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POINT-PUSHING PSEUDO-ANOSOV MAPPING CLASSES AND THEIR ACTIONS ON THE CURVE COMPLEX

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Abstract

Let S be an analytically finite Riemann surface of type (p, n) with $3p + n > 4$, which is equipped with a hyperbolic metric and contains at least one puncture x. Let $\mathcal{C}(S)$ be the curve complex endowed with a path metric $d_{\mathcal{C}}$. It is known that a point-pushing pseudo-Anosov mapping class f on S determines a filling closed geodesic c on $S \cup \{x\}$, and that every non-preperipheral vertex u in $\mathcal{C}(S)$ determines a simple curve $\tilde{u}$ on $S \cup \{x\}$. In this paper, we consider the action of f on $\mathcal{C}(S)$. We describe all geodesic segments in $\mathcal{C}(S)$ that connect u and $f(u)$ when $d_{\mathcal{C}}(u, f(u)) = 2$. We also give sufficient conditions with respect to the intersection points between c and $\tilde{u}$ for the path distance $d_{\mathcal{C}}(u, f(u))$ to be larger.

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1. Introduction

Let S be an analytically finite Riemann surface of type (p, n) with $3p + n > 4,$ where p is the genus and n is the number of punctures on S. Assume that S is equipped with a hyperbolic metric and contains at least one puncture x. In [5], Harvey introduced a curve complex $\mathcal{C}(S)$ on S, which is a simplicial complex where vertices are simple closed geodesics and a kth dimensional simplex, denoted by $\mathcal{C}_k(S),$ is a collection of $k + 1$ disjoint simple closed geodesics on S.

Let $\mathcal{C}'_0(S)$ be the subset of $\mathcal{C}_0(S)$ consisting of boundary geodesics of twice punctured disks enclosing x ($\mathcal{C}'_0(S)$ is not empty if and only if $n \geq 2$). Let $\hat{\mathcal{C}}(S)$ denote the subcomplex of $\mathcal{C}(S)$ which consists of non-preperipheral simplexes, where a simplex $\{u_0, ..., u_k\}$ is called *non-preperipheral* if none of u_i belongs to $\mathcal{C}'_0(S)$. It is easily seen that $\mathcal{C}_0(S) = \hat{\mathcal{C}}_0(S)$ if S is of type (p, n) with $p \geq 2$ and $n = 0, 1$. Otherwise, we have $\mathcal{C}_0(S) \backslash \hat{\mathcal{C}}_0(S) = \mathcal{C}'_0(S)$. Note that any two vertices in $\mathcal{C}'_0(S)$ intersect, which means that $\mathcal{C}'_0(S)$ is totally disconnected.

Write $\widetilde{S} = S \cup \{x\},$ and let $\widetilde{S}$ be equipped with a hyperbolic metric. The curve complex $\mathcal{C}(\widetilde{S})$ can be similarly defined so that there is a natural projection:

$$\varepsilon : \hat{\mathcal{C}}(S) \to \mathcal{C}(\widetilde{S}). \tag{1.1}$$

According to Birman and Series [3], we may choose a point x that misses every simple closed geodesic on $\widetilde{S}$, which means that a vertex in $\mathcal{C}(\widetilde{S})$ can also be regarded as a vertex on $\mathcal{C}(S)$ (by simply removing the point x). Hence (1.1) admits a global section.

For any $u, v \in \mathcal{C}_0(S),$ the distance $d_{\mathcal{C}}(u, v)$ between u and v is defined as the minimum number of edges in $\mathcal{C}_1(S)$ joining u and v, thereby $\mathcal{C}(S)$ is equipped with the path metric $d_{\mathcal{C}}$. Clearly, $d_{\mathcal{C}}(u, v) = 1$ if and only if u,

v are disjoint, and $d_{\mathcal{C}}(u, v) \geq 3$ if and only if (u, v) fills S in the sense that every closed geodesic intersects u or v. It is well-known that $\mathcal{C}(S)$ is connected and is δ-hyperbolic in the sense of Gromov [4]. See Masur and Minsky [8] for a detailed explanation.

