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SOME PROPERTIES ON OPERATIONS OF FUZZY GRAPH

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Abstract

This paper discusses about the properties between complete fuzzy graph and some operations on fuzzy graph like Cartesian product, symmetric difference, complement, conjunction, disjunction and rejection.

1. Introduction

Zadeh [12] introduced fuzzy sets in 1965 to represent/manipulate data and information possessing non-statistical uncertainties. It was, particularly, designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. But it was Rosenfeld [10] who considered fuzzy relations on fuzzy

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sets and developed the theory of fuzzy graphs in 1975. Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. There are several operations on G_1 and G_2 which result in a graph G whose set of points is the Cartesian product $V_1 \times V_2$, where $V_1 \times V_2$ is the point set of $G_1 \times G_2$. These include the Cartesian product, the composition and the tensor product. Other operations of this form are developed in Harary and Wilcox [3] and they investigated some invariant properties of them. Operations on (crisp) graphs such as conjunction, disjunction, rejection and symmetric difference were extended to fuzzy graphs and a methodology is proposed to find the resulting fuzzy graphs of the same operations using adjacency matrices of G_1 and G_2 [7]. Bhutani [1] introduced the notion of weak isomorphism and isomorphism between fuzzy graphs. Nagoorgani and Malarvizhi [8] discussed the order, size and degree of the vertices of the isomorphic fuzzy graphs. Nagoorgani and Latha [6] introduced neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs and a comparative study between them had been analyzed. In the paper, properties between complete fuzzy graph and some operations like Cartesian product, symmetric difference, complement conjunction, disjunction and rejection of fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are discussed.

2. Preliminaries

A fuzzy subset of a nonempty set S is a mapping $\sigma : S \to [0, 1]$. A fuzzy relation on S is a fuzzy subset of $S \times S$. If μ and ν are fuzzy relations, then $\mu \circ \nu(u, w) = Sup\{\mu(u, v) \Lambda \nu(v, w) : v \in S\}$ and

$$\mu^k(u, v) = Sup\{\mu(u, u_1) \Lambda \mu(u_1, u_2)$$

$$\Lambda \mu(u_2, u_3) \Lambda \ldots \Lambda \mu(u_{k-1}, v) : u_1, u_2, ..., u_{k-1} \in S\},$$

where ‘Λ’ stands for minimum. A fuzzy graph is a pair of functions

$\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, where for all u, v in V, we have $\mu(u, v) \leq \sigma(u) \Lambda \sigma(v)$.

Definition 2.1 [4]**.** The underlying crisp graph of a fuzzy graph $G = (\sigma, \mu)$ is denoted by $G^* = (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V / \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V / \mu(u, v) > 0\}$.

Definition 2.2 [4]**.** A fuzzy graph $G = (\sigma, \mu)$ is a complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in \sigma^*$.

Definition 2.3 [4]**.** A fuzzy graph G is said to be a *strong fuzzy graph* if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all (x, y) in μ^*.

Definition 2.4 [5]**.** Let $G = (\sigma, \mu)$ be a fuzzy graph. Then the degree of a vertex u is $d_G(u) = d(u) = \sum_{u \neq v} \mu(u, v) = \sum_{(u, v) \in E} \mu(u, v)$.

Definition 2.5 [5]**.** Let $G = (\sigma, \mu)$ be a fuzzy graph. Then G is irregular, if there is a vertex which is adjacent to vertices with distinct degree.

Definition 2.6 [5]**.** Let $G = (\sigma, \mu)$ be a connected fuzzy graph. Then G is said to be a *neighbourly irregular fuzzy graph* if every two adjacent vertices of G have distinct degree.

Definition 2.7 [11]**.** The complement of a fuzzy graph $G : (\sigma, \mu)$ is a fuzzy graph $\overline{G} : (\overline{\sigma}, \overline{\mu})$, where $\overline{\sigma} = \sigma$ and $\overline{\mu}(u, v) = \sigma(u) \Lambda \sigma(v) - \mu(u, v)$, $\forall u, v \in V$.

Definition 2.8 [6]**.** An isomorphism of neighbourly irregular fuzzy graphs $h : G \to G'$ is a map $h : V \to V'$ which is bijective that satisfies (i) $\sigma(u) = \sigma'(h(u))$, $\forall u \in V$ and (ii) $\mu(u, v) = \mu'(h(u), h(v))$, $\forall u, v \in V$. It is denoted by $G \cong G'$.

Definition 2.9 [5]**.** Let $G = (\sigma, \mu)$ be a connected fuzzy graph. Then G is said to be a *highly irregular fuzzy graph* if every vertex of G is adjacent to vertices with distinct degrees.

Definition 2.10 [4]**.** Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of X_i, $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \times \sigma_2$ of V and $\mu_1 \times \mu_2$ of X as follows:

(i) $(\sigma_1 \times \sigma_2)(u_1, u_2) = \min\{\sigma_1(u_1), \sigma_2(u_2)\}, \forall (u_1, u_2) \in V$,

(ii) $(\mu_1 \times \mu_2)((u, u_2), (u, v_2)) = \min\{\sigma_1(u), \mu_2(u_2, v_2)\}, \forall u \in V_1$ and $(u_2, v_2) \in X_2$,

(iii) $(\mu_1 \times \mu_2)((u_1, w), (v_1, w)) = \min\{\sigma_2(w), \mu_1(u_1, v_1)\}, \forall w \in V_2$ and $(u_1, v_1) \in X_1$.

Then the fuzzy graph $G = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is said to be the *Cartesian product* of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

Definition 2.11 [4]**.** Let $G_1[G_2]$ denote the composition of graph $G_1 = (V_1, X_1)$ with $G_2 = (V_2, X_2)$. Now $G_1[G_2] = (V_1 \times V_2, X^0)$, where $X^0 = X \cup \{((u_1, u_2), (v_1, v_2))/(u_1, v_1) \in X_1, u_2 \neq v_2\}$ and where X is defined as in the case for $G_1 \times G_2$. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of X_i, $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \circ \sigma_2$ and $\mu_1 \circ \mu_2$ of $V_1 \times V_2$ and X^0, respectively, as follows:

(i) $(\sigma_1 \circ \sigma_2)(u_1, u_2) = \min\{\sigma_1(u_1), \sigma_2(u_2)\}, \forall (u_1, u_2) \in V_1 \times V_2$,

(ii) $(\mu_1 \circ \mu_2)((u, u_2), (u, v_2)) = \min\{\sigma_1(u), \mu_2(u_2, v_2)\}, \forall u \in V_1$ and $(u_2, v_2) \in X_2$,

(iii) $(\mu_1 \circ \mu_2)((u_1, u_2), (v_1, v_2)) = \min\{\sigma_2(u_2), \sigma_2(v_2), \mu_1(u_1, v_1)\}$, $\forall ((u_1, u_2), (v_1, v_2)) \in X^0 - X$.

