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REGULARLY OPEN SETS AND NORMAL AND T_4 SPACES

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Abstract

Within this paper, recent discoveries concerning regularly open and regularly closed sets are used to further investigate and characterize normal and T_4 spaces.

1. Introduction

Within this paper, all spaces are topological spaces.

Normal and T_4 spaces were introduced by Tietze in 1923 [7].

Definition 1.1. A space (X, T) is normal iff for disjoint closed sets C and D, there exist disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. A normal T_1 space is denoted by T_4.

Since their introductions, each of normal and T_4 spaces have been further investigated and characterized, and today are separation axioms included in the study of classical topology. As a result, each of normal and T_4 spaces

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continue to be a focus of interest in the continued study and investigation of topological spaces.

Regularly open sets were introduced by Stone in 1937 [6].

Definition 1.2. Let (X, T) be a space and let $A \subseteq X$. Then A is regularly open, denoted by $A \in RO(X, T)$, iff $A = Int(Cl(A))$.

Within the 1937 paper [6], it was shown that the set of regularly open sets is a base for a topology Ts on X coarser than T, and the space (X, Ts) was called the *semiregularization space* of (X, T).

Following the introduction of regularly open sets was regularly closed sets [8].

Definition 1.3. Let (X, T) be a space and let $A \subseteq X$. Then A is regularly closed, denoted by $A \in RC(X, T)$, iff one of the following equivalent conditions is satisfied: (a) $A = Cl(Int(A))$ or (b) $X \setminus A \in RO(X, T)$.

Within a recent paper [1], the question of what happens if the semiregularization process is repeated was resolved, showing that at most one new topology can be obtained by use of the semiregularization process. To accomplish the stated objective, it was proven that for a space (X, T), $RC(X, T) = \{Cl(O) | O \in T\}$ [1], which was combined with the fact that for a space (X, T), $RO(X, T) = \{Int(Cl(O)) | O \in T\}$ [2] to achieve the objective.

The semiregularization process has been useful in the continued investigation of regular spaces: for a regular space (X, T), $T = Ts$ [8]. Thus, the question of whether or not regularly open sets, regularly closed sets, and the semiregularization process can be used to gain additional insights into normal and T_4 spaces naturally arises. Within this paper, it is proven that, in fact, that is the case.

Given below are known results that will be used in the work below.

Theorem 1.1. *Let* (X, T) *be a space and let* $C(T)$ *denote the closed sets in* (X, T). *Then*

$$RO(X, T) = \{Int(Cl(O)) \mid O \in T\}[2] = \{Ext(O) \mid O \in T\}[3]$$
$$= \{Int(C) \mid C \in C(T)\}[1],$$

and

$$RC(X, T) = \{Cl(Int(C)) \mid C \in C(T)\} = \{Cl(U) \mid U \in RO(X, T)\}$$
$$= \{Cl(V) \mid V \in Ts\}[4].$$

Given below are additional results in the paper [4] that will be used in this paper.

Theorem 1.2. *Let* (X, T) *be a space and let* $C \in C(T)$. *Then* $X = Int(C) \cup Fr(Int(C)) \cup Ext(Int(C))$, *where* $Int(Fr(Int(C))) = \phi$, *and*

$$Fr(Int(C)) = Cl(Int(C)) \cap Cl(Ext(Int(C))).$$

Theorem 1.3. *Let* (X, T) *be a space, let* $C \in C(T)$, *and let* $D = Cl(Int(C))$. *Then D is regularly closed*, $Int(D) = Int(C)$, $Ext(Int(D)) = Ext(Int(C))$, *and* $Fr(Int(D)) = Fr(Int(C))$.

Theorem 1.4. *Let* (X, T) *be a space and let D and E be disjoint closed sets. Then* $X = Int(D) \cup Int(E) \cup Fr(Int(D \cup E)) \cup Ext(Int(D \cup E))$, *where* $Int(Fr(Int(D \cup E))) = \phi$.

2. New Characterizations of Normal and T_4 Spaces

What would be the circumstances if in the definition of normal, open was replaced by regularly open? Within this paper, the use of open, closed, regularly open, and regularly closed all are for the space (X, T).

