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GLOBAL DIFFERENTIAL INVARIANTS OF AFFINE CURVES IN $\mathbb{R}^2$

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Abstract

In this paper, the complete system of affine global differential invariants of affine curves in $\mathbb{R}^2$ is obtained. The problem of affine equivalence of affine curves is reduced to that of paths. Conditions of affine-equivalence of curves are given in terms of the affine invariants. It is obtained that the generating affine invariants are independent.

1. Introduction

Invariants in differential geometry have received a great deal of attention from the mathematical communities since earlier times. Invariant functions are absolute invariant if the derivatives of the included functions are taken with respect to an invariant parameter.

The theory of affine differential invariants dates back to the 1920's, see [3, 4, 11], for further historical remarks and a modern exposition

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[2, 5, 7, 8, 12]. In recent years, geometrical image processing applications based on curvature and arc length have received a lot of attention.

The theory of curves in centroequi-affine, equi-affine and centro-affine (respectively, $SL(n, \mathbb{R})$, $EA(n, \mathbb{R})$, $GL(n, \mathbb{R})$) was studied in [5, 9, 10, 13]. In most papers named affine theory of curves was studied equi-affine theory of curves.

The construction of affine invariants of curves was studied in [1, 2]. But, in these papers, they used equi-affine invariant parameter instead of affine. They worked with this parametrization and formed the affine theory of curves in $\mathbb{R}^2$. In our approach, the affine arc length parametrization must be affine invariant under the full affine group. In [2], it was introduced invariant parametrization, but not affine invariant and examined local theory of affine curves, but we globally.

This paper is concerned with the basic theory of affine geometry of curves in $\mathbb{R}^2$ and related questions of affine differential invariants. Motivated such questions, we introduced the affine invariant parametrization which is invariant under the full affine group. This parametrization is quite useful for the study of affine invariants of curves and congruence of two affine curves in $\mathbb{R}^2$. In the second part of this paper, we give the definitions of affine curve, affine arc length and affine types of an affine curve. And we show that the type of an affine curve is an affine invariant of a curve. Later, we introduce affine invariant parametrization of an affine curve which is invariant under the affine group. Then we reduced the problem of the affine equivalence of curves to that of paths. In the third part, we give the complete system of affine differential invariants for affine plane curves and show that these invariants are independent. So this is the smallest generating system of $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$. The conditions of equivalence of two affine curves are obtained in terms of affine differential invariants.

2. Affine Curve and Affine Arc Length of a Affine Curve in $\mathbb{R}^2$

Let $\mathbb{R}$ be the field of real numbers and $I = (a, b)$ be an open interval of $\mathbb{R}$.

We denote the group $\{F : \mathbb{R}^2 \to \mathbb{R}^2 : Fx = gx + b,\ g \in GL(2, \mathbb{R}),\ b \in \mathbb{R}^2\}$ of all affine transformations of $\mathbb{R}^2$ by $A(2, \mathbb{R})$, where $GL(2, \mathbb{R})$ is the general linear group.

Definition 2.1. A C^∞-map $x : I \to \mathbb{R}^2$ will be called an *I-path* in $\mathbb{R}^2$.

Definition 2.2. An I_1-path $x(t)$ and an I_2-path $y(r)$ in $\mathbb{R}^2$ will be called *D-equivalent* if there exists a C^∞-diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. A class of *D*-equivalent paths in $\mathbb{R}^2$ will be called a *curve* in $\mathbb{R}^2$. A path $x \in \alpha$ will be called a *parametrization* of a curve α.

Remark 2.1. This definition is the same as the definition of a curve in [9, p. 2]. There exist different definitions of a curve [6, p. 2], [8].

If $x(t)$ is an *I*-path in $\mathbb{R}^2$, then $Fx(t)$ is an *I*-path in $\mathbb{R}^2$ for any $F \in A(2, \mathbb{R})$.

Definition 2.3. Two *I*-paths x and y in $\mathbb{R}^2$ will be called $A(2, \mathbb{R})$-*equivalent* and written $x \overset{A(2,\mathbb{R})}{\sim} y$ if there exists $F \in A(2, \mathbb{R})$ such that $y(t) = Fx(t)$.

Let $\alpha = \{h_\tau, \tau \in Q\}$ be a curve in $\mathbb{R}^2$, where h_τ is a parametrization of α. Then $F\alpha = \{Fh_\tau, \tau \in Q\}$ is a curve in R^2 for any $F \in A(2, \mathbb{R})$.

Definition 2.4. Two curves α and β in $\mathbb{R}^2$ will be called $A(2, \mathbb{R})$-*equivalent* if $\beta = F\alpha$ for some $F \in A(2, \mathbb{R})$.