The fuzzy graph $G = (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ is the composition of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

Definition 2.12 [7]**.** Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of X_i, $i = 1, 2.$ Define the fuzzy subsets $\sigma_1 \wedge \sigma_2$ of $V = V_1 \times V_2$ and $\mu_1 \wedge \mu_2$ of X as follows:

(i) $(\sigma_1 \wedge \sigma_2)(u_1, v_1) = \min\{\sigma_1(u_1), \sigma_2(v_1)\}, \forall u_1, v_1 \in V,$

(ii) $(\mu_1 \wedge \mu_2)((u_1, v_1), (u_2, v_2)) = \min\{\mu_1(u_1, u_2), \mu_2(v_1, v_2)\},$
$\forall u_1, u_2 \in V_1, (u_1, u_2) \in X_1$ and $v_1, v_2 \in V_2, (v_1, v_2) \in X_2.$

Then $G_1 \wedge G_2 = (\sigma_1 \wedge \sigma_2, \mu_1 \wedge \mu_2)$ is said to be the *conjunction* of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2).$

Definition 2.13 [7]**.** Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of X_i, $i = 1, 2.$ Define the fuzzy subsets $\sigma_1 | \sigma_2$ of $V = V_1 \times V_2$ and $\mu_1 | \mu_2$ of

$$\begin{aligned} X = &\{(u_1, v_1), (u_1, v_2)/u_1 \in V_1, (v_1, v_2) \notin X_2\} \\ &\cup \{(u_1, v_1), (u_2, v_1)/v_1 \in V_2, (u_1, u_2) \notin X_1\} \\ &\cup \{(u_1, v_1), (u_2, v_2)/u_1, u_2 \in V_1, u_1 \neq u_2, v_1, v_2 \in V_2, \\ &\quad v_1 \neq v_2, (u_1, u_2) \notin X_1, (v_1, v_2) \notin X_2\} \end{aligned}$$

as follows:

(i) $(\sigma_1 | \sigma_2)(u_1, v_1) = \min\{\sigma_1(u_1), \sigma_2(v_1)\}, \forall u_1 \in V_1, v_1 \in V_2,$

(ii) $(\mu_1 | \mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \sigma_2(v_1), \sigma_2(v_2)\}, \forall u_1 \in V_1,$
$(v_1, v_2) \notin X_2,$

(iii) $(\mu_1 | \mu_2)((u_1, v_1), (u_2, v_1)) = \min\{\sigma_2(v_1), \sigma_1(u_1), \sigma_1(u_2)\}, \quad \forall v_1 \in V_2,$
$(u_1, u_2) \notin X_1,$

(iv) $(\mu_1 | \mu_2)((u_1, v_1), (u_2, v_2)) = \min\{\sigma_1(u_1), \sigma_1(u_2), \sigma_2(v_1), \sigma_2(v_2)\},$
$\forall (u_1, u_2) \notin X_1$ and $(v_1, v_2) \notin X_2.$

Then $G_1 | G_2 = (\sigma_1 | \sigma_2, \mu_1 | \mu_2)$ is said to be the *rejection* of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2).$

Definition 2.14 [7]**.** Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of X_i, $i = 1, 2.$ Define the fuzzy subsets $\sigma_1 \oplus \sigma_2$ of $V = V_1 \times V_2$ and $\mu_1 \oplus \mu_2$ of

$$\begin{aligned} X = & \{(u_1, v_1), (u_1, v_2)/u_1 \in V_1, (v_1, v_2) \in X_2\} \\ & \cup \{(u_1, v_1), (u_2, v_1)/v_1 \in V_2, (u_1, u_2) \in X_1\} \\ & \cup \{(u_1, v_1), (u_2, v_2)/u_1, u_2 \in V_1, u_1 \neq u_2, \\ & \quad v_1, v_2 \in V_2, v_1 \neq v_2 \text{ either } (u_1, u_2) \in X_1 \text{ or } (v_1, v_2) \in X_2\} \end{aligned}$$

as follows:

(i) $(\sigma_1 \oplus \sigma_2)(u_1, v_1) = \min\{\sigma_1(u_1), \sigma_2(v_1)\}, \forall u_1 \in V_1, v_1 \in V_2,$

(ii) $(\mu_1 \oplus \mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\}, \forall u_1 \in V_1,$ $(v_1, v_2) \in X_2,$

(iii) $(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1)) = \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\}, \forall v_1 \in V_2,$ $(u_1, u_2) \in X_1,$

(iv) $(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2))$

$$= \begin{cases} \min(\sigma_1(u_1), \sigma_1(u_2), \mu_2(v_1, v_2)), (u_1, u_2) \notin X_1, (v_1, v_2) \in X_2 \\ \text{or} \\ \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)), (u_1, u_2) \in X_1, (v_1, v_2) \notin X_2. \end{cases}$$

Then $G_1 \oplus G_2 = (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)$ is said to be the *symmetric difference* of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

3. Properties on Some Operations of Fuzzy Graphs

Theorem 3.1. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are strong fuzzy graphs, then* $G_1 \wedge G_2$ *is also a strong fuzzy graph.*

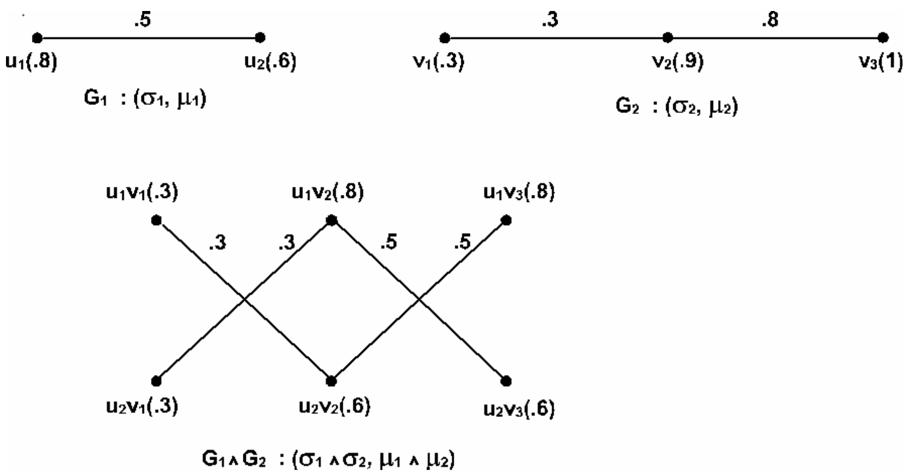
Proof. If $\forall u_1, u_2 \in V_1$, $(u_1, u_2) \in X_1$ and $v_1, v_2 \in V_2$, $(v_1, v_2) \in X_2$, then

