in $[2n, 3n]$ by El Bachraoui [4], in $(3n, 4n)$ by Loo [7]). There has been a race for proving existence of primes in smaller intervals. In 1952, Nagura proved that for $n \geq 25$, there is always a prime in $(n, (1 + 1/5)n)$ [8]. Schoenfeld [15] proved that $(n, (1 + 1/16597)n)$ contains a prime for $n \geq 2010760$. Ramaré and Saouter [10] showed that $[n(1 - 1/28314000), n]$ contains a prime for $n \geq 10726905041$. There are also results involving intervals whose sizes are not a constant fraction of n, but a small function of n. Existence of primes is shown in $(n, (1 + 1/(2 \ln^2 n))n)$ for $n \geq 3275$ by Dusart [1]. Currently, the best result in this direction belongs to Dusart [2]: for $n \geq 396738$, there is at least one prime between n and $(1 + 1/(25 \ln^2 n))n$.

The Prime Number Theorem implies that for every real $\varepsilon > 0$, there exists a positive integer n_0 such that there is always a prime between n and $(1 + \varepsilon)n$ for all $n > n_0$. We use the following corollary of this statement.

Corollary 1. *For every real $K > 1$, there exists a positive integer $n_0(K)$ such that there is always a prime in $(n, (1 + 1/K)n)$ for all integers $n \geq n_0(K)$.*

3. Expressing an Integer as the Sum of Many Distinct Primes

Definition 2. Let $L(n) = \max\{r \mid n$ can be partitioned into $r > 1$ distinct primes, i.e. n can be written as the sum of $r > 1$ distinct primes$\}$ for all sufficiently large positive integers n. For a given positive integer n, $L(n)$ would be undefined if n cannot be partitioned into two or more distinct primes.

We note that $L(n)$ is not defined for some $n < 12$. For example, 6 cannot be partitioned into distinct prime numbers. The prime number 11 can only be partitioned into one prime (itself) when the summands have to be distinct. $L(12) = 3$ since 12 can be partitioned into maximum three distinct primes $(12 = 2 + 3 + 7)$, and $L(13) = 2$ since 13 can be partitioned into maximum two distinct primes $(13 = 2 + 11)$. All integers $n \geq 14$ can be expressed as the sum of at least three distinct primes. We state this claim in the following lemma.

Lemma 1. *Let* $\ell_1(x) = \lceil \log_{5/2} x + (3 - \log_{5/2} 50)\rceil$ *for all real* $x \geq 14$. *For all integers* $n \geq 14$, $L(n)$ *is defined, and* $L(n) \geq \ell_1(n)$. *That is, for all integers* $n \geq 14$, n *can be partitioned into at least* $\ell_1(n) > 1$ *distinct primes.*

Proof. We prove this by induction on *n*. We also use the fact that for any real $x \geq 14$,

$$\ell_1(\lceil x \rceil) = \lceil \log_{5/2}\lceil x \rceil + (3 - \log_{5/2} 50)\rceil \geq \lceil \log_{5/2} x + (3 - \log_{5/2} 50)\rceil.$$

For all integers $n \in [14, 50]$, from Table 1, we see that $L(n)$ is defined (i.e. n can be partitioned into two or more distinct primes), and $L(n) \geq 3$ and $3 \geq \ell_1(n) \geq \lceil 3 + (\log_{5/2} n - \log_{5/2} 50)\rceil$ because $n \leq 50$ in this case. Also, in this interval,

$$\ell_1(n) \geq \lceil 3 + (\log_{5/2} n - \log_{5/2} 50)\rceil \geq \lceil 3 + (\log_{5/2} 14 - \log_{5/2} 50)\rceil$$

$$= 2 > 1.$$

This proves the claims in the lemma for these special (base) cases (i.e. when $n \in [14, 50]$). Consider $n > 50$, and assume that for all integers $m \in [14, n)$, $L(m)$ is defined, and $L(m) \geq \ell_1(m)$, i.e. m can be expressed as the sum of at least $\ell_1(m) = \lceil \log_{5/2} m + (3 - \log_{5/2} 50)\rceil > 1$ distinct primes (no prime is used more than once in the sum). Due to Nagura [8], for $n \geq 25$, there is always a prime between $(n, (1 + 1/5)n)$. We apply this result by using $\lfloor n/2 \rfloor$