Let $x(t)$ be an I-path in $\mathbb{R}^2$ and $[x'(t)x''(t)]$ be the determinant of the vectors $x'(t), x''(t)$.

Definition 2.5. An I-path $x(t)$ in $\mathbb{R}^2$ will be called *affine regular* if for all $t \in I$, $[x'(t)x''(t)] \neq 0$ and

$$3[x'(t)x^{(iv)}(t)][x'(t)x''(t)] + 12[x'(t)x''(t)][x''(t)x'''(t)] - 5[x'(t)x'''(t)]^2 \neq 0.$$

A curve will be called *regular* if it contains a regular path.

Example 2.1. $x : (0, 1) \to \mathbb{R}^2$, $x(t) = (t, t^3)$ is affine regular.

For $I(a, b)$, $p, q \in I$, put (see [11, p. 54]):

$$l_x(p, q) = \int_p^q \left| \frac{3[x'(t)x^{(iv)}(t)][x'(t)x''(t)] + 12[x'(t)x''(t)][x''(t)x'''(t)] - 5[x'(t)x'''(t)]^2}{3[x'(t)x''(t)]^2} \right|^{\frac{1}{2}} dt$$

and $l_x(a, q) = \lim_{p \to a} l_x(p, q)$, $l_x(p, b) = \lim_{q \to b} l_x(p, q)$. There are only four possible cases:

(i) $l_x(a, q) < \infty$, $l_x(p, b) < \infty$,

(ii) $l_x(a, q) < \infty$, $l_x(p, b) = \infty$,

(iii) $l_x(a, q) = \infty$, $l_x(p, b) < \infty$,

(iv) $l_x(a, q) = \infty$, $l_x(p, b) = \infty$.

In case of (i), we say that x *belongs* to the finite affine type, in case of (ii), (iii) and (iv), we say that x *belongs* to the affine types of $(0, \infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively. There exist paths of all types. The affine type of a path x will be denoted by $T(x)$.

Example 2.2. We consider the I-path $x : I \to \mathbb{R}^2$, $x(t) = (t, e^t)$, $I = (a, b)$, $a, b \in \mathbb{R}$. Then $l_x(p, q) = \left(\frac{2}{3}\right)^{\frac{1}{2}} \cdot (q - p)$. So $x(t)$ has the finite affine type. If we take $I = (0, \infty)$, $(-\infty, 0)$ and $(-\infty, \infty)$, respectively, in this example, then $x(t)$ has the $(0, \infty)$, $(-\infty, 0)$ and $(-\infty, \infty)$-affine types, respectively.

Proposition 2.1. (i) *If* $x \overset{A(2,\mathbb{R})}{\sim} y$, *then* $T(x) = T(y)$.

(ii) *Let* α *be a curve and* $x, y \in \alpha$. *Then* $T(x) = T(y)$.

Proof. (i) is obvious. We show (ii). Let $x, y \in \alpha$. Then there exists a $\varphi : (c, d) \to (a, b)$, $\varphi'(r) > 0$, C^∞-diffeomorphism such that $y(r) = x(\varphi(r))$ for all $r \in (c, d)$. Let x has the finite affine type. So we get

$$l_y(c, \bar{q})$$

$$= \lim_{\bar{p} \to c} l_y(\bar{p}, \bar{q})$$

$$= \lim_{\bar{p} \to c} \int_{\bar{p}}^{\bar{q}} \left| \frac{\begin{array}{c} 3[y'(r)y^{(iv)}(r)][y'(r)y''(r)] \\ + 12[y'(r)y''(r)][y''(r)y'''(r)] - 5[y'(r)y'''(r)]^2 \end{array}}{3[y'(r)y''(r)]^2} \right|^{\frac{1}{2}} dr$$

$$= \lim_{\bar{p} \to c} \int_{\bar{p}}^{\bar{q}} \left| \frac{\begin{array}{c} 3[(x(\varphi))'(x(\varphi))^{(iv)}][(x(\varphi))'(x(\varphi))''] \\ + 12[(x(\varphi))'(x(\varphi))''][(x(\varphi))''(x(\varphi))'''] - 5[(x(\varphi))'(x(\varphi))''']^2 \end{array}}{3[(x(\varphi))'(x(\varphi))'']^2} \right|^{\frac{1}{2}} dr.$$

We put in order

$$l_y(c, \overline{q})$$

$$= \lim_{\overline{p}\to c} \int_{\overline{p}}^{\overline{q}} \varphi'(r) \left| \frac{\begin{array}{l} 3[x'(\varphi)x^{(iv)}(\varphi)][x'(\varphi)x''(\varphi)] \\ + 12[x'(\varphi)x''(\varphi)][x''(\varphi)x'''(\varphi)] - 5[x'(\varphi)x'''(\varphi)]^2 \end{array}}{3[x'(\varphi)x''(\varphi)]^2} \right|^{\frac{1}{2}} dr.$$