$$\begin{aligned}(\mu_1 \wedge \mu_2)((u_1, v_1)(u_2, v_2)) &= \min\{\mu_1(u_1, u_2), \mu_2(v_1, v_2)\} \\ &= \min\{\sigma_1(u_1) \wedge \sigma_1(u_2), \sigma_2(v_1) \wedge \sigma_2(v_2)\}, \\ &\quad \text{since } G_1 \text{ and } G_2 \text{ are strong fuzzy graphs} \\ &= \min\{\sigma_1(u_1) \wedge \sigma_2(v_1), \sigma_1(u_2) \wedge \sigma_2(v_2)\} \\ &= \min\{(\sigma_1 \wedge \sigma_2)(u_1, v_1), (\sigma_1 \wedge \sigma_2)(u_2, v_2)\}.\end{aligned}$$

Suppose $(u_1, u_2) \notin X_1$ and $(v_1, v_2) \in X_2$ (or) $(u_1, u_2) \in X_1$ and $(v_1, v_2) \notin X_2$. Then in both cases, $(\mu_1 \wedge \mu_2)((u_1, v_1)(u_2, v_2)) = 0$. Thus, $G_1 \wedge G_2$ is a strong fuzzy graph.

In the following example, it is shown that if G_1 and G_2 are not strong fuzzy graphs, then $G_1 \wedge G_2$ is not a strong fuzzy graph.

Example 3.2.



Theorem 3.3. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $G_1 \oplus G_2$ *is a strong fuzzy graph.*

Proof. Let $G_1 \oplus G_2 = G = (\sigma, \mu)$, where $\sigma = \sigma_1 \oplus \sigma_2$ and $\mu = \mu_1 \oplus \mu_2$ and $G^* : (V, X)$, where $V = V_1 \times V_2$ and

$$
\begin{aligned}
X = &\{((u_1, v_1), (u_1, v_2))/u_1 \in V_1, (v_1, v_2) \in X_2\} \\
&\cup \{((u_1, v_1), (u_2, v_1))/v_1 \in V_2, (u_1, u_2) \in X_1\} \\
&\cup \{((u_1, v_1), (u_2, v_2))/u_1, u_2 \in V_1, u_1 \neq u_2, \\
&\quad v_1, v_2 \in V_2, v_1 \neq v_2 \text{ either } (u_1, u_2) \in X_1 \text{ or } (v_1, v_2) \in X_2\}.
\end{aligned}
$$

Case (i) Let $e = ((u_1, v_1), (u_1, v_2))$, $\forall u_1 \in V_1$, $(v_1, v_2) \in X_2$. Then

$$
\begin{aligned}
(\mu_1 \oplus \mu_2)((u_1, v_1), (u_1, v_2)) &= \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\} \\
&= \sigma_1(u_1) \wedge [\sigma_2(v_1) \wedge \sigma_2(v_2)], \\
&\quad \text{since } G_2 \text{ is a complete fuzzy graph} \\
&= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_1) \wedge \sigma_2(v_2)] \\
&= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_1, v_2).
\end{aligned}
$$

Case (ii) Let $e = ((u_1, v_1), (u_2, v_1))$, $\forall v_1 \in V_2$, $(u_1, u_2) \in X_1$. Then

$$
\begin{aligned}
(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1)) &= \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\} \\
&= \sigma_2(v_1) \wedge [\sigma_1(u_1) \wedge \sigma_1(u_2)], \\
&\quad \text{since } G_1 \text{ is a complete fuzzy graph} \\
&= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_2) \wedge \sigma_2(v_1)] \\
&= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_1).
\end{aligned}
$$

Case (iii) Let $e = ((u_1, v_1), (u_2, v_2))$, $\forall u_1, u_2 \in V_1$, $v_1, v_2 \in V_2$.

(a) Suppose $(u_1, u_2) \notin X_1$ and $(v_1, v_2) \in X_2$. Then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_1(u_1), \sigma_1(u_2), \mu_2(v_1, v_2))$$

$$= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \{\sigma_2(v_1) \wedge \sigma_2(v_2)\},$$

since G_2 is a complete fuzzy graph

$$= \{\sigma_1(u_1) \wedge \sigma_2(v_1)\} \wedge \{\sigma_1(u_2) \wedge \sigma_2(v_2)\}$$

$$= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_2).$$

(b) Suppose $(u_1, u_2) \in X_1$, $(v_1, v_2) \notin X_2$. Then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2))$$

$$= \sigma_2(v_1) \wedge \sigma_2(v_2) \wedge \{\sigma_1(u_1) \wedge \sigma_1(u_2)\},$$

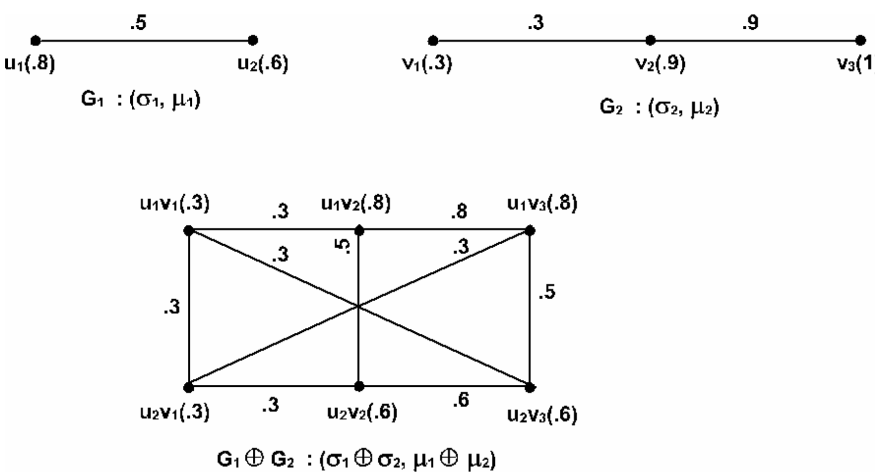
since G_1 is a complete fuzzy graph

$$= \{\sigma_1(u_1) \wedge \sigma_2(v_1)\} \wedge \{\sigma_1(u_2) \wedge \sigma_2(v_2)\}$$

$$= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_2).$$

Thus, in all cases, it is true that $G_1 \oplus G_2$ is a strong fuzzy graph.