Let $\mathscr{F}$ be the set of pseudo-Anosov mapping classes of S onto itself that fix the puncture x and are isotopic to the identity on $\widetilde{S}$ (see Thurston [10] for the definition and basic properties of pseudo-Anosov maps). Let $I : \widetilde{S} \times [0, 1] \to \widetilde{S}$ denote the associated isotopy between an element $f \in \mathscr{F}$ and the identity, i.e., $I(\cdot, 0) = f$ and $I(\cdot, 1) = \text{id}$. Then $I(x, t)$, $t \in [0, 1]$, defines an oriented closed curve c on $\widetilde{S}$ passing through x. It was shown in Kra [7] that c is freely homotopic to an oriented filling closed geodesic (call it c also) and every such a geodesic determines a conjugacy class $K(c)$ of c in $\mathscr{F}$. Let $\mathscr{S}$ denote the set of primitive oriented filling closed geodesics on $\widetilde{S}$. Obviously, $\mathscr{F}$ can be partitioned into conjugacy classes and there is a bijection between the set of these conjugacy classes and the set $\mathscr{S}$.

By Proposition 3.6 of [8], there exists a constant $a > 0$, which depends only on (p, n) such that for all pseudo-Anosov maps $f : S \to S$, all positive integers m, and all $u \in \mathcal{C}_0(S)$, it holds that $d_{\mathcal{C}}(u, f^m(u)) \geq am$. By contrast, it was shown in [14, 16] that if $f \in \mathscr{F}$, then $d_{\mathcal{C}}(u, f^m(u)) \geq m$ for $1 \leq m \leq 4$. In [15], we showed that $d_{\mathcal{C}}(u, f(u)) \geq 3$ provided that $u \in \mathcal{C}'_0(S)$.

The purpose of this paper is to study some questions on the distance $d_{\mathcal{C}}(u, f(u))$ for $u \in \hat{\mathcal{C}}_0(S)$, which is related to the set $\#\{\widetilde{u}, c\}$ of intersection points between $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$ and $c \in \mathscr{S}$, where and throughout the rest of the paper, we use the symbol $\widetilde{u}$ to denote $\varepsilon(u)$, which is the geodesic homotopic to u on $\widetilde{S}$ if u is also viewed as a curve on $\widetilde{S}$. The main results will be stated in the next section.

This paper is organized as follows: In Section 2, we state our main results. In Section 3, we collect some background information on the fibration $\varepsilon : \mathcal{C}(S) \to \mathcal{C}_0(\widetilde{S})$ between curve complexes; we then investigate the fibers $F_{\widetilde{u}}$ of the fibration and discuss some properties by means of a tessellation of **H**. We show that there is an intimate relationship between $F_{\widetilde{u}}$ and a tessellation of **H**. In Section 4, we study the distance between vertices in each fiber $F_{\widetilde{u}}$ in terms of the intersection numbers between the filling geodesics and vertices, and prove Theorem 2.1 and Theorem 2.2. The proof of Theorem 2.3 is given in Section 5.

2. Main Theorems

Fix $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$ and $c \in \mathscr{S}$. The geometric intersection number $i(c, \widetilde{u})$ between c and $\widetilde{u}$ is defined as the number of points in $\#\{\widetilde{u}, c\}$, which is also given by

$$i(c, \widetilde{u}) = \min| c' \cap \widetilde{u}' |,$$

where c' and $\widetilde{u}'$ are in the homotopy classes of c and $\widetilde{u}$, respectively.

It is known that $i(c, \widetilde{u}) = 1$ if and only if $d_{\mathcal{C}}(u, f(u)) = 1$ for some $f \in K(c)$ and $u \in \{\varepsilon^{-1}(\widetilde{u})\}$ (Lemma 4.3).

Consider the case where $d_{\mathcal{C}}(u, f(u)) = 2$ for $u \in \hat{\mathcal{C}}_0(S)$. Denote by

$$V(u, f) = \{v \in \mathcal{C}_0(S) : d_{\mathcal{C}}(u, v) = 1 \text{ and } d_{\mathcal{C}}(f(u), v) = 1\}.$$

For $\widetilde{u} = \varepsilon(u)$, we define a subset $W(\widetilde{u}, c)$ of $\mathcal{C}_0(\widetilde{S})$ as follows. Let $k = i(c, \widetilde{u})$ and let $\{Q_1, ..., Q_k\}$ be the intersection points between $\widetilde{u}$ and c. Let $\widetilde{v} \in \mathcal{C}_0(\widetilde{S})$ be such that $\widetilde{v} \neq \widetilde{u}$. Let $\{P_1, ..., P_r\}$ denote the intersection points between c and $\widetilde{v}$. With the aid of a parametrization $c = c(\tau)$ for $\tau \in [0, 1]$, we can write $P_i = c(\tau_i^1)$ and $Q_j = c(\tau_j^2)$ for some $\tau_i^1, \tau_j^2 \in (0, 1)$. We call $\widetilde{u}$ and $\widetilde{v}$ are separated by c if there is a parametrization $c = c(\tau)$ such that $\tau_i^1 < \tau_j^2$ for all $1 \leq i \leq r$ and $1 \leq j \leq k$. Let $W^*(\widetilde{u}, c)$

denote the set of vertices $\tilde{v} \in \mathcal{C}_0(\tilde{S})$ such that $\tilde{v}$ and $\tilde{u}$ are separated by c. It is clear that $W^*(\tilde{u}, c) = \mathcal{C}_0(\tilde{S})$ if $i(\tilde{u}, c) = 1$. Also, if $i(\tilde{v}, c) = 1$, then $\tilde{v} \in W^*(\tilde{u}, c)$.