If we take $\varphi(r) = t,\ \varphi'(r)dr = dt,$ then we obtain

$$l_y(c, \overline{q}) = \lim_{\varphi(\overline{p})\to a} \int_{\varphi(\overline{p})}^{\varphi(\overline{q})} \left| \frac{\begin{array}{l} 3[x'(t)x^{(iv)}(t)][x'(t)x''(t)] \\ + 12[x'(t)x''(t)][x''(t)x'''(t)] - 5[x'(t)x'''(t)]^2 \end{array}}{3[x'(t)x''(t)]^2} \right|^{\frac{1}{2}} dt$$

$$= \lim_{\varphi(\overline{p})\to a} l_x(\varphi(\overline{p}), \varphi(\overline{q})) = l_x(a, \varphi(\overline{q})).$$

We get $l_y(\overline{p}, d) = l_x(\varphi(\overline{p}), b)$ similarly. Therefore,

$$l_y(c, \overline{q}) = l_x(a, \varphi(\overline{q})) < \infty, \quad l_y(\overline{p}, d) = l_x(\varphi(\overline{p}), b) < \infty.$$

So y has the finite affine type. Then $T(x) = T(y)$. The proofs of other types are proved similarly. □

The affine type of a path $x \in \alpha$ will be called the *affine type* of the curve α and denoted by $T(\alpha)$. According to Proposition 2.1, $T(\alpha)$ is an affine invariant of a curve α.

Let $x(t)$ be a regular I-path in $\mathbb{R}^2$. We define the affine arc length function $s_x(t)$ for each affine type as follows. We put $s_x(t) = l_x(a, t)$ for the cases finite affine type and $T(x) = (0, \infty)$ and $s_x(t) = -l_x(t, b)$ for the case $T(x) = (-\infty, 0)$. We choose a fixed point in every interval $I = (a, b)$

of $\mathbb{R}$ and denote it by a_I. Let $a_I = 0$ for $I = (-\infty, +\infty)$. We set $s_x(t) = l_x(a_I, t)$.

Since $s'_x(t) \neq 0$ for all $t \in I$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $T(x)$ and $t'_x(s) > 0$ for all $s \in T(x)$.

Proposition 2.2. *Let* $I = (a, b)$, $J = (c, d)$ *and x be a regular I-path in* $\mathbb{R}^2$. *Then*

(i) $s_{Fx}(t) = s_x(t)$, $t_{Fx}(s) = t_x(s)$ *for all* $F \in A(2, \mathbb{R})$.

(ii) *The equalities* $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$ *and* $\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s)$ *hold for any* C^∞*-diffeomorphism* $\varphi : J \to I$ *such that* $\varphi'(r) > 0$ *for all* $r \in J$, *where* $s_0 = 0$ *for* $T(x) \neq (-\infty, +\infty)$ *and* $s_0 = l_x(\varphi(a_J), a_I)$ *for* $T(x) = (-\infty, +\infty)$.

Proof. The proof (i) is obvious. So we prove (ii). Let $T(x) = (-\infty, +\infty)$. Then we have

$$s_{x(\varphi)}(r)$$

$$= \int_{a_J}^{r} \left| \frac{\begin{array}{l} 3[(x(\varphi))'(x(\varphi))^{(iv)}][(x(\varphi))'(x(\varphi))''] \\ +12[(x(\varphi))'(x(\varphi))''][(x(\varphi))''(x(\varphi))'''] - 5[(x(\varphi))'(x(\varphi))''']^2 \end{array}}{3[(x(\varphi))'(x(\varphi))'']^2} \right|^{\frac{1}{2}} dr$$

$$= l_x(\varphi(a_J), \varphi(r)) = l_x(a_I, \varphi(r)) + l_x(\varphi(a_J), a_I).$$

So $s_{x(\varphi)}(r) = s_x(\varphi(r)) + s_0$, where $s_0 = l_x(\varphi(a_J), a_I)$. This implies that

$$\varphi(t_{x(\varphi)}(s + s_0)) = t_x(s).$$

It is easy to see that $s_0 = 0$ for $T(x) \neq (-\infty, +\infty)$. $\square$

Remark 2.2. Let α be a regular curve and $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 2.6. The parametrization $x(t_x(s))$ of a regular curve α will be called an *invariant parametrization* of α. We denote the set of all invariant parametrizations of α by ϕ_α.