Example 3.4. This example illustrates that if G_1 and G_2 are not complete fuzzy graphs, then $G_1 \oplus G_2$ is not a strong fuzzy graph.



Theorem 3.5. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $G_1 \circ G_2$ *is also a complete fuzzy graph.*

Proof. Let $G_1 \circ G_2 = G = (\sigma, \mu)$, where $\sigma = \sigma_1 \circ \sigma_2$ and $\mu = \mu_1 \circ \mu_2$ and $G^* : (V, X)$, where $V = V_1 \times V_2$ and

$$\begin{aligned} X = {} & \{((u_1, v_1), (u_1, v_2))/u_1 \in V_1, (v_1, v_2) \in X\} \\ & \cup \{((u_1, v_1), (u_2, v_1))/v_1 \in V_2, (u_1, u_2) \in X_1\} \\ & \cup \{((u_1, v_1), (u_2, v_2))/(u_1, u_2) \in X_1, v_1 \neq v_2\}. \end{aligned}$$

Case (i)

$$\begin{aligned} (\mu_1 \circ \mu_2)((u_1, v_1), (u_1, v_2)) &= \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\} \\ &= \sigma_1(u_1) \wedge [\sigma_2(v_1) \wedge \sigma_2(v_2)], \\ &\quad \text{since } G_2 \text{ is complete} \\ &= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_1) \wedge \sigma_2(v_2)] \\ &= (\sigma_1 \circ \sigma_2)(u_1, v_1) \wedge (\sigma_1 \circ \sigma_2)(u_1, v_2). \end{aligned}$$

Case (ii)

$$\begin{aligned} (\mu_1 \circ \mu_2)((u_1, v_1), (u_2, v_1)) &= \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\} \\ &= \sigma_2(v_1) \wedge [\sigma_1(u_1) \wedge \sigma_1(u_2)], \\ &\quad \text{since } G_1 \text{ is complete} \\ &= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_2) \wedge \sigma_2(v_1)] \\ &= (\sigma_1 \circ \sigma_2)(u_1, v_1) \wedge (\sigma_1 \circ \sigma_2)(u_2, v_1). \end{aligned}$$

Case (iii)

$$\begin{aligned} (\mu_1 \circ \mu_2)((u_1, v_1), (u_2, v_2)) &= \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)) \\ &= \{\sigma_2(v_1) \wedge \sigma_2(v_2)\} \wedge \{\sigma_1(u_1) \wedge \sigma_1(u_2)\}, \\ &\quad \text{since } G_1 \text{ is complete} \end{aligned}$$

$$= \{\sigma_1(u_1) \wedge \sigma_2(v_1)\} \wedge \{\sigma_1(u_2) \wedge \sigma_2(v_2)\}$$

$$= (\sigma_1 \circ \sigma_2)(u_1, v_1) \wedge (\sigma_1 \circ \sigma_2)(u_2, v_2).$$

Thus, it is clear that $G_1 \circ G_2$ is a complete fuzzy graph.

Theorem 3.6. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $\overline{G_1 \circ G_2} \cong \overline{G_1} \circ \overline{G_2}$.

Proof. Let $G_1 \circ G_2 = G = (\sigma, \mu)$, where $\sigma = \sigma_1 \circ \sigma_2$ and $\mu = \mu_1 \circ \mu_2$ and $G^* : (V, X)$, where $V = V_1 \times V_2$ and

$$X = \{((u_1, v_1), (u_1, v_2))/u_1 \in V_1, (v_1, v_2) \in X_2\}$$

$$\cup \{((u_1, v_1), (u_2, v_1))/v_1 \in V_2, (u_1, u_2) \in X_1\}$$

$$\cup \{((u_1, v_1), (u_2, v_2))/(u_1, u_2) \in X_1, v_1 \neq v_2\}.$$

To prove $(\overline{\sigma_1 \circ \sigma_2})(u_1v_1) = (\overline{\sigma_1} \circ \overline{\sigma_2})(u_1v_1)$,

$$(\overline{\sigma_1 \circ \sigma_2})(u_1v_1) = (\sigma_1 \circ \sigma_2)(u_1v_1), \text{ by definition of complement}$$

$$= \sigma_1(u_1) \wedge \sigma_2(v_1)$$

$$= \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1)$$

$$= (\overline{\sigma_1} \circ \overline{\sigma_2})(u_1v_1).$$

Now to prove $\overline{G_1 \circ G_2} = \overline{G_1} \circ \overline{G_2}$ for all edges X, obtained by joining the vertices of V_1 and V_2.

Case (i) If for all $u_1 \in V_1$, $(v_1, v_2) \in X_2$, then

$$(\mu_1 \circ \mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\}.$$

Therefore,

$$(\overline{\mu_1 \circ \mu_2})((u_1v_1), (u_1v_2))$$

$$= (\sigma_1 \circ \sigma_2)(u_1v_1) \wedge (\sigma_1 \circ \sigma_2)(u_1v_2) - (\mu_1 \circ \mu_2)((u_1v_1), (u_1v_2))$$

$$= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_1) \wedge \sigma_2(v_2)] - [\sigma_1(u_1) \wedge \mu_2(v_1, v_2)]$$

$$= [\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)] - [\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)]$$

$$= 0,$$

$(\overline{\mu_1} \circ \overline{\mu_2})((u_1v_1), (u_1v_2))$

$$= \overline{\sigma_1}(u_1) \wedge \overline{\mu_2}(v_1, v_2) = 0, \text{ since } G_2 \text{ is complete.}$$

Case (ii) If for all $v_1 \in V_2$, $(u_1, u_2) \in X_1$, then $(\mu_1 \circ \mu_2)((u_1, v_1), (u_2, v_1)) = \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\}$,

$(\overline{\mu_1 \circ \mu_2})((u_1v_1), (u_2v_1))$

$$= (\sigma_1 \circ \sigma_2)(u_1v_1) \wedge (\sigma_1 \circ \sigma_2)(u_2v_1) - (\mu_1 \circ \mu_2)((u_1v_1), (u_2v_1))$$

$$= [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_2) \wedge \sigma_2(v_1)] - [\sigma_2(v_1) \wedge \mu_1(u_1, u_2)]$$

$$= [\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1)] - [\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1)],$$

since G_1 is complete

$$= 0,$$

$(\overline{\mu_1} \circ \overline{\mu_2})((u_1, v_1), (u_2, v_1))$

$$= \overline{\sigma_2}(v_1) \wedge \overline{\mu_1}(u_1, u_2) = 0, \text{ since } G_1 \text{ is complete.}$$