for n. If n is even, we choose a prime p in $\left(\frac{n}{2}, \frac{3}{5}n\right)$ that we know exists when $n > 50$. In this case, $n - p$ is in $\left[\frac{2}{5}n, \frac{n}{2}\right]$, and since it is an integer, $n - p$ is in $\left[\left\lceil\frac{2}{5}n\right\rceil, \frac{n}{2}\right]$. The prime number p that we chose in this case is strictly larger than all integers in this interval for $n - p$ (therefore distinct) since p is in $\left(\frac{n}{2}, \frac{3}{5}n\right)$. If n is odd, we choose a prime p that we know exists in $\left(\frac{n-1}{2}, \frac{3}{5}(n-1)\right)$ when $n > 50$. Since p is an integer, the smallest possible p in this interval is $\frac{n+1}{2}$ for odd n. In this case, $n - p$ is in $\left[\frac{2n+3}{5}, \frac{n+1}{2}\right]$, and since $n - p$ is an integer, $n - p$ is in $\left[\left\lceil\frac{2n+3}{5}\right\rceil, \frac{n+1}{2}\right]$. The prime number p that we chose in this case is either larger than all integers in the interval $\left[\left\lceil\frac{2n+3}{5}\right\rceil, \frac{n-1}{2}\right]$ containing possible values for $n - p$, or $n - p = p = \frac{n+1}{2}$. However, the latter is not possible because $n - p = p$ implies that n is even, which is a contradiction. In this case, $n - p$ must be in $\left[\left\lceil\frac{2n+3}{5}\right\rceil, \frac{n-1}{2}\right]$, and the prime number p must be larger (therefore distinct) than all the integers in this interval. An alternate way to see this is that by the induction hypothesis, $n - p$ $(\leq p$ in this case$)$ is partitioned into two or more distinct primes. Each of these primes must be strictly less than $n - p$ (therefore, they are also strictly less than p since $n - p \leq p$). That is, in all cases (whether n is even or odd), p is distinct from all the integers used in partitioning $n - p$, and from the interval limits, we see that $n - p \geq \left\lceil\frac{2}{5}n\right\rceil$. If $n - p \leq 50$, then since $n - p \geq \lceil 2n/5 \rceil$, it must be the case that $n - p \geq 21$ since $n > 50$, and therefore, $n - p \in [21, 50]$. That is, the integer $n - p$ in this case falls in $[14, 50]$, and we already proved the claims of Lemma 1 using Table 1 for these special cases. Therefore, in this case $L(n - p)$ is defined (i.e. $n - p$ can be partitioned into two or more distinct primes), and

$$L(n - p) \geq \ell_1(n - p) \geq \ell_1(\lceil 2n/5 \rceil) \geq \lceil \log_{5/2}(2n/5) + (3 - \log_{5/2} 50) \rceil > 1.$$

If $n - p > 50$, by our induction hypothesis, $L(n - p)$ is defined (i.e. $n - p$ can be partitioned into two or more distinct primes), and $L(n - p) \geq \ell_1(n - p) \geq \ell_1(\lceil 2n/5 \rceil) \geq \lceil \log_{5/2}(2n/5) + (3 - \log_{5/2} 50) \rceil > 1$. In other words, in all possible cases for $n - p$, $L(n - p)$ is defined and $n - p$ can be expressed as the sum of at least $\ell_1(n - p) \geq \ell(\lceil 2n/5 \rceil) \geq \lceil \log_{5/2}(2n/5) + (3 - \log_{5/2} 50) \rceil > 1$ distinct primes. Hence, including *p*, *n* can be expressed as the sum of at least

$$\begin{aligned} \ell_1(\lceil 2n/5 \rceil) + 1 &\geq \lceil \log_{5/2}(2n/5) + (3 - \log_{5/2} 50) \rceil + 1 \\ &= \lceil \log_{5/2} n + \log_{5/2}(2/5) + (3 - \log_{5/2} 50) + 1 \rceil \\ &= \lceil \log_{5/2} n - 1 + (3 - \log_{5/2} 50) + 1 \rceil \\ &= \lceil \log_{5/2} n + (3 - \log_{5/2} 50) \rceil > 1 \end{aligned}$$

distinct primes since $n > 50$. Therefore, the claims in Lemma 1 are correct. When we consider the partitioning of *n* done recursively in the above proof, we note that by a series of found primes *p*, and updates of *n* to $n - p$, eventually *n* will become an integer in the special cases, i.e. in $[14, 50]$. Then summations in Table 1 will complete the partitioning of *n* into distinct primes.