Proposition 2.3. *Let* α *be a regular curve*, $x \in \alpha$ *and* x *be an I-path*, *where* $I = T(\alpha)$. *Then the following conditions are equivalent*:

(i) *x is an invariant parametrization of* α;

(ii) $$\left(\frac{\begin{array}{l} 3[x'(s)x^{(iv)}(s)][x'(s)x''(s)] \\ +12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2 \end{array}}{3[x'(s)x''(s)]^2}\right)^2 = 1 \textit{ for all } s \in T(\alpha);$$

(iii) $s_x(s) = s$ *for all* $s \in T(\alpha)$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in \phi_\alpha$. Then there exists $y \in \alpha$ such that $x(s) = y(t_y(s))$. By Proposition 2.2, $s_x(s) = s_{y(t_y)}(s) + s_0 = s + s_0$, where s_0 is as in Proposition 2.2. Since s_0 does not depend on s,

$$s_x'(s) = \left|\left(\frac{\begin{array}{l} 3[x'(s)x^{(iv)}(s)][x'(s)x''(s)] \\ +12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2 \end{array}}{3[x'(s)x''(s)]^2}\right)\right|^{\frac{1}{2}} = 1.$$

Hence, for all $s \in T(\alpha)$, we have

$$\left(\frac{\begin{array}{l} 3[x'(s)x^{(iv)}(s)][x'(s)x''(s)] \\ +12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2 \end{array}}{3[x'(s)x''(s)]^2}\right)^2 = 1.$$

(ii) $\Rightarrow$ (iii) By the definition of $s_x(t)$, for the affine type $(0, \infty)$, we have

$$s_x(s) = l_x(a, s)$$

$$= \lim_{p \to a} \int_p^s \left| \left(\frac{\begin{array}{c} 3[x'(s)x^{(iv)}(s)][x'(s)x''(s)] \\ + 12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2 \end{array}}{3[x'(s)x''(s)]^2} \right) \right|^{\frac{1}{2}} ds$$

$$= \lim_{p \to a} \int_p^s ds = \lim_{p \to a} (s - p) = s - a.$$

We put $a = 0$, it is obtained that $s_x(s) = s$. The proofs of the other types are proved similarly.

(iii) $\Rightarrow$ (i) The equality $s_x(s) = s$ implies $t_x(s) = s$. Therefore, $x(s) = x(t_x(s)) \in \phi_\alpha$. □

Proposition 2.4. *Let* α *be a regular curve and* $T(\alpha) \neq (-\infty, +\infty)$. *Then there exists the unique invariant parametrization of* α.

Proof. Let $x, y \in \alpha$, x be an I_1-path and y be an I_2-path. Then there exists a C^∞-diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. By Proposition 2.2 and $T(\alpha) \neq (-\infty, +\infty)$, we obtain $y(t_y(s)) = x(\varphi(t_{x(\varphi)}(s))) = x(t_x(s))$. □

Proposition 2.5. *Let* α *be a regular curve*, $T(\alpha) = (-\infty, +\infty)$ *and* $x \in \phi_\alpha$. *Then*

$$\phi_\alpha = \{y : y(s) = x(s + s'),\ s' \in (-\infty, +\infty)\}.$$

In particular, *the set* ϕ_α *is not countable.*

Proof. Let $x, y \in \phi_\alpha$. Then there exists $h, k \in \alpha$ such that $x(s) =$

$h(t_h(s)),\ y(s) = k(t_k(s)),$ where h be an I_1-path and k be an I_2-path. Since $h, k \in \alpha,$ there exists $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $k(r) = h(\varphi(r))$ for all $r \in I_2$. By Proposition 2.2,

$$y(s) = k(t_k(s)) = h(\varphi(t_{h(\varphi)}(s))) = h(t_h(s - s_0)) = x(s - s_0).$$

Let $x \in \phi_\alpha,\ s' \in (-\infty, +\infty)$ and $z(s) = x(s + s') \in \phi_\alpha$. By Proposition 2.3,

$$\left(\frac{3[x'(s)x^{(iv)}(s)][x'(s)x''(s)] + 12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2}{3[x'(s)x''(s)]^2}\right)^2 = 1$$

for all $s \in T(\alpha)$. For $s' \in (-\infty, +\infty)$, since $s + s' \in (-\infty, +\infty)$, we have

$$\left(\frac{3[z'(s)z^{(iv)}(s)][z'(s)z''(s)] + 12[z'(s)z''(s)][z''(s)z'''(s)] - 5[z'(s)z'''(s)]^2}{3[z'(s)z''(s)]^2}\right)^2 = 1$$

for all $s \in T(\alpha)$. By Proposition 2.3, $z(s) \in \phi_\alpha$. □

Theorem 2.1. *Let* α, β *be regular curves and* $x \in \phi_\alpha,\ y \in \phi_\beta$. *Then*

(i) *for* $T(\alpha) = T(\beta) \neq (-\infty, +\infty),$ $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$ *if and only if* $x(s) \overset{A(2,\mathbb{R})}{\sim} y(s)$;

(ii) *for* $T(\alpha) = T(\beta) = (-\infty, \infty),$ $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$ *if and only if* $x(s) \overset{A(2,\mathbb{R})}{\sim} y(s + s')$ *for some* $s' \in (-\infty, +\infty)$.