Case (iii) If for all $(u_1, u_2) \in X_1$, $v_1 \neq v_2$, then $(\mu_1 \circ \mu_2)((u_1v_1), (u_2v_2)) = \min\{\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1u_2)\}$,

$(\overline{\mu_1 \circ \mu_2})((u_1v_1), (u_2v_2))$

$$= (\sigma_1 \circ \sigma_2)(u_1v_1) \wedge (\sigma_1 \circ \sigma_2)(u_2v_2) - (\mu_1 \circ \mu_2)((u_1v_1), (u_2v_2))$$

$$= \{\sigma_1(u_1) \wedge \sigma_2(v_1)\} \wedge \{\sigma_1(u_2) \wedge \sigma_2(v_2)\}$$

$$- \{\sigma_2(v_1) \wedge \sigma_2(v_2) \wedge \sigma_1(u_1) \wedge \sigma_1(u_2)\}, \text{ since } G_1 \text{ is complete}$$

$$= 0,$$

$$(\overline{\mu_1} \circ \overline{\mu_2})((u_1v_1), (u_2v_2))$$

$$= \overline{\sigma_2}(v_1) \wedge \overline{\sigma_2}(v_2) \wedge \overline{\mu_1}(u_1, u_2) = 0, \text{ since } G_1 \text{ is complete.}$$

Thus, in all cases, it follows that $\overline{G_1 \circ G_2} \cong \overline{G_1} \circ \overline{G_2}$.

In the above discussion of all the cases, it is clear that the vertices in $\overline{G_1 \circ G_2}$ and $\overline{G_1} \circ \overline{G_2}$ are isolated. This implies that no two vertices are adjacent. Therefore, they are neighbourly irregular fuzzy graphs.

Theorem 3.7. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $\overline{G_1 | G_2} \cong \overline{G_1} | \overline{G_2}$.

Proof. Let $G_1 | G_2 = (\sigma, \mu)$, where $\sigma = \sigma_1 | \sigma_2$, $\mu = \mu_1 | \mu_2$, $G^* : (V, X)$, where $V = V_1 \times V_2$,

$$X = \{(u_1, v_1), (u_1, v_2)/u_1 \in V_1, (v_1, v_2) \notin X_2\}$$

$$\cup \{(u_1, v_1), (u_2, v_1)/v_1 \in V_2, (u_1, u_2) \notin X_1\}$$

$$\cup \{(u_1, v_1), (u_2, v_2)/u_1, u_2 \in V_1, u_1 \neq u_2,$$

$$v_1, v_2 \in V_2, v_1 \neq v_2, (u_1, u_2) \notin X_1, (v_1, v_2) \notin X_2\}.$$

To prove $(\overline{\sigma_1 | \sigma_2})(u_1v_1) = (\overline{\sigma_1} | \overline{\sigma_2})(u_1v_1)$,

$$(\overline{\sigma_1 | \sigma_2})(u_1v_1) = (\sigma_1 | \sigma_2)(u_1v_1), \text{ by definition of complement}$$

$$= \sigma_1(u_1) \wedge \sigma_2(v_1)$$

$$= \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1)$$

$$= (\overline{\sigma_1} | \overline{\sigma_2})(u_1v_1).$$

Now to prove $\overline{G_1 | G_2} = \overline{G_1} | \overline{G_2}$ for all edges X, obtained by joining the vertices of V_1 and V_2.

Case (i) By definition, $\forall u_1 \in V_1$, $(v_1, v_2) \notin X_2$,

$$(\mu_1|\mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \sigma_2(v_1), \sigma_2(v_2)\}.$$

Since G_2 is complete, $(v_1, v_2) \in X_2$. Therefore, $(\mu_1|\mu_2)((u_1, v_1), (u_1, v_2)) = 0$.

Hence,

$$\begin{aligned}
&\overline{(\mu_1|\mu_2)}((u_1v_1), (u_1v_2)) \\
&\quad = (\sigma_1|\sigma_2)(u_1v_1) \wedge (\sigma_1|\sigma_2)(u_1v_2) - (\mu_1|\mu_2)((u_1v_1), (u_1v_2)) \\
&\quad = [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_1) \wedge \sigma_2(v_2)] - 0 \\
&\quad = [\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)],
\end{aligned}$$

$$\begin{aligned}
&(\overline{\mu_1}|\overline{\mu_2})((u_1v_1), (u_1v_2)) \\
&\quad = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1) \wedge \overline{\sigma_2}(v_2) = \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2),
\end{aligned}$$

since G_2 is complete.

Case (ii) By definition, $\forall v_1 \in V_2$, $(u_1, u_2) \notin X_1$,

$$(\mu_1|\mu_2)((u_1v_1), (u_2v_1)) = \min\{\sigma_1(u_1), \sigma_1(u_2), \sigma_2(v_1)\}.$$

Since G_1 is complete, $(u_1, u_2) \in X_1$. Therefore, $(\mu_1|\mu_2)((u_1v_1), (u_2v_1)) = 0$.

Hence,

$$\begin{aligned}
&\overline{(\mu_1|\mu_2)}((u_1v_1), (u_2v_1)) \\
&\quad = (\sigma_1|\sigma_2)(u_1v_1) \wedge (\sigma_1|\sigma_2)(u_2v_1) - (\mu_1|\mu_2)((u_1v_1), (u_2v_1)) \\
&\quad = [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_2) \wedge \sigma_2(v_1)] - 0 \\
&\quad = [\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1)],
\end{aligned}$$

$$\begin{aligned}
&(\overline{\mu_1}|\overline{\mu_2})((u_1v_1), (u_2v_1)) \\
&\quad = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_1}(u_2) \wedge \overline{\sigma_2}(v_1) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1),
\end{aligned}$$

since G_1 is complete.

Case (iii) By definition, $\forall (u_1, u_2) \notin X_1$ and $(v_1, v_2) \notin X_2$,

$$(\mu_1|\mu_2)((u_1, v_1), (u_2, v_2)) = \min\{\sigma_1(u_1), \sigma_1(u_2), \sigma_2(v_1), \sigma_2(v_2)\}.$$

Since G_1 and G_2 are complete, $\forall u_1, u_2 \in V_1$, $(u_1, u_2) \in X_1$ and $v_1, v_2 \in V_2$, $(v_1, v_2) \in X_2$. Therefore, $(\mu_1|\mu_2)((u_1, v_1), (u_2, v_2)) = 0$.