Let $W(\tilde{u}, c)$ be the subset of $W^*(\tilde{u}, c)$ consisting of vertices $\tilde{v}$ disjoint from $\tilde{u}$. We first prove the following result.

Theorem 2.1. *Let* $f \in \mathscr{F}$ *and let* $c \in \mathscr{S}$ *be determined by f. Let* $u \in \hat{\mathcal{C}}_0(S)$. *Assume that* $d_{\mathcal{C}}(u, f(u)) = 2$. *Then* $V(u, f) \cap \mathcal{C}'_0(S) = \varnothing$ *and the puncture-forgetting projection* (1.1) *restricts to a bijection* $V(u, f) \simeq W(\tilde{u}, c)$.

We now study the problem of when $d_{\mathcal{C}}(u, f(u)) \geq 3$. It seems likely that the distance between u and $f(u)$ is proportional to the number of intersections between $\tilde{u}$ and c. Unfortunately, this is not true. In fact, from the discussion in Section 3, there exist geodesics $c \in \mathscr{S}$ that intersect some $\tilde{u}$ as many times as we expect, yet the distance $d_{\mathcal{C}}(u, f(u))$ remains small.

Nevertheless, with respect to a vertex $\tilde{u} \in \mathcal{C}_0(\tilde{S})$, any $c \in \mathscr{S}$ with $i(c, \tilde{u}) > 1$ can be written as a curve concatenation $c_1 \cdot c_2$ (from left to right), where c_1 and c_2 are constructed as follows. Given an orientation for $\tilde{u}$ and recall that $\#\{c \cap \tilde{u}\} = \{Q_1, ..., Q_k\}$. Let $c = c(\tau)$, $\tau \in [0, 1]$, be a parametrization and let $Q_i = c(\tau_i^2)$. Assume that $\tau_1^2 < \tau_2^2 < \cdots < \tau_k^2$. Let α be the path in c connecting Q_1 and Q_k, and let β be the path in $\tilde{u}$ joining Q_k and Q_1. Then the complement $\gamma = c \backslash \alpha$ is also a path in c that joins Q_k and Q_1. Let c_1 be the path α followed by β, and c_2 be the path β^{-1} followed by γ. Obviously, we have $c = \alpha \cdot \gamma = (\alpha \cdot \beta) \cdot (\beta^{-1} \cdot \gamma) = c_1 \cdot c_2$.

Let δ_1, δ_2 be the geodesics freely homotopic to c_1 and c_2, respectively, and let $\mathscr{S}^*$ be the subset of $\mathscr{S}$ consisting of geodesics that intersect each

$\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$ at least twice. It is easy to see that $\delta_1, \delta_2 \in \mathscr{S}$ implies $c \in \mathscr{S}^*$, this particularly implies that $\mathscr{S}^*$ is not empty. Let $F_{\widetilde{u}}$ denote the set $\{\varepsilon^{-1}(\widetilde{u})\}$ which consists of vertices u in $\mathcal{C}_0(S)$ with $\varepsilon(u) = \widetilde{u}$. Theorem 2.1 leads to the following result.

Theorem 2.2. *Let $c \in \mathscr{S}$ be determined by an element $f \in \mathscr{F}$. Let $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$. Suppose that $i(c, \widetilde{u}) \geq 3$ and with respect to $\widetilde{u}$, the filling geodesic c can be expressed as $c_1 \cdot c_2$, where c_1 and c_2 are freely homotopic to nontrivial geodesics δ_1 and δ_2 which satisfies the condition that $\delta_1 \in \mathscr{S}$. Then $d_{\mathcal{C}}(u, f(u)) \geq 3$ for all $u \in F_{\widetilde{u}}$.*

Remark. In the case where $i(\widetilde{u}, c) = 2$, Lemma 3.3 together with Lemma 4.1 asserts that $d_{\mathcal{C}}(u, f(u)) \geq 3$ for most $u \in F_{\widetilde{u}}$, but for the remaining ones in $F_{\widetilde{u}}$, we have $d_{\mathcal{C}}(u, f(u)) = 2$.