Proof. (i) Let $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$ and $h \in \alpha$. Then there exists $F \in A(2, \mathbb{R})$ such that $\beta = F\alpha$. This implies $Fh \in \beta$. Using Proposition 2.2 and

Proposition 2.4, we get $x(s) = h(t_h(s))$, $y(s) = (Fh)(t_{Fh}(s))$ and $Fx(s) = F(h(t_h(s))) = (Fh)(t_{Fh}(s)) = y(s)$. Thus, $x(s) \overset{A(2, \mathbb{R})}{\sim} y(s)$. Conversely, let $x(s) \overset{A(2, \mathbb{R})}{\sim} y(s)$, that is, there exists $F \in A(2, \mathbb{R})$ such that $Fx = y$. Then $\alpha \overset{A(2, \mathbb{R})}{\sim} \beta$.

(ii) Let $\alpha \overset{A(2, \mathbb{R})}{\sim} \beta$. Then there exist I-paths $h \in \alpha$, $k \in \beta$ and $F \in A(2, \mathbb{R})$ such that $k(t) = Fh(t)$. We have $k(t_k(s)) = k(t_{Fh}(s)) = k(t_h(s)) = (Fh)(t_h(s))$. By Proposition 2.5, $x(s) = k(t_k(s + s_1))$, $y(s) = h(t_h(s + s_2))$ for some $s_1, s_2 \in (-\infty, +\infty)$. Therefore, $x(s - s_1) = Fy(s - s_2)$. This implies that $x(s) \overset{A(2, \mathbb{R})}{\sim} y(s + s')$, where $s' = s_1 - s_2$. Conversely, let $x(s) \overset{A(2, \mathbb{R})}{\sim} y(s + s')$ for some $s' \in (-\infty, +\infty)$. Then there exists $F \in A(2, \mathbb{R})$ such that $y(s + s') = Fx(s)$. Since $y(s + s') \in \beta$, $\alpha \overset{A(2, \mathbb{R})}{\sim} \beta$. □

Theorem 2.1 reduces the problem of the $A(2, \mathbb{R})$-equivalence of regular curves to that of paths.

3. The Solution of Equivalence Problem of Paths

Let $x(t)$ be an I-path in $\mathbb{R}^2$.

Definition 3.1. A polynomial $p(x, x', ..., x^{(k)})$ of x and a finite number of derivatives $x', ..., x^{(k)}$ of x with the coefficients from $\mathbb{R}$ will be called a *differential polynomial* of x. It will be denoted by $p\{x\}$.

We denote the set of all differential polynomials of x by $\mathbb{R}\{x\}$. It is a differential $\mathbb{R}$-algebra.

Proposition 3.1. *The differential* $\mathbb{R}$*-algebra* $\mathbb{R}\{x\}$ *is an integral domain.*

Proof. In [14, p. 11]. □

Since $\mathbb{R}\{x\}$ is an integral domain, there exists a division field containing it. This division field will be denoted by $\mathbb{R}\langle x\rangle$. An element $f\langle x\rangle$ of the $\mathbb{R}\langle x\rangle$ has the form $f\langle x\rangle = \dfrac{p\{x\}}{q\{x\}}$, $q\{x\} \neq 0$, $p\{x\}, q\{x\} \in \mathbb{R}\{x\}$. The elements of the $\mathbb{R}\langle x\rangle$ will be called *differential rational functions*. $\mathbb{R}\langle x\rangle$ is a differential field.

Definition 3.2. A differential rational function $f\langle x\rangle$ will be called $A(2, \mathbb{R})$-*invariant* if $f\langle Fx\rangle = f\langle x\rangle$ for all $F \in A(2, \mathbb{R})$.

The set of all $A(2, \mathbb{R})$-invariant differential rational functions of x will be denoted by $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$. It is a differential $\mathbb{R}$-subalgebra of $\mathbb{R}\langle x\rangle$.

Definition 3.3. A subset S of $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$ will be called a *generating system of* $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$ if the smallest differential subfield containing S is $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$.

Theorem 3.1. *The system*

$$\frac{[x'\ x'']'}{[x'\ x'']}, \frac{[x''\ x''']}{[x'\ x'']} \tag{3.1}$$

is a generating system of $\mathbb{R}\langle x\rangle^{A(2, \mathbb{R})}$.