Hence,

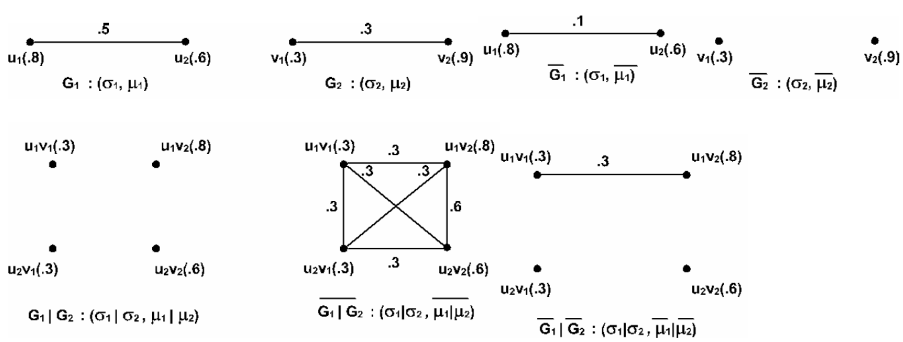
$$\begin{aligned}
&(\overline{\mu_1|\mu_2})((u_1, v_1), (u_2, v_2)) \\
&\quad = (\sigma_1|\sigma_2)(u_1v_1) \wedge (\sigma_1|\sigma_2)(u_2v_2) - (\mu_1|\mu_2)((u_1v_1), (u_2v_1)) \\
&\quad = [\sigma_1(u_1) \wedge \sigma_2(v_1)] \wedge [\sigma_1(u_2) \wedge \sigma_2(v_2)] - 0 \\
&\quad = [\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)],
\end{aligned}$$

$$\begin{aligned}
&(\overline{\mu_1}|\overline{\mu_2})((u_1v_1), (u_2v_2)) \\
&\quad = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_1}(u_2) \wedge \overline{\sigma_2}(v_1) \wedge \overline{\sigma_2}(v_2) \\
&\quad = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2), \text{ since } G_1 \text{ and } G_2 \text{ are complete.}
\end{aligned}$$

Thus, for all the cases, it follows that $\overline{G_1|G_2} \cong \overline{G_1}|\overline{G_2}$.

In general, $\overline{G_1|G_2} \neq \overline{G_1}|\overline{G_2}$. This is shown in the following example where G_1 or G_2 is not complete.

Example 3.8.



Theorem 3.9. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $\overline{G_1} \oplus \overline{G_2} \cong G_1 | G_2$.

Proof. Let $G_1 \oplus G_2 = (\sigma, \mu)$, where $\sigma = \sigma_1 \oplus \sigma_2$, $\mu = \mu_1 \oplus \mu_2$ and $G^* : (V, X)$, where $V = V_1 \times V_2$ and

$$X = \{(u_1, v_1), (u_1, v_2) / u_1 \in V_1, (v_1, v_2) \in X_2\}$$

$$\cup \{(u_1, v_1), (u_2, v_1) / v_1 \in V_2, (u_1, u_2) \in X_1\}$$

$$\cup \{(u_1, v_1), (u_2, v_2) / u_1, u_2 \in V_1, u_1 \neq u_2, v_1, v_2 \in V_2,$$

$$v_1 \neq v_2 \text{ either } (u_1, u_2) \in X_1 \text{ or } (v_1, v_2) \in X_2\}.$$

Since G_1 and G_2 are complete fuzzy graphs,

$$\overline{\mu_1}(u_1, u_2) = 0, \forall (u_1, u_2) = X_1 \quad \text{and} \quad \overline{\mu_2}(v_1, v_2) = 0, \forall (v_1, v_2) = X_2.$$

To prove $(\overline{\sigma_1} \oplus \overline{\sigma_2})(u_1 v_1) = (\sigma_1 | \sigma_2)(u_1 v_1)$,

$$(\overline{\sigma_1} \oplus \overline{\sigma_2})(u_1 v_1) = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1)$$

$$= \sigma_1(u_1) \wedge \sigma_2(v_1), \text{ by definition of complement}$$

$$= (\sigma_1 | \sigma_2)(u_1 v_1).$$

Now to prove $\overline{G_1} \oplus \overline{G_2} = G_1 | G_2$ for all edges joining the vertices of V_1 and V_2.

Case (i)

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\},$$

$$\forall u_1 \in V_1, \ (v_1, v_2) \in X_2.$$

Therefore, $(\overline{\mu_1} \oplus \overline{\mu_2})((u_1 v_1), (u_1 v_2)) = \overline{\sigma_1}(u_1) \wedge \overline{\mu_2}(v_1, v_2) = 0$, since G_2 is complete and $(\mu_1 | \mu_2)((u_1, v_1), (u_1, v_2)) = 0$, since $(v_1, v_2) \in X_2$.

Case (ii)

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1)) = \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\},$$

$$\forall v_1 \in V_2, \ (u_1, u_2) \in X_1.$$

Therefore, $(\overline{\mu_1} \oplus \overline{\mu_2})((u_1, v_1), (u_2, v_1)) = \overline{\sigma_2}(v_1) \wedge \overline{\mu_1}(u_1, u_2) = 0,$ since G_1 is complete and $(\mu_1 | \mu_2)((u_1, v_1), (u_2, v_1)) = 0,$ since $(u_1, u_2) \in X_1.$

Case (iii) If $(u_1, u_2) \notin X_1, \ (v_1, v_2) \in X_2,$ then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_1(u_1), \sigma_1(u_2), \mu_2(v_1, v_2)).$$

Therefore, $(\overline{\mu_1} \oplus \overline{\mu_2})((u_1, v_1), (u_2, v_2)) = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1) \wedge \overline{\mu_2}(v_1, v_2) = 0$ and $(\mu_1 | \mu_2)((u_1, v_1), (u_2, v_2)) = 0,$ since $(v_1, v_2) \in X_2.$