Let f, c and $\widetilde{u}$ be as in Theorem 2.2. Then $c = c_1 \cdot c_2$, where c_1 can further be written as the concatenation $c_1 = c_{11} \cdots c_{1,k-1}$, where c_{11} is obtained from the path in c joining Q_1 and Q_2, followed by the path β_1 in $\widetilde{u}$ joining Q_2 and Q_1. Inductively, for each $1 \leq i \leq k-1$, let c_{1i} be the closed curve obtained from the path β_i^{-1} in $\widetilde{u}$ joining Q_1 and Q_i, followed by the path in c joining Q_i and Q_{i+1}, then followed by the path β_{i+1} in $\widetilde{u}$ joining Q_{i+1} and Q_1. Let δ_{1i} be the geodesic representative in the homotopy class of c_{1i}.

Theorem 2.3. *Let f, c and $\widetilde{u}$ be as above. Assume that $k = i(c, \widetilde{u}) \geq 4$. If c can be expressed as $(c_{11} \cdots c_{1,k-1}) \cdot c_2$, where at least two curves c_{1i} and c_{1j}, $1 \leq i, j \leq k-1$ and $|i - j| \geq 2$, are homotopic to filling closed geodesics, then there are $u \in F_{\widetilde{u}}$ such that $d_{\mathcal{C}}(u, f(u)) \geq 4$.*

There arises the following question:

Question. Is there a primitive $f \in \mathscr{F}$ such that $d_{\mathcal{C}}(u, f(u))$ is arbitrarily large for $u \in \mathcal{C}(S)$?

3. Background and Tree Structures of Fibers in the Curve Complex

Let $\mathbf{H}$ denote the upper half plane that is endowed with the hyperbolic metric $\rho(z)|dz| = |dz|/y$. Let $\varrho : \mathbf{H} \to \widetilde{S}$ be the universal covering map with the covering group G. Then G is a finitely generated Fuchsian group of the first kind and acts on $\mathbf{H}$ as a group of isometries with respect to $\rho(z)$. The group G is torsion free and contains infinitely many hyperbolic Möbius transformations. G also contains parabolic Möbius transformations if and only if $\widetilde{S}$ contains punctures $(n > 1)$.

Choose $\hat{x} \in \mathbf{H}$ with $\varrho(\hat{x}) = x$. Let $\mathscr{A} = \{h(\hat{x}) : h \in G\} \subset \mathbf{H}$ and let $\dot{G}$ be the covering group of a universal covering map $\varrho' : \mathbf{H} \to S$. Then $\mathbf{H}/\dot{G} \cong S \cong (\mathbf{H}/G)\backslash\{x\}$ and there exists an exact sequence

$$1 \to \Gamma \to \dot{G} \to G \to 1,$$

where Γ is the covering group of a universal covering map $v : \mathbf{H} \to \mathbf{H}\backslash\mathscr{A}$.

Let $Q(G)$ (resp. $Q(\dot{G})$) be the group of quasiconformal automorphisms w of $\mathbf{H}$ with $wGw^{-1} = G$ (resp. $w\dot{G}w^{-1} = \dot{G}$). Two elements $w, w_1 \in Q(G)$ are said to be *equivalent* if $w|_{\mathbf{R}} = w_1|_{\mathbf{R}}$. Let $[w]$ be the equivalence class of w. The x-pointed mapping class group, denoted by Mod_S^x, is the subgroup of the ordinary mapping class group $\mathrm{Mod}(S)$ that consists of mapping classes fixing x. In [1], Bers explicitly constructed an isomorphism $\varphi^* : Q(G)/\sim \to \mathrm{Mod}_S^x$ (Theorem 10 of [1]), which is outlined below.

Let $Q_0(\dot{G}) \subset Q(\dot{G})$ be the subgroup of $Q(\dot{G})$ consisting of maps projecting (under ϱ') to maps on S leaving the puncture x fixed. For any $[w] \in Q(G)/\sim$, there is a map $w_0 \in Q(G)$, which is obtained by performing a local quasiconformal deformation within each fundamental region D leaving the boundary ∂D fixed, such that $w \sim w_0$ and $w_0(\hat{x}) \in \mathscr{A}$. Clearly, as a map of $\mathbf{H}\backslash\mathscr{A}$, w_0 can be lifted (through ϱ') to a map $\omega_0 \in Q_0(\dot{G})$. Hence $\varphi^*([w])$ can be defined as the mapping class on S represented by the projection of ω_0 under ϱ'.