Proof. Let $\{x_\tau, \tau \in \Delta\}$ be vectors in $\mathbb{R}^2$. For the group $G = GL(n, \mathbb{R})$, the generating system of $\mathbb{R}(x_\tau, \tau \in \Delta)^G$ is

$$\frac{[x_1 \cdots x_{i-1}\ x_\tau\ x_{i+1} \cdots x_n]}{[x_1 \cdots x_n]}, \ i = 1, ..., n, \ \tau \in \Delta/\{1, ..., n\}$$

in [14]. So for $n = 2$, this generating system is the form

$$\frac{[x_1\ x_\tau]}{[x_1\ x_2]}, \frac{[x_\tau\ x_2]}{[x_1\ x_2]}, \ \tau \in \Delta/\{1, 2\}.$$

Therefore, if we take $\mathbb{R}\langle x\rangle^G = \mathbb{R}(x', \ldots, x^{(n)}, \ldots)^{A(2,\mathbb{R})}$, then we obtain

$$\frac{[x'\ x^{(\tau)}]}{[x'\ x'']}, \frac{[x^{(\tau)}\ x'']}{[x'\ x'']}, \tau \in \Delta/\{1, 2\}.$$

But this set is generated by the formulas (3.1). We want to show this. Let $\tau = 3$. Since $[x'\ x'']' = [x'\ x''']$, we have $\frac{[x'\ x'']'}{[x'\ x'']} = \frac{[x'\ x''']}{[x'\ x'']}$.

For $\tau = 3$, the generating system is $\frac{[x'\ x''']}{[x'\ x'']}, \frac{[x''\ x''']}{[x'\ x'']}$ and the assumption is true for $\tau = 3$.

Let the system (3.1) be the generating system for $\tau - 1$ and we show this for τ. $[x'\ x^{(\tau-1)}]' = [x''\ x^{(\tau-1)}] + [x'\ x^{(\tau)}]$ and so $[x'\ x^{(\tau)}] = [x'\ x^{(\tau-1)}]' - [x''\ x^{(\tau-1)}]$. Dividing this equation into $[x'\ x'']$, we obtain

$$\frac{[x'\ x^{(\tau)}]}{[x'\ x'']} = \frac{[x'\ x^{(\tau-1)}]'}{[x'\ x'']} - \frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']}.$$

Therefore, $\frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']}$ is generated by the formulas (3.1) by induction. Since

$$\left(\frac{[x'\ x^{(\tau-1)}]}{[x'\ x'']}\right)' = \frac{([x''\ x^{(\tau-1)}] + [x'\ x^{(\tau)}]) \cdot [x'\ x''] - [x'\ x'''] \cdot [x'\ x^{(\tau-1)}]}{[x'\ x'']^2}$$

$$= \frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']} + \frac{[x'\ x^{(\tau)}]}{[x'\ x'']} - \frac{[x'\ x''']}{[x'\ x'']} \cdot \frac{[x'\ x^{(\tau-1)}]}{[x'\ x'']},$$

we get

$$\frac{[x'\ x^{(\tau)}]}{[x'\ x'']} = \left(\frac{[x'\ x^{(\tau-1)}]}{[x'\ x'']}\right)' - \frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']} + \frac{[x'\ x''']}{[x'\ x'']} \cdot \frac{[x'\ x^{(\tau-1)}]}{[x'\ x'']}$$

and similarly, since

$$\frac{[x''\ x^{(\tau)}]}{[x'\ x'']} = \left(\frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']}\right)' - \frac{[x'''\ x^{(\tau-1)}]}{[x'\ x'']} + \frac{[x'\ x''']}{[x'\ x'']} \cdot \frac{[x''\ x^{(\tau-1)}]}{[x'\ x'']},$$

$\frac{[x'\ x^{(\tau)}]}{[x'\ x'']}$, $\frac{[x''\ x^{(\tau)}]}{[x'\ x'']}$ are generated by the formulas (3.1). According to induction hypothesis, the system (3.1) is the generating system of $\mathbb{R}\langle x\rangle^{A(2,\mathbb{R})}$. □

Theorem 3.2. *Let* α, β *be regular curves in* $\mathbb{R}^2$ *and* $x \in \phi_\alpha$, $y \in \phi_\beta$. *Then*