Case (iv) If $(u_1, u_2) \in X_1, \ (v_1, v_2) \notin X_2,$ then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)),$$

$$(\overline{\mu_1} \oplus \overline{\mu_2})((u_1, v_1), (u_2, v_2)) = \overline{\sigma_2}(v_1), \overline{\sigma_2}(v_2) \wedge \overline{\mu_1}(u_1, u_2) = 0$$

and

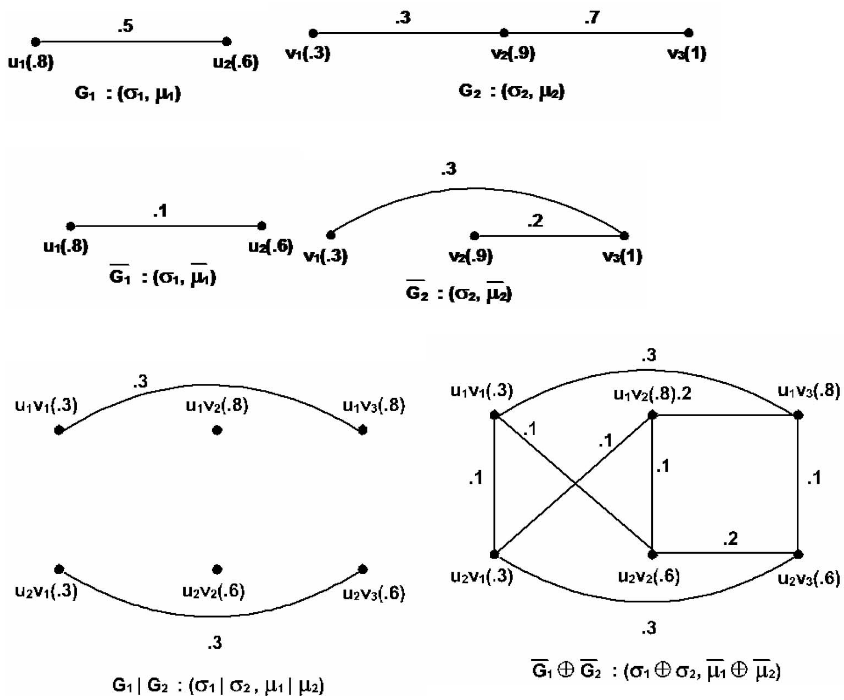
$$(\mu_1 | \mu_2)((u_1, v_1), (u_2, v_2)) = 0, \text{ since } (u_1, u_2) \in X_1.$$

Thus, in all cases, it follows that $\overline{G_1} \oplus \overline{G_2} \cong G_1 | G_2.$

In the discussion of the above theorem, it is clear that the vertices in $\overline{G_1} \oplus \overline{G_2}$ and $G_1 | G_2$ are isolated. Therefore, they are neighbourly irregular fuzzy graphs as no two vertices are adjacent.

In general, $\overline{G_1} \oplus \overline{G_2} \neq G_1 | G_2.$ This is illustrated in the following example:

Example 3.10.



Theorem 3.11. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $G_1 \oplus G_2 \cong G_1 \times G_2$.

Proof. To prove $(\sigma_1 \oplus \sigma_2)(u_1v_1) = (\sigma_1 \times \sigma_2)(u_1v_1)$,

$$(\sigma_1 \oplus \sigma_2)(u_1v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1),\ \forall u_1 \in V_1,\ v_1 \in V_2$$

$$= (\sigma_1 \times \sigma_2)(u_1v_1).$$

Now, for all edges (e), obtained by joining the vertices of V_1 and V_2, it suffices to prove that $(\mu_1 \oplus \mu_2)(e) = (\mu_1 \times \mu_2)(e)$.

Case (i) If for all $u_1 \in V_1$, $(v_1, v_2) \in X_2$, then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_1, v_2)) = \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\}$$

$$= (\mu_1 \times \mu_2)((u_1, v_1), (u_1, v_2)).$$

Case (ii) If for all $v_1 \in V_2,\ (u_1, u_2) \in X_1,$ then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1)) = \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\}$$

$$= (\mu_1 \times \mu_2)((u_1, v_1), (u_2, v_1)).$$

Case (iii) Now consider the edge $e = ((u_1, v_1), (u_2, v_2)),\ \ u_1, u_2 \in V_1,$ $u_1 \neq u_2,\ v_1, v_2 \in V_2,\ v_1 \neq v_2.$

(a) If $(u_1, u_2) \notin X_1,\ (v_1, v_2) \in X_2,$ then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_1(u_1), \sigma_1(u_2), \mu_2(v_1, v_2)) = 0,$$

since $(u_1, u_2) \in X_1$ as G_1 is complete.

(b) If $(u_1, u_2) \in X_1,\ (v_1, v_2) \notin X_2,$ then

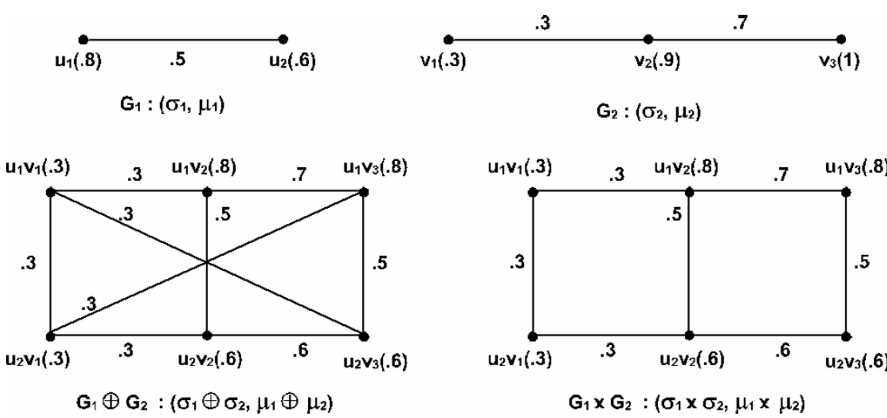
$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)) = 0,$$

since $(v_1, v_2) \in X_2$ as G_2 is complete.

Thus, by all means, $G_1 \oplus G_2 \cong G_1 \times G_2.$

$G_1 \oplus G_2 \cong G_1 \times G_2$ is not true always. This is shown in the following example:

Example 3.12.



Theorem 3.13. *If* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *are complete fuzzy graphs, then* $\overline{G_1 \oplus G_2} \cong G_1 \wedge G_2$.

Proof. To prove $(\overline{\sigma_1 \oplus \sigma_2})(u_1v_1) = (\sigma_1 \wedge \sigma_2)(u_1v_1)$,

$$(\overline{\sigma_1 \oplus \sigma_2})(u_1v_1) = \overline{\sigma_1}(u_1) \wedge \overline{\sigma_2}(v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1) = (\sigma_1 \wedge \sigma_2)(u_1v_1).$$

Now to prove $(\overline{\mu_1 \oplus \mu_2})(e) = (\mu_1 \wedge \mu_2)(e)$, for all edges (e), obtained by joining the vertices of V_1 and V_2.

Case (i) If $\forall u_1 \in V_1$, $(v_1, v_2) \in X_2$, then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2))$$

$$= \min\{\sigma_1(u_1), \mu_2(v_1, v_2)\},$$

$$(\overline{\mu_1 \oplus \mu_2})((u_1, v_1), (u_1, v_2))$$

$$= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_1, v_2)$$

$$- (\mu_1 \oplus \mu_2)((u_1, v_1), (u_1, v_2))$$

$$= (\sigma_1(u_1) \wedge \sigma_2(v_1)) \wedge (\sigma_1(u_1) \wedge \sigma_2(v_2)) - (\sigma_1(u_1) \wedge \mu_2(v_1, v_2))$$

$$= (\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)) - (\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2))$$

$= 0$, since G_2 is a complete fuzzy graph.