Definition 2.1. A space (X, T) is regularly open normal iff for disjoint closed sets *C* and *D*, there exist disjoint regularly open sets *U* and *V* such that $C \subseteq U$ and $D \subseteq V$.

Theorem 2.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is normal*, (b) *for disjoint closed sets C and D, there exists a* *closed set E such that* $C \subseteq Int(E)$ *and* $D \subseteq (X \backslash E)$, (c) *for disjoint closed sets C and D, there exists a closed set E such that* $X = Int(E) \bigcup Fr(Int(E)) \bigcup Ext(Int(E))$, *where* $Int(Fr(Int(E))) = \phi$, $C \subseteq Int(E)$, *and*

$$D \subseteq Ext(Int(E)),$$

(d) *for disjoint closed sets C and D, there exists a regularly closed set F such that* $X = Int(F) \bigcup Fr(Int(F)) \bigcup Ext(Int(F))$, *where* $C \subseteq Int(F)$, $D \subseteq Ext(Int(F))$, *and* $Int(Fr(Int(F))) = \phi$, (e) *for disjoint closed sets C and D, there exists a regularly closed set F such that* $C \subseteq Int(F)$ *and* $D \subseteq Ext(Int(F))$, (f) *for disjoint closed sets C and D, there exists a regularly open set U such that* $C \subseteq U$ *and* $D \subseteq Ext(U)$, (g) (X, T) *is regularly open normal*, *and* (h) *for disjoint closed sets C and D, there exist disjoint Ts-open sets U and V such that* $C \subseteq U$ *and* $D \subseteq V$.

Proof. (a) implies (b): Let C and D be disjoint closed sets. Let U and V be disjoint open sets such that $C \subseteq U$ and $D \subseteq V$. Then $E = Cl(U)$ is closed and $D \bigcap V = \phi$. Since $C \subseteq U \subseteq Cl(U) = E$, $C \subseteq Int(E)$ and $D \subseteq V \subseteq (X \backslash D)$.

(b) implies (c): Let C and D be disjoint closed sets. Let E be closed such that $C \subseteq Int(E)$ and $D \subseteq (X \backslash E)$. The remainder of the proof is straightforward using Theorem 1.2 and is omitted.

(c) implies (d): Let C and D be disjoint closed sets. Let E be closed such that $X = Int(E) \bigcup Fr(Int(E)) \bigcup Ext(Int(E))$, where $Int(Fr((Int(E)))) = \phi$, $C \subseteq Int(E)$, and $D \subseteq Ext(Int(E))$. Let $F = Cl(Int(E))$. By Theorem 1.1, $E \in RC(X, T)$ and the remainder of the proof follows immediately from Theorem 1.3.

Clearly, (d) implies (e).

(e) implies (f): Let C and D be disjoint closed sets. Let F be regularly closed such that $C \subseteq Int(F)$ and $D \subseteq Ext(Int(F))$. Let $U = Int(F)$. Since

F is regularly closed, F is closed and, by Theorem 1.1, $U \in RO(X, T)$. Thus, $C \subseteq Int(f) = U$ and $D \subseteq Ext(Int(F)) = Ext(U)$.

(f) implies (g): Let C and D be disjoint closed sets. Let U be regularly open such that $C \subseteq U$ and $D \subseteq Ext(U)$. Since $RO(X, T) \subseteq T$, by Theorem 1.1, $Ext(U) \in RO(X, T)$ and $U \cap Ext(U) = \phi$. Hence (X, T) is regularly open normal.

Since $RO(X, T) \subseteq Ts$, (h) implies (g), and since $Ts \subseteq T$, (h) implies (a).

Theorem 2.2. *Let* (X, T) *be a space. Then* (a) (X, T) *is* T_4 *iff* (b) (X, T) *is* T_1 *and for disjoint closed sets C and D, there exists a continuous function* $f : (X, Ts) \to ([0, 1], W)$ *such that* $f(C) = 0$ *and* $f(D) = 1$, *where W is the usual subspace topology on* $[0, 1]$.

Proof. (a) implies (b): Since (X, T) is T_4, (X, T) is regular [8] and $T = Ts$. Let C and D be disjoint closed sets. Let $f : (X, T) \to ([0, 1], W)$ be continuous such that $f(C) = 0$ and $f(D) = 1$. Since $T = Ts$, f is the desired function.