Alternatively, let $T(\widetilde{S})$ denote the Teichmüller space of $\widetilde{S}$ and let $F(\widetilde{S})$ be the fiber space over $T(\widetilde{S})$ (see [1] for the definitions of $T(\widetilde{S})$ and $F(\widetilde{S})$). By Theorem 9 of [1], there is a biholomorphic map $\varphi : F(\widetilde{S}) \to T(S)$ that respects the forgetting map of $T(S)$ onto $T(\widetilde{S})$. By Theorem 1 of [1], $Q(G)/\sim$ is considered a group of holomorphic automorphisms of $F(\widetilde{S})$. Via φ it thus defines a group of holomorphic automorphisms of $T(S)$, which is, by Royden's theorem [9], identified with the group Mod_S^x.

Note that G is centerless, for any $g, g' \in G$, $g = g'$ if and only if $ghg^{-1} = g'h(g')^{-1}$ for all $h \in G$. We see that G can be regarded as a normal subgroup of $Q(G)/\sim$. Thus, φ^* restricts to an isomorphism of G onto $\varphi^*(G) \subset \mathrm{Mod}_S^x$. Write $[w]^* = \varphi^*([w])$ for $[w] \in Q(G)/\sim$ and $g^* = \varphi^*(g)$ for $g \in G$.

We now construct some special elements of $Q(G)\backslash G$. Let $\mathscr{N}$ be the disjoint union of small crescent neighborhoods of all geodesics in $\{\varrho^{-1}(\widetilde{u})\}$. Let $\widetilde{\mathscr{N}} = \varrho(\mathscr{N})$ and $t_{\widetilde{u}}$ the positive Dehn twist along $\widetilde{u}$ which is supported in $\widetilde{\mathscr{N}}$. Let $\tau : \mathbf{H} \to \mathbf{H}$ be a lift of $t_{\widetilde{u}}$. That is, τ satisfies the two conditions (i) $\tau G \tau^{-1} = G$ and (ii) $\varrho\tau = t_{\widetilde{u}}\varrho$. One can easily verify that $\tau \in Q(G)$ and thus that $[\tau]^* \in \mathrm{Mod}_S^x$.

Fix $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$. Let $\{\varrho^{-1}(\widetilde{u})\}$ denote the collection of all (disjoint) geodesics $\hat{u} \subset \mathbf{H}$ such that $\varrho(\hat{u}) = \widetilde{u}$. Denote by $\mathscr{R}_{\widetilde{u}}$ the collection of all components of $\mathbf{H}\backslash\{\varrho^{-1}(\widetilde{u})\}$ and by $F_{\widetilde{u}}$ the subset of $\mathcal{C}_0(S)$ that projects to $\widetilde{u}$ under the projection (1.1). For any $\Omega \in \mathscr{R}_{\widetilde{u}}$, there exists a unique hyperbolic element $g \in G$ such that $g^{-1}\tau|_{\Omega\backslash\mathscr{N}} = \mathrm{id}$ (see [13] for more explanations). Let $\tau_\Omega = g^{-1}\tau$. Then $\tau_\Omega \in Q(G)$ and satisfies conditions (i) and (ii) above, which tells us that $[\tau_\Omega]^* \in \mathrm{Mod}_S^x$.

The complement of Ω in $\mathbf{H}$ consists of infinitely many half-planes, which are invariant half-planes under the action of τ and are called *maximal* (or *first order*) *elements* determined by τ. Each maximal element contains infinitely many second order half-planes which are not invariant by the action of τ, and so on (the actions of τ on higher order half-planes were investigated in [11]). Let $\mathscr{U}$ be the collection of all these half-planes of different orders. Then $\mathscr{U}$ is partially ordered defined by inclusion. That is, for any $n \geq 1$ and any element Δ_{n+1} of $\mathscr{U}$ with $(n+1)$th order, there is a unique element $\Delta_n \in \mathscr{U}$ with nth order, such that $\Delta_{n+1} \subset \Delta_n$.