$(\mu_1 \wedge \mu_2)((u_1, v_1), (u_1, v_2)) = 0$, by definition of operation 'conjunction'.

Case (ii) If $\forall v_1 \in V_2$, $(u_1, u_2) \in X_1$, then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1))$$

$$= \min\{\sigma_2(v_1), \mu_1(u_1, u_2)\},$$

$$(\overline{\mu_1 \oplus \mu_2})((u_1, v_1), (u_2, v_1))$$

$$= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_1)$$

$$- (\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_1))$$

$= (\sigma_1(u_1) \wedge \sigma_2(v_1)) \wedge (\sigma_1(u_2) \wedge \sigma_2(v_1)) - (\sigma_2(v_1) \wedge \mu_1(u_1, u_2))$

$= (\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1)) - (\sigma_2(v_1) \wedge \sigma_1(u_1) \wedge \sigma_1(u_2))$

(as G_1 is complete)

$= 0.$

$(\mu_1 \wedge \mu_2)((u_1, v_1), (u_2, v_1)) = 0$, by definition of operation 'conjunction'.

Case (iii) (a) Suppose $(u_1, u_2) \notin X_1$, $(v_1, v_2) \in X_2$. Then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_1(u_1), \sigma_1(u_2), \mu_2(v_1, v_2)),$$

$$\begin{aligned}(\overline{\mu_1 \oplus \mu_2})((u_1, v_1), (u_2, v_2)) &= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_2) \\ &\quad - (\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) \\ &= (\sigma_1(u_1) \wedge \sigma_2(v_1)) \wedge (\sigma_1(u_2) \wedge \sigma_2(v_2)) - 0 \\ &\quad \text{since } (u_1, u_2) \in X_1 \text{ as } G_1 \text{ is complete} \\ &= (\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_2)), \\ &\quad (\mu_1 \wedge \mu_2)((u_1, v_1), (u_2, v_2)) \\ &= \min\{\mu_1(u_1, u_2), \mu_2(v_1, v_2)\} \\ &= \min\{\sigma_1(u_1) \wedge \sigma_1(u_2), \sigma_2(v_1) \wedge \sigma_2(v_2)\} \\ &\quad \text{since } G_1 \text{ and } G_2 \text{ are complete} \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2).\end{aligned}$$

(b) Suppose $(u_1, u_2) \in X_1, (v_1, v_2) \notin X_2$. Then

$$(\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2)) = \min(\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)),$$

$$\begin{aligned}(\overline{\mu_1 \oplus \mu_2})((u_1, v_1), (u_2, v_2)) &= (\sigma_1 \oplus \sigma_2)(u_1, v_1) \wedge (\sigma_1 \oplus \sigma_2)(u_2, v_2) \\ &\quad - (\mu_1 \oplus \mu_2)((u_1, v_1), (u_2, v_2))\end{aligned}$$

$$= (\sigma_1(u_1) \wedge \sigma_2(v_1)) \wedge (\sigma_1(u_2) \wedge \sigma_2(v_2)) - 0$$

since $(v_1, v_2) \in X_2$ as G_2 is complete

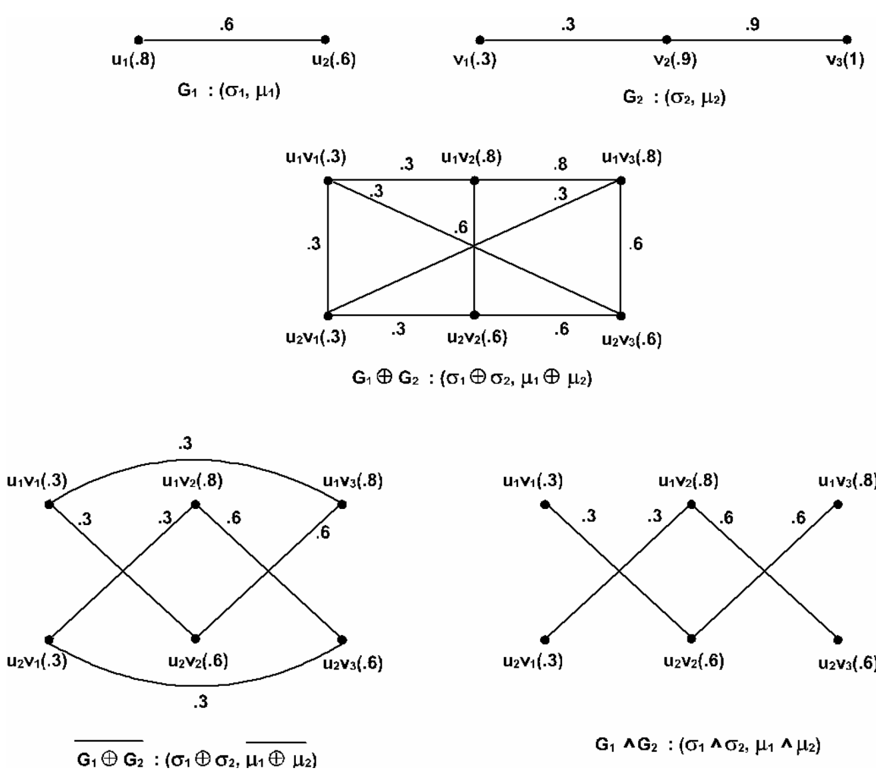
$$= (\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_2)),$$

$(\mu_1 \wedge \mu_2)((u_1, v_1), (u_2, v_2)) = (\sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_2))$ (by Case (iii)(a)).

Thus, in all cases, $\overline{G_1 \oplus G_2} \cong G_1 \wedge G_2$.

In general, $\overline{G_1 \oplus G_2} \neq G_1 \wedge G_2$. This is shown in the following example:

Example 3.14.



4. Conclusion

The theory on properties between some operations of fuzzy graphs G_1 and G_2 has been analyzed in this context. It is proved that these properties exist when G_1 and G_2 are complete fuzzy graphs. It is established that when G_1 and G_2 are complete, the operation symmetric difference is strong and when G_1 and G_2 are strong, the operation conjunction is strong. From Theorem 3.6 and Theorem 3.9, it is quite evident that the complement of composition of G_1 and G_2 and the symmetric difference of the complement of G_1 and G_2 are neighbourly irregular fuzzy graphs.

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