Table 1. A list of summation expressions that we use to prove that for all *n* in $[14, 50]$, $L(n)$ is defined (i.e. *n* can be partitioned into two or more distinct primes), and $L(n) \geq 3 \geq \lceil 3 + (\log_{5/2} n - \log_{5/2} 50) \rceil > 1$

n	$L(n) \geq$	*summation*	n	$L(n) \geq$	*summation*
14	3	2+5+7	33	4	2+3+5+23
15	3	3+5+7	34	5	2+3+5+7+17
16	3	2+3+11	35	4	2+3+7+23
17	4	2+3+5+7	36	5	2+3+5+7+19
18	3	2+3+13	37	4	2+7+11+17
19	3	3+5+11	38	5	2+5+7+11+13
20	3	2+7+11	39	5	3+5+7+11+13
21	4	2+3+5+11	40	5	2+3+5+7+23
22	3	2+7+13	41	6	2+3+5+7+11+13
23	4	2+3+5+13	42	5	2+3+5+13+19
24	3	2+3+19	43	5	3+5+7+11+17
25	4	2+3+7+13	44	5	2+3+5+11+23
26	4	3+5+7+11	45	6	2+3+5+7+11+17
27	4	2+3+5+17	46	5	2+3+5+7+29
28	5	2+3+5+7+11	47	6	2+3+5+7+11+19
29	4	2+3+5+19	48	5	2+3+5+7+31
30	5	2+3+5+7+13	49	6	2+3+5+7+13+19
31	4	2+3+7+19	50	5	2+3+5+11+29
32	4	3+5+7+17			

The following corollary is due to Lemma 1 and Table 1.

Corollary 2. *Every integer* $n \geq 14$ *can be expressed as the sum of at least* 3 *distinct prime numbers* (*no prime is used more than once in the sum*). *That is,* $L(n)$ *is defined and* $L(n) \geq 3$ *for all integers* $n \geq 14$.

Since we know that $L(n)$ is defined for all integers $n \geq 12$, we focus on finding a better lower bound $\ell(n)$ for $L(n)$.

Theorem 1. *For any real* $K > 5$, *let* $\ell(x) = \ell_K(x) = \lceil \log_{\frac{2K}{K-1}} x + (T - \log_{\frac{2K}{K-1}}(2n_0(K))) \rceil$ *for all real* $x \geq \frac{K-1}{K} n_0(K)$, *where*

$$T = \ell_1\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil\right) = \left\lceil \log_{5/2}\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil\right) + (3 - \log_{5/2} 50) \right\rceil.$$

Then n can be partitioned into two or more distinct primes, and $L(n) \geq \ell(n) > 1$ *for all integers* $n \geq \frac{K-1}{K} n_0(K)$, *where K and* $n_0(K)$ *are the numbers as described in Corollary* 1, *and* $K > 5$, $n_0(K) \geq 25$.