(b) implies (a): Let C and D be disjoint closed sets. Let $f : (X, Ts) \to ([0, 1], W)$ be continuous such that $f(C) = 0$ and $f(D) = 1$. Then $U = f^{-1}\left(\left[0, \frac{1}{3}\right)\right)$ and $V = f^{-1}\left(\left(\frac{2}{3}, 1\right]\right)$ are disjoint Ts-open sets such that $C \subseteq U$ and $D \subseteq V$. Since $Ts \subseteq T$, U and V are disjoint T-open sets containing C and D, respectively. Thus, (X, T) is normal and since (X, T) is T_1, (X, T) is T_4.

Corollary 2.1. *Let* (X, T) *be a space. Then* (X, T) *is* T_4 *iff* (X, T) *is* T_1 *and for disjoint closed sets C and D, there exists a closed set E such that* $C \subseteq Int(E)$ *and* $D \subseteq (X \setminus E)$.

In a similar manner, the other characterizations of normal in Theorem 2.1 can be extended to T_4 spaces.

3. More Characterizations of Normal and T_4 Spaces

In the search for additional characterizations of normal and T_4 spaces, the following result proved useful.

Theorem 3.1. *Let (X, T) be a space and let C and D be disjoint closed sets. Then* $Int(C \cup D) = Int(C) \cup Int(D)$, $Fr(Int(C \cup D)) = Fr(Int(C)) \cup Fr(Int(D))$, *and* $Ext(Int(C \cup D)) = Ext(Int(C)) \cap Ext(Int(D))$ [5].

Theorem 3.2. *Let (X, T) be a space. Then the following are equivalent*: (a) (X, T) *is normal*, (b) *for each* $O \in T$ *and each closed set* $C \subseteq O$, *there exist disjoint open sets U and V such that* $C \subseteq U \subseteq O$ *and* $Fr(O) \subseteq V$, (c) *for disjoint closed sets C and D, there exist disjoint closed sets E and F such that* $C \subseteq Int(E)$ *and* $D \subseteq Int(F)$, (d) *for disjoint closed sets C and D, there exist disjoint closed sets E and F such that* $X = Int(E) \cup Int(F) \cup Fr(Int(E \cup F)) \cup Ext(Int(E \cup F))$, *where* $C \subseteq Int(E)$, $D \subseteq Int(F)$, *and* $Int(Fr(Int(E \cup F))) = \phi$, (e) *for disjoint closed sets C and D, there exist disjoint closed sets E and F such that* $X = Int(E) \cup Int(F) \cup Fr((Int(E))) \cup Fr(Int(F)) \cup (Ext(Int(E)) \cap Ext(Int(F)))$, *where* $C \subseteq Int(E)$, $D \subseteq Int(F)$, *and* $Int(Fr(Int(E))) = Int(Fr(Int(F))) = \phi$, (f) *for disjoint closed sets C and D, there exist disjoint regularly closed sets G and H such that* $X = Int(G) \cup Int(H) \cup Fr(Int(G)) \cup Fr(Int(H)) \cup (Ext(Int(G)) \cap Ext(Int(H)))$, *where* $C \subseteq Int(G)$, $D \subseteq Int(H)$, *and* $Int(Fr(Int(G))) = Int(Fr(Int(H))) = \phi$, (g) *for disjoint closed sets C and D, there exist disjoint regularly open sets U and V such that* $C \subseteq U$ *and* $D \subseteq V$, (h) *for disjoint closed sets C and D, there exist disjoint Ts-open sets U and V such that* $C \subseteq U$ *and* $D \subseteq V$, (i) *for each* $O \in T$ *and each closed set* $C \subseteq O$, *there exist disjoint regularly open sets U and V such that* $C \subseteq U$ *and* $Fr(O) \subseteq V$, *and* (j) *for each* $O \in T$ *and each* $C \subseteq O$, *there exist disjoint Ts-open sets U and V such that* $C \subseteq U$ *and* $Fr(O) \subseteq V$.

Proof. Clearly, (a) implies (b).