(i) *for* $T(\alpha) = T(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$ *if and only if*

$$\operatorname{sgn}\left(\frac{\begin{array}{l}3[x'(s)x^{(iv)}(s)][x'(s)x''(s)]\\ +12[x'(s)x''(s)][x''(s)x'''(s)] - 5[x'(s)x'''(s)]^2\end{array}}{3[x'(s)x''(s)]^2}\right)$$

$$= \operatorname{sgn}\left(\frac{\begin{array}{l}3[y'(s)y^{(iv)}(s)][y'(s)y''(s)]\\ +12[y'(s)y''(s)][y''(s)y'''(s)] - 5[y'(s)y'''(s)]^2\end{array}}{3[y'(s)y''(s)]^2}\right),$$

$$\frac{[x'(s)x'''(s)]}{[x'(s)x''(s)]} = \frac{[y'(s)y'''(s)]}{[y'(s)y''(s)]}, \quad \frac{[x''(s)x'''(s)]}{[x'(s)x''(s)]} = \frac{[y''(s)y'''(s)]}{[y'(s)y''(s)]}$$

for all $s \in T(\alpha) = T(\beta)$;

(ii) *for* $T(\alpha) = T(\beta) = (-\infty, +\infty)$, $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$ *if and only if there exists* $a \in (-\infty, +\infty)$ *such that*

$$\operatorname{sgn}\left(\frac{\begin{array}{l}3[x'(s)x^{(iv)}(s)][x'(s)x''(s)]\\ +12[x'(s)x''(s)][x''(s)x'''(s)]-5[x'(s)x'''(s)]^2\end{array}}{3[x'(s)x''(s)]^2}\right)$$

$$=\operatorname{sgn}\left(\frac{\begin{array}{l}3[y'(s')y^{(iv)}(s')][y'(s')y''(s')]\\ +12[y'(s')y''(s')][y''(s')y'''(s')]-5[y'(s')y'''(s')]^2\end{array}}{3[y'(s')y''(s')]^2}\right),$$

where $s' = s + a$,

$$\frac{[x'(s)x'''(s)]}{[x'(s)x''(s)]} = \frac{[y'(s+a)y'''(s+a)]}{[y'(s+a)y''(s+a)]},$$

$$\frac{[x''(s)x'''(s)]}{[x'(s)x''(s)]} = \frac{[y''(s+a)y'''(s+a)]}{[y'(s+a)y''(s+a)]}$$

for all $s \in T(\alpha) = T(\beta)$.

Proof. (i) Let $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$. By claim (i) of Theorem 2.1, $x \overset{A(2,\mathbb{R})}{\sim} y$. So $y = gx + b$ for some $g \in GL(2, \mathbb{R})$, $b \in \mathbb{R}^2$. Therefore, using this equality, we obtain above formulas easily.

Conversely, the above equalities hold for all $s \in T(\alpha) = T(\beta)$. Let

$$A_x = \begin{bmatrix} x_1'(s) & x_1''(s) \\ x_2'(s) & x_2''(s) \end{bmatrix}, \quad A_x' = \begin{bmatrix} x_1''(s) & x_1'''(s) \\ x_2''(s) & x_2'''(s) \end{bmatrix}, \tag{3.2}$$

for all $s \in T(\alpha) = T(\beta)$. Since $[x'(s)x''(s)] \neq 0$ for all $s \in T(\alpha) = T(\beta)$, there exists the inverse A_x^{-1} of the matrix A_x. Let $A_x^{-1} \cdot A_x' = C$ and $C = \| c_{ij} \|$, $i, j = 1, 2$. Thus, $A_x' = A_x \cdot C$. That is,

$$\begin{bmatrix} x_1''(s) & x_1'''(s) \\ x_2''(s) & x_2'''(s) \end{bmatrix} = \begin{bmatrix} x_1'(s) & x_1''(s) \\ x_2'(s) & x_2''(s) \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Therefore,

$$x_1''(s) = c_{11} \cdot x_1'(s) + c_{21} \cdot x_1''(s),$$

$$x_2''(s) = c_{11} \cdot x_2'(s) + c_{21} \cdot x_2''(s),$$

$$x_1'''(s) = c_{12} \cdot x_1'(s) + c_{22} \cdot x_1''(s),$$

$$x_2'''(s) = c_{12} \cdot x_2'(s) + c_{22} \cdot x_2''(s).$$

We obtain

$$c_{11} = 0, \quad c_{21} = 1, \quad c_{12} = \frac{[x'''(s)x''(s)]}{[x'(s)x''(s)]}, \quad c_{22} = \frac{[x'(s)x'''(s)]}{[x'(s)x''(s)]}.$$

In view of above equalities $A_y^{-1} \cdot A_y' = C$, thus $A_x^{-1} \cdot A_x' = A_y^{-1} \cdot A_y' = C$. So

$$(A_y \cdot A_x^{-1})' = A_y' \cdot A_x^{-1} + A_y \cdot (A_x^{-1})' = A_y \cdot A_x^{-1} \cdot (A_y^{-1} A_y' - A_x^{-1} \cdot A_x') = 0,$$

that is, $A_y \cdot A_x^{-1} = g$, g constant. Since $\det(A_y \cdot A_x^{-1}) \neq 0$, $\det g \neq 0$. Therefore, $A_y = g \cdot A_x$, $g \in GL(2, \mathbb{R})$. We get $y'(s) = gx'(s)$ for all $s \in T(\alpha) = T(\beta) \neq (-\infty, +\infty)$. Integrating we obtain $y(s) = g \cdot x(s) + b$ for some $b \in \mathbb{R}^2$. Thus, $x \overset{A(2,\mathbb{R})}{\sim} y$. This, in view of claim (i) of Theorem 3.1, it implies $\alpha \overset{A(2,\mathbb{R})}{\sim} \beta$.