Proof. We prove this theorem by induction on *n*. It is important for the correctness of implications to note that for any real $x \geq \frac{K-1}{K} n_0(K)$, $\ell(\lceil x \rceil) = \lceil \log_{\frac{2K}{K-1}} \lceil x \rceil + (T - \log_{\frac{2K}{K-1}} (2n_0(K))) \rceil \geq \lceil \log_{\frac{2K}{K-1}} x + (T - \log_{\frac{2K}{K-1}} (2n_0(K))) \rceil$. The proofs of Theorem 1 and Lemma 1 are similar. For all integers $n \in \left[\frac{K-1}{K} n_0(K), 2n_0(K)\right]$, by Lemma 1, and by the definition of T, $L(n) \geq \ell_1(n) \geq \ell_1\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil\right) = T \geq \lceil T + (\log_{\frac{2K}{K-1}} n - \log_{\frac{2K}{K-1}} (2n_0(K))) \rceil = \ell(n) \geq \lceil T - 1 \rceil$ because $\log_{\frac{2K}{K-1}} n - \log_{\frac{2K}{K-1}} (2n_0(K)) \geq \log_{\frac{2K}{K-1}} \left(\frac{K-1}{K} n_0(K)\right) - \log_{\frac{2K}{K-1}} (2n_0(K)) = \log_{\frac{2K}{K-1}} \left(\frac{K-1}{2K}\right) = -1$ for $n \in \left[\frac{K-1}{K} n_0(K), 2n_0(K)\right]$, $K > 5$, and $n_0(K) \geq 25$. Also, by Lemma 1, by the definition of *T*, and since $K > 5$, $\frac{25(K-1)}{K} > 20$, $\left\lceil \frac{25(K-1)}{K} \right\rceil \geq 21$, and $T = \ell_1\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil\right) = \left\lceil \log_{5/2}\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil\right) + (3 - \log_{5/2} 50) \right\rceil \geq \lceil \log_{5/2} 21 + (3 - \log_{5/2} 50) \rceil = 3$. In this interval, since we found that $\ell(n) \geq \lceil T - 1 \rceil$, and since $T \geq 3$, we conclude that $\ell(n) > 1$. These verify the inequality in the theorem for these special (base) cases. Consider $n > 2n_0(K) \geq 50$, and assume that for all $m \in \left[\frac{K-1}{K} n_0(K), n\right]$, $L(m) \geq \ell(m)$, i.e. *m* can be expressed as the sum of at least $\ell(m) = \lceil \log_{\frac{2K}{K-1}} m + (T - \log_{\frac{2K}{K-1}} (2n_0(K))) \rceil > 1$ distinct primes (no prime is used more than once in the sum). Corollary 1 states that there is always a prime in $\left(n, \frac{K+1}{K} n\right)$ when $n \geq n_0(K)$. We apply this result by using $\lfloor n/2 \rfloor$ for n. If n is even, we choose a prime p in $\left(\frac{n}{2}, \frac{K+1}{2K} n\right)$ that we know exists since $n > 2n_0(K) \geq 50$. In this case, $n - p$ is in $\left[\frac{K-1}{2K} n, \frac{n}{2}\right]$, and since it is an integer, $n - p$ is in $\left[\left\lceil \frac{K-1}{2K} n \right\rceil, \frac{n}{2}\right]$. The prime number p that we chose in this case is strictly larger than all integers in this interval for $n - p$ (therefore distinct) since p is in $\left(\frac{n}{2}, \frac{K+1}{2K} n\right)$. If n is odd, then we choose a prime p that we know exists in $\left(\frac{n-1}{2}, \frac{K+1}{2K} (n-1)\right)$ since $n > 2n_0(K) \geq 50$. Since p is an integer, the smallest possible p in this interval is $\frac{n+1}{2}$ for odd n. In this case, $n - p$ is in $\left[\frac{(K-1)n + K + 1}{2K}, \frac{n+1}{2}\right]$, and since $n - p$ is an integer, $n - p$ is in $\left[\left\lceil \frac{(K-1)n + K + 1}{2K} \right\rceil, \frac{n+1}{2}\right]$. The prime number p that we chose in this case is either larger than all integers in the interval $\left[\left\lceil \frac{(K-1)n + K + 1}{2K} \right\rceil, \frac{n-1}{2}\right]$ containing possible values for $n - p$, or $n - p = p = \frac{n+1}{2}$. However, the latter is not possible because $n - p = p$ implies that n is even, which is a contradiction. In this case, $n - p$ must be in $\left[\left\lceil \frac{(K-1)n + K + 1}{2K} \right\rceil, \frac{n-1}{2}\right]$, and the prime number p must be larger (therefore distinct) than all the integers in this interval. An alternate way to see this is that by the induction hypothesis, $n - p$ ($\leq p$ in this case) is partitioned into two or more distinct primes. Each of these primes must be strictly less than $n - p$ (therefore, they are also strictly less than p since $n - p \le p$). That is, in all cases (whether n is even or odd), p is distinct from all integers used in partitioning $n - p$, and from the interval limits, we see that $n - p \ge \left\lceil \frac{K-1}{2K} n \right\rceil$. If $n - p \le 2n_0(K)$, then since $n - p \ge \left\lceil \frac{K-1}{2K} n \right\rceil$, it must be the case that $n - p > \frac{K-1}{K} n_0(K)$ since $n > 2n_0(K) \ge 50$, and therefore, $n - p \in \left(\frac{K-1}{K} n_0(K), 2n_0(K) \right]$. That is, the integer $n - p$ falls in the set of the special cases (i.e. integers in $\left[\frac{K-1}{K} n_0(K), 2n_0(K) \right]$) for which we verified the claims of Theorem 1 by using Lemma 1. Therefore, in this case,

$$L(n - p) \ge \ell(n - p) \ge \ell\left(\left\lceil \frac{K-1}{2K} n \right\rceil \right)$$

$$\ge \left\lceil \log_{\frac{2K}{K-1}} \left(\frac{K-1}{2K} n \right) + \left(T - \log_{\frac{2K}{K-1}} (2n_0(K))\right) \right\rceil > 1.$$