(b) implies (c): Let C and D be disjoint closed sets. Let $O = X\backslash D$. Then $C \subseteq O \in T$. Let U and V be disjoint open sets such that $C \subseteq U \subseteq O$ and $Fr(O) \subseteq V$. Let $F = Cl(V)$. Then F is closed, $U \cap F = \phi$, and $D \subseteq V \subseteq F$. Thus, $D \subseteq Int(F)$. Since $C \subseteq U \in T$, let W and Z be disjoint open sets such that $C \subseteq W \subseteq U$ and $Fr(U) \subseteq Z$. Then $(X\backslash U) \cup Fr(U) = Int(X\backslash U) \cup Fr(X\backslash U) \cup Fr(U) = Int(X\backslash U) \cup Fr(U) \subseteq Int(X\backslash U) \cup Z = Y \in T$ and $Y \cap W = \phi$. Let $E = Cl(W)$. Then E such that $C \subseteq Int(E)$ and $E \cap F = \phi$.

(c) implies (d): By Theorem 1.4, (c) implies (d). Combining (d) with Theorem 3.1 gives (e).

(e) implies (f): Let C and D be disjoint closed sets. Let E and F be disjoint closed sets such that $X = Int(E) \cup Int(F) \cup Fr(Int(E) \cup Fr(Int(F) \cup (Ext(Int(E)) \cap Ext(Int(F)))))$, where $C \subseteq Int(E)$, $D \subseteq Int(F)$, and $Int(Fr(Int(E))) = Int(Fr(Int(F))) = \phi$. Let $G = Cl(Int(E))$ and let $H = Cl(Int(F))$. Then, by Theorem 1.3, F and G are disjoint regularly closed sets. Thus, combining the statement of (e) given above in this proof with Theorem 1.3 gives (f).

(f) implies (g): Let C and D be disjoint closed sets. Let G and H be disjoint regularly closed sets such that $X = Int(G) \cup Int(H) \cup Fr(Int(G)) \cup Fr(Int(H)) \cup (Ext(Int(G)) \cap Ext(Int(H)))$, where $C \subseteq Int(G)$, $D \subseteq Int(H)$, and $Int(Fr(Int(G))) = Int(Fr(Int(H))) = \phi$. Since regularly closed sets are closed, by Theorem 1.1, $Int(G)$ and $Int(H)$ are disjoint regularly open sets such that $C \subseteq Int(G)$ and $D \subseteq Int(H)$.

Since $RO(X, T) \subseteq Ts$, (g) implies (h).

(h) implies (i): Since $Ts \subseteq T$, (X, T) is normal. Let $O \in T$ and let C be closed such that $C \subseteq O$. Then $Fr(O)$ is closed and $C \cap Fr(O) = \phi$, and by the arguments above, there exist disjoint regularly open sets U and V such that $C \subseteq U$ and $Fr(O) \subseteq V$.

Since $RO(X, T) \subseteq Ts \subseteq T$, (i) implies (j) and (j) implies (a).

Corollary 3.1. *Let* (X, T) *be a space. Then* (X, T) *is* T_4 *iff* (X, T) *is* T_1 *and for disjoint closed sets C and D, there exists a closed set E such that* $C \subseteq Int(E)$ *and* $D \subseteq X \setminus E$.

In a similar manner, each of the characterizations of normal spaces given in Theorem 3.2 can be extended to T_3 spaces.

Thus, regularly open and regularly closed sets proved to be important in the continued investigation of both normal and T_4 spaces.

References

[1] C. Dorsett, Regularly open sets, the semiregularization process, and subspaces, Pioneer J. Math. Math. Sci. 12(1) (2014), 29-33.

[2] C. Dorsett, Regular open sets and *r*-topological properties, Nat. Acad. Sci. Lett. 10(1) (1987), 17-21.

[3] C. Dorsett, New characterizations of regular open sets, extremally disconnectedness, and *RS*-compactness, J. Sci. Research 8(2-3) (1986), 95-99.

[4] C. Dorsett, Regularly open set separations and new characterizations of classical separation axioms, Universal J. Math. Math. Sci. 6(1) (2014), 53-63.

[5] C. Dorsett, Regularly open sets and regular spaces, Pioneer J. Math. Math. Sci. (accepted).

[6] M. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 374-481.

[7] H. Tietze, Beiträge zur allgemeinen Topologie I (German), Math. Ann. 88(3-4) (1923), 290-312.

[8] S. Willard, General Topology, Addison-Wesley, 1970.