The proof of (ii) follows similarly from claim (ii) of Theorem 3.1. □

Theorem 3.3. *Let* $h_1(s)$, $h_2(s)$ *be* C^∞*-functions on* T, *where*

$$\left(h_1'(s) + 3h_2(s) - \frac{2}{3} h_1^2(s)\right)^2 = 1$$

for all $s \in T$. *Then there exists an invariant parametrization* y *of a regular curve such that*

$$\frac{[y'(s)\,y'''(s)]}{[y'(s)\,y''(s)]} = h_1(s)\,\frac{[y''(s)\,y'''(s)]}{[y'(s)\,y''(s)]} = h_2(s)$$

for all $s \in T$.

Proof. Put $y'(s) = x(s)$. Then we have

$$\frac{[x(s)x''(s)]}{[x(s)x'(s)]} = h_1(s), \quad \frac{[x'(s)x''(s)]}{[x(s)x'(s)]} = h_2(s) \tag{3.3}$$

for all $s \in T$. We take

$$A_x = \begin{bmatrix} x_1(s) & x_1'(s) \\ x_2(s) & x_2'(s) \end{bmatrix}, \quad A_x' = \begin{bmatrix} x_1'(s) & x_1''(s) \\ x_2'(s) & x_2''(s) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -h_2(s) \\ 1 & h_1(s) \end{bmatrix}.$$

Then $A_x^{-1} \cdot A_x' = B$ and so $A_x' = A_x \cdot B$. That is,

$$\begin{bmatrix} x_1'(s) & x_1''(s) \\ x_2'(s) & x_2''(s) \end{bmatrix} = \begin{bmatrix} x_1(s) & x_1'(s) \\ x_2(s) & x_2'(s) \end{bmatrix} \cdot \begin{bmatrix} 0 & -h_2(s) \\ 1 & h_1(s) \end{bmatrix}$$

$$= \begin{bmatrix} x_1'(s) & -h_2(s)x_1(s) + h_1(s)x_1'(s) \\ x_2'(s) & -h_2(s)x_2(s) + h_1(s)x_2'(s) \end{bmatrix}.$$

Hence, we get the following differential equation system:

$$x_1''(s) + h_2(s)x_1(s) - h_1(s)x_1'(s) = 0,$$

$$x_2''(s) + h_2(s)x_2(s) - h_1(s)x_2'(s) = 0.$$

Here, $x_1(s)$, $x_2(s)$ are solutions of the following differential equation:

$$z'' + h_2 z - h_1 z' = 0. \tag{3.4}$$

It is known from the theory of differential equations that there exists a solution of the differential equation (3.4) which these solutions are k_1, k_2, where $\det\begin{bmatrix} k_1(s) & k_1'(s) \\ k_2(s) & k_2'(s) \end{bmatrix} \neq 0$. Put $x(s) = \begin{bmatrix} k_1(s) \\ k_2(s) \end{bmatrix}$. Then the curve $x(s)$ provides equations (3.3) and $[x(s)x'(s)] \neq 0$. Since $y'(s) = x(s)$, integrating we have for some $b \in \mathbb{R}^2$,

$$y(s) = \int_{s_0}^{s} x(s)ds + b.$$

Therefore,

$$\left(h_1'(s) + 3h_2(s) - \frac{2}{3}h_1^2(s)\right)^2$$

$$= \left(\frac{\begin{matrix} 3[y'(s)y^{(IV)}(s)][y'(s)y''(s)] \\ +12[y'(s)y''(s)][y''(s)y'''(s)] - 5[y'(s)y'''(s)]^2 \end{matrix}}{3[y'(s)y''(s)]^2}\right)^2 = 1.$$

$[y'(s)y''(s)] = [x(s)x'(s)] \neq 0$ and

$$3[y'(s)y^{(IV)}(s)][y'(s)y''(s)] + 12[y'(s)y''(s)][y''(s)y'''(s)] - 5[y'(s)y'''(s)]^2 \neq 0.$$

Then by Proposition 2.3, $y \in \phi_\alpha$ for some regular curve α. □

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