By Lemma 3.2 of [13], $[\tau_\Omega]^* = t_u$ for a $u \in F_{\widetilde{u}}$, where t_u denotes the positive Dehn twist along u on S. Conversely, for every $u \in F_{\widetilde{u}}$, there is a component $\Omega \in \mathscr{R}_{\widetilde{u}}$ such that $[\tau_\Omega]^* = t_u$. In what follows, we write $\tau = \tau_\Omega$ and call the *triple* $(\tau, \Omega, \mathscr{U})$ the configuration corresponding to the vertex $u \in F_{\widetilde{u}}$. It is clear that for each $h \in G$, $\tau_{h(\Omega)} = h(g^{-1}\tau)h^{-1}$. Hence $[\tau_{h(\Omega)}]^* = [h(g^{-1}\tau)h^{-1}]^* = h^*[\tau_\Omega]^*(h^*)^{-1} = t_{h^*(u)}$. We thus obtain a map

$$\chi : \mathscr{R}_{\widetilde{u}} \to F_{\widetilde{u}} \tag{3.1}$$

defined by sending Ω to u. It is not difficult to prove that χ is surjective. To see that χ is also injective, we note that for any two regions $\Omega, \Omega' \in \mathscr{R}_{\widetilde{u}}$ with $\Omega \neq \Omega'$, they are either adjacent, by which we mean that Ω and Ω' share a common geodesic boundary e for $e \in \{\varrho^{-1}(\widetilde{u})\}$, or Ω and Ω' are disjoint. If Ω and Ω' are adjacent, then by Lemma 2.1 of [17],

$d_{\mathcal{C}}(\chi(\Omega_1), \chi(\Omega_2)) = 1$, which occurs if and only if $\{\chi(\Omega_1), \chi(\Omega_2)\}$ are boundary components of an x-punctured cylinder on S. If Ω and Ω' are disjoint, then there are maximal elements $\Delta \in \mathscr{U}$ and $\Delta' \in \mathscr{U}'$ such that $\Delta \bigcup \Delta' = \mathbf{H}$ and $\partial\Delta \bigcap \partial\Delta' = \varnothing$. By examining the action of τ and τ' on $\mathbf{S}^1 = \partial\mathbf{H}$, we conclude that $\tau^n\tau'^m \neq \tau'^m\tau^n$ for all large integers m, n. See Lemma 4 of [11] for more details. This implies that $t^n_{\chi(\Omega)}t^m_{\chi(\Omega')} \neq t^m_{\chi(\Omega')}t^n_{\chi(\Omega)}$. In particular, $\chi(\Omega') \neq \chi(\Omega)$.

We have thus proved the following lemma:

Lemma 3.1. *The map* χ *defined as* (3.1) *is a bijection which satisfies the equivariant condition* $\chi(g(\Omega)) = g^*(\chi(\Omega))$ *for each* $g \in G$ *and each* $\Omega \in \mathscr{R}_{\widetilde{u}}$.

Remark. The bijection (3.1) can also be obtained from topological terms by Theorem 7.1 of Kent et al. [6]. In fact, Theorem 4.1 and Theorem 4.2 of Birman [2] state that there exists an exact sequence

$$1 \to \pi_1(\widetilde{S}, x) \to \mathrm{Mod}_S^x \to \mathrm{Mod}(\widetilde{S}) \to 1, \tag{3.2}$$

where $\mathrm{Mod}(\widetilde{S})$ is the mapping class group and $\pi_1(\widetilde{S}, x)$ is the fundamental group of $\widetilde{S}$. Thus, there defines an injective map of $\pi_1(\widetilde{S}, x)$ into Mod_S^x. Note that an isomorphism between G and $\pi_1(\widetilde{S}, x)$ is obtained by choosing a lift $\hat{x}$ of x in $\mathbf{H}$ to serve as the base point. As such, we obtain an injective map $\psi : G \to \mathrm{Mod}_S^x$ so that the image $\psi(G)$ consists of mapping classes projecting to the trivial mapping class on $\widetilde{S}$ as x is filled in. Since G keeps $\varrho^{-1}\{\widetilde{u}\}$ invariant, G naturally acts on $\mathscr{R}_{\widetilde{u}}$. Hence $\psi(G)$ acts on $F_{\widetilde{u}}$. As such, the bijection χ in Lemma 3.1 can be given by identifying a region Ω that is the stabilizer of an $h \in G$ with a vertex $u \in F_{\widetilde{u}}$ that is the stabilizer of $\psi(h)$.

By a similar discussion to the above, we know that $\mathscr{R}_{\tilde{u}}$ naturally inherits a structure of a tree. From the bijection (3.1), we know that $F_{\tilde{u}}$ is also a tree. Hence $F_{\tilde{u}}$ is path connected. We formulate the result as the following lemma:

Lemma 3.2. *For each* $\tilde{u} \in \mathcal{C}_0(\widetilde{S})$, *the fiber* $F_{\tilde{u}}$ *is path connected in* $F_{\tilde{u}}$. *Moreover, for any* $u, v \in F_{\tilde{u}}$, *there is one and only one path in* $F_{\tilde{u}}$ *connecting u and v.*

For $\Omega, \Omega' \in \mathscr{R}_{\tilde{u}}$, we can introduce the distance $D(\Omega, \Omega')$ between Ω and Ω' as follows. If there are elements $\Omega_1, ..., \Omega_{n-1} \in \mathscr{R}_{\tilde{u}}$ which are obtained from Lemma 3.2, we then declare $D(\Omega, \Omega') = n$.