If $n - p > 2n_0(K)$, by using our induction hypothesis, we see that

$$L(n - p) \ge \ell(n - p) \ge \ell\left(\left\lceil \frac{K-1}{2K} n \right\rceil \right)$$

$$\ge \left\lceil \log_{\frac{2K}{K-1}} \left(\frac{K-1}{2K} n \right) + \left(T - \log_{\frac{2K}{K-1}} (2n_0(K))\right) \right\rceil > 1.$$

In other words, in all possible cases, $n - p$ can be expressed as the sum of at least

$$\ell(n - p) \ge \ell\left(\left\lceil \frac{K-1}{2K} n \right\rceil \right)$$

$$\ge \left\lceil \log_{\frac{2K}{K-1}} \left(\frac{K-1}{2K} n \right) + \left(T - \log_{\frac{2K}{K-1}} (2n_0(K))\right) \right\rceil > 1$$

distinct primes. Hence, including p, n can be expressed as the sum of at least

$$\ell\left(\left\lceil \frac{K-1}{2K} n \right\rceil\right) + 1$$

$$\geq \left\lceil \log_{\frac{2K}{K-1}} n + \log_{\frac{2K}{K-1}}\left(\frac{K-1}{2K}\right) + (T - \log_{\frac{2K}{K-1}}(2n_0(K))) \right\rceil + 1$$

$$= \lceil \log_{\frac{2K}{K-1}} n - 1 + (T - \log_{\frac{2K}{K-1}}(2n_0(K))) + 1 \rceil$$

$$= \lceil \log_{\frac{2K}{K-1}} n + (T - \log_{\frac{2K}{K-1}}(2n_0(K))) \rceil > 1$$

distinct primes as claimed since $n > 2n_0(K)$ and $T > 1$. Considering the recursive partitioning of n in the above proof, we note that by a series of found primes p, and updates of n to $n - p$, eventually n will become an integer in the special cases, i.e. in $\left[\frac{K-1}{K} n_0(K), 2n_0(K)\right]$. Then the successful completion of the partitioning of n into distinct primes is guaranteed by Lemma 1 when n is in this interval.

We also note that in Theorem 1, the constant T can be replaced by any value less than or equal to the minimum $\ell_{K'}(n)$ value where n is an integer in $\left[\frac{K'-1}{K'} n_0(K'), 2n_0(K')\right]$ and $5 < K' < K$. We can choose

$$T = \ell_{K'}\left(\left\lceil \frac{K'-1}{K'} n_0(K') \right\rceil\right).$$

The value of T can be improved by applying Theorem 1 using larger parameters K' (approaching, from below, the actual value of K in Theorem 1).

If we use Schoenfeld's [15] proof of existence of a prime in $(n, (1 + 1/16597)n)$ for $n \geq 2010760$ in Theorem 1, with $K = 16597$, and $2n_0(K) = 4021520$, then we obtain the following:

$$T = \left\lceil \log_{5/2}\left(\left\lceil \frac{16596}{16597} 2010760 \right\rceil\right) + (3 - \log_{5/2} 50) \right\rceil$$

$$= \lceil 15.83991 + (3 - 4.269413) \rceil = 15,$$

and using this as a lower bound value for T,

$$L(n) \geq \ell(n) = \lceil \log_{\frac{33194}{16596}} n + (15 - \log_{\frac{33194}{16596}} 4021520) \rceil$$

$$\geq \lceil \log_{2.000120511} n + 15 - 21.9375 \rceil$$

$$= \lceil \log_{2.000120511} n - 6.9375 \rceil.$$

We summarize this result in the following corollary:

Corollary 3. *Every integer* $n \geq 4021520$ *can be expressed as the sum of at least* $\lceil \log_{2.000120511} n - 6.9375 \rceil$ *distinct primes* (*no prime is used more than once in the sum*).