Lemma 3.3. *We have* $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) \leq D(\Omega, \Omega')$, *and if* $D(\Omega, \Omega') = 2$, *then* $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) = 2$.

Proof. From Lemma 3.2, there is a sequence $\Omega_0 = \Omega$, $\Omega_1, ..., \Omega_n = \Omega'$ in $\mathscr{R}_{\tilde{u}}$ such that for $0 \leq i \leq n-1$, Ω_i is adjacent to Ω_{i+1}. By Lemma 2.1 of [17], we get $d_{\mathcal{C}}(\chi(\Omega_i), \chi(\Omega_{i+1})) = 1$. It follows from the triangle inequality that $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) \leq \sum_{i=0}^{n-1} d_{\mathcal{C}}(\chi(\Omega_i), \chi(\Omega_{i+1})) = n$. Hence $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) \leq D(\Omega, \Omega')$.

If $D(\Omega, \Omega') = 2$, then $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) \leq 2$. Assume that $d_{\mathcal{C}}(\chi(\Omega), \chi(\Omega')) = 1$, which says that $\chi(\Omega)$ and $\chi(\Omega')$ are *disjoint*. Thus, $\{\chi(\Omega), \chi(\Omega')\}$ are boundary components of an x-punctured cylinder on S. By Lemma 2.1 of [17], Ω and Ω' are adjacent. But this means that $D(\Omega, \Omega') = 1$. This is a contradiction. □

4. Distances between Vertices in Fibers in the Curve Complex

For each $\tilde{u} \in \mathcal{C}_0(\widetilde{S})$, the group G naturally acts on $\mathscr{R}_{\tilde{u}}$. Let $g \in G$ be an essential hyperbolic element, i.e., $g^* \in \mathscr{F}$. Let $\text{axis}(g)$ be the axis of g; that is, $\text{axis}(g) \subset \mathbf{H}$ is the unique geodesic so that $g(\text{axis}(g)) = \text{axis}(g)$. Let $\Omega \in \mathscr{R}_{\tilde{u}}$ and denote by $u = \chi(\Omega)$. We first investigate the situation where $\text{axis}(g) \cap \Omega = \varnothing$. We have

Lemma 4.1. *Suppose that* $\Omega \cap \mathit{axis}(g) = \varnothing$. *Then* $d_{\mathcal{C}}(u, g^*(u)) \geq 3$.

Proof. The proof can be found in Section 4 of [14]. $\square$

Observe that for any $\Omega \in \mathscr{R}_{\tilde{u}}$, there always exists an essential hyperbolic element g (and hence there are infinitely many such elements) in the conjugacy class of g for which $\text{axis}(g) \cap \Omega \neq \varnothing$, and conversely, for any essential hyperbolic element $g \in G$, there are infinitely many $\Omega \in \mathscr{R}_{\tilde{u}}$ such that $\text{axis}(g) \cap \Omega \neq \varnothing$. We now proceed to explore the relationship between $D(\Omega, g(\Omega))$ and the intersection number between $\text{axis}(g)$ and $\tilde{u}$.

Lemma 4.2. *Let* $g \in G$ *be an essential hyperbolic element*, *and let* $\Omega \subset \mathscr{R}_{\tilde{u}}$ *be a region so that* $\mathit{axis}(g) \cap \Omega \neq \varnothing$. *Then* $i(\tilde{u}, \varrho(\mathit{axis}(g))) = D(\Omega, g(\Omega))$.

Proof. Write $n = i(\tilde{u}, \varrho(\text{axis}(g)))$. Let $\Delta \in \mathscr{U}_u$ be the maximal element that covers the repelling fixed point of g. By Lemma 2.1 of [14], there is a maximal element $\Delta_1 \in \mathscr{U}_u$ that contains $g(\mathbf{H} \backslash \Delta)$. Δ_1 is disjoint from Δ and covers the attracting fixed point of g. Thus, $\Omega \subset \mathbf{H} \backslash (\Delta \cup \Delta_1)$. Parametrize $c(t) = \text{axis}(g)$, $-\infty < t < +\infty$, so that $c(t) \cap \partial\Delta = c(t_0)$. Also, write $z = c(t_0)$. Let $c(t_1) = c(t) \cap \partial\Delta_1$. Then $c(t)$ continues to travel and enters a second-order element $\Delta_2 \in \mathscr{U}_u$. Set $c(t_2) = c(t) \cap \partial\Delta_2$, and so on.