Theorem 1 gives $\lceil \log_{2.000120511} n - 6.9375 \rceil$ as a lower bound for $L(n)$ for all $n \geq 4021520$. With larger values of K in Theorem 1, this lower bound approaches $\log_2 n - c$ for some positive real constant c for all sufficiently large n. Naturally the next question we ask is how good a lower bound this is for $L(n)$. We note that the number of distinct naturals yielding the sum n cannot be more than $\sqrt{2n}$ because $\sum_{i=1}^{\sqrt{2n}} i > n$. Therefore, $L(n) < \sqrt{2n}$. For every integer $n \geq 17$, there are more than $n/\ln n$ primes smaller than n [11]. This makes us believe that there is hope for improving the lower bound $\ell(n)$ in Theorem 1 for $L(n)$ to functions asymptotically larger than $\log_2 n$.

4. Conclusion

We have introduced $L(n) = \max\{r \mid n$ can be partitioned into $r > 1$ distinct primes$\}$ for all sufficiently large positive integers n. We showed that $L(n)$ is defined for all integers $n \geq 12$, and for every real $K > 5$,

$L(n) \geq \ell(n) =$

$$\left\lceil \log_{\frac{2K}{K-1}} n + \left(\log_{5/2}\left(\left\lceil \frac{K-1}{K} n_0(K) \right\rceil \right) + 3 - \log_{5/2} 50 - \log_{\frac{2K}{K-1}} (2n_0(K)) \right) \right\rceil$$

for all integers $n \geq \dfrac{K-1}{K} n_0(K)$, where $n_0(K) \geq 25$. In particular, one corollary of our results is that every integer $n \geq 14$ can be partitioned into at least 3 distinct prime numbers (without repeating primes). Another one is that every integer $n \geq 4021520$ can be expressed as the sum of at least $\lceil \log_{2.000120511} n - 6.9375 \rceil$ distinct primes. Finding an optimal lower bound $\ell(n)$ for $L(n)$ will be an interesting challenge that calls for further research. The distinctness requirement of the primes in partitioning n is one major source of difficulty in attaining an optimal bound.

References

[1] P. Dusart, Autour de la fonction qui compte le nombre de nombres premiers, 1998 (in French).

[2] P. Dusart, Estimates of some functions over primes without R.H., 2010, arXiv:1002.0442.

[3] T. O. e Silva, Goldbach conjecture verification, Retrieved 20 July 2013.

[4] M. El Bachraoui, Primes in the interval $[2n, 3n]$, Int. J. Contemp. Math. Sci. 1(13) (2006), 617-621.

[5] H. A. Helfgott, Major arcs for Goldbach's theorem, 2013, arXiv:1305.2897.

[6] H. A. Helfgott, Minor arcs for Goldbach's problem, 2012, arXiv:1205.5252.

[7] A. Loo, On the primes in the interval $(3n, 4n)$, Int. J. Contemp. Math. Sci. 6(38) (2011), 1871-1882.

[8] J. Nagura, On the interval containing at least one prime number, Proceedings of the Japan Academy, Series A 28 (1952), 177-181.

[9] S. Ramanujan, A proof of Bertrand's postulate, J. Indian Math. Society 11 (1919), 181-182.

[10] O. Ramaré and Y. Saouter, Short effective primes containing primes, J. Number Theory 98 (2003), 10-33.

[11] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), 6494.

[12] L. G. Schnirelmann, On the additive properties of numbers, First Published in Proceedings of the Don Polytechnic Institute in Novocherkassk (in Russian), Vol. XIV (1930), 3-27, and reprinted in Uspekhi Matematicheskikh Nauk (in Russian), (6) (1939), 9-25 (1930).

[13] L. G. Schnirelmann, Über additive Eigenschaften von Zahlen, Mathematische Annalen (in German), Vol. 107 (1933), 649-690, and also reprinted as, On the additive properties of numbers, Uspekhi Matematicheskikh Nauk (in Russian) 7 (1940), 7-46 (1933).

[14] M. K. Sinisalo, Checking the Goldbach conjecture up to 4×10^{11}, Math. Comp. 61(204) (1993), 931-934.

[15] L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, II, Math. Comp. 30(134) (1976), 337-360.

[16] T. Tao, Every odd number greater than 1 is the sum of at most five primes, 2012, arXiv:1201.6656v4 [math.NT]. Bibcode 2012arXiv1201.6656T.