When $c(t)$ reaches the point $g(z)$ at time t_n, $c(t_n) = g(z)$ lies in $\partial\Delta_n$ for an nth order element $\Delta_n \in \mathscr{U}_u$. We thus obtain a finite sequence $-\infty < t_0 < t_1 < t_2 < \cdots < t_n < +\infty$, and see that $c(t)$, $t \in [t_0, t_n]$ meets n regions of $\mathscr{R}_{\tilde{u}}$. It is easy to check that $g(\Omega)$ lies in $\Delta_n \in \mathscr{U}_u$. It follows that $n = D(\Omega, g(\Omega))$, as claimed. $\square$

Consider the case in which $d_{\mathcal{C}}(u, g^*(u)) = 1$.

Lemma 4.3. *Let $g \in G$ be an essential hyperbolic element. Let $\tilde{u} \in \mathcal{C}_0(\tilde{S})$. Then $i(\tilde{u}, \varrho(axis(g))) = 1$ if and only if there is $u \in F_{\tilde{u}}$ such that $d_{\mathcal{C}}(u, g^*(u)) = 1$, in which case $u = \chi(\Omega)$ for an $\Omega \in \mathscr{R}_{\tilde{u}}$ with $\Omega \cap axis(g) \neq \emptyset$.*

Proof. If $i(\tilde{u}, \varrho(\text{axis}(g))) = 1$, then by Lemma 4.2, $D(\Omega, g(\Omega)) = 1$ for an $\Omega \subset \mathscr{R}_{\tilde{u}}$ with $\Omega \cap \text{axis}(g) \neq \emptyset$. This means that Ω and $g(\Omega)$ are adjacent. By Lemma 2.1 of [17], $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) = 1$. Therefore, if we let $u = \chi(\Omega)$, then

$$d_{\mathcal{C}}(u, f(u)) = d_{\mathcal{C}}(\chi(\Omega), g^*(\chi(\Omega))) = d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) = 1.$$

Conversely, suppose $d_{\mathcal{C}}(u, f(u)) = 1$. Let $\Omega \subset \mathscr{R}_{\tilde{u}}$ be such that $u = \chi(\Omega)$. By Lemma 3.1, we have $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) = d_{\mathcal{C}}(\chi(\Omega), g^*(\chi(\Omega))) = 1$. By Lemma 2.1 of [17] again, Ω and $g(\Omega)$ are adjacent. If $\Omega \cap \text{axis}(g) = \emptyset$, then by Lemma 4.1, $d_{\mathcal{C}}(\chi(\Omega), g^*(\chi(\Omega))) \geq 3$, which leads to a contradiction. We thus conclude that $\Omega \cap \text{axis}(g) \neq \emptyset$. By Lemma 4.2, $i(\tilde{u}, \varrho(\text{axis}(g))) = 1$. $\square$

The case where $d_{\mathcal{C}}(u, f(u)) = 2$ is more involved. Let $c = \varrho(\text{axis}(g))$. If $i(\tilde{u}, c) = 1$, then by Lemma 4.1 and Lemma 4.3, either $d_{\mathcal{C}}(u, f(u)) \geq 3$ or $d_{\mathcal{C}}(u, f(u)) = 1$. So it must be the case that $i(\tilde{u}, c) \geq 2$. It is worthwhile pointing out that the number $i(\tilde{u}, c)$ can be arbitrarily large. In fact, one can easily construct a geodesic $c \in \mathscr{S} \backslash \mathscr{S}^*$ on a genus p surface for $p > 1$ such that a vertex $\widetilde{v} \in \mathcal{C}_0(\widetilde{S})$ can be found with the property that $i(c, \widetilde{v}) = 1$ and $\widetilde{v}$ is disjoint from $\widetilde{u}$. Let $f \in \mathscr{F}$ be defined by c by a point-pushing deformation, let $g \in G$ be such that $g^* = f$. In this case, $d_{\mathcal{C}}(u, f(u)) = 2$ for some $f \in K(c)$ and some $u \in F_{\widetilde{u}}$ (the author thanks the referee for his comments).

Figure 1 depicts such a filling curve c which intersects $\widetilde{u}$ and $\widetilde{v}$, and goes around a hole as many times as possible. This illustrates that $i(\widetilde{u}, c) = i(\widetilde{u}, \varrho(\text{axis}(g)))$ can be arbitrarily large, whereas $d_{\mathcal{C}}(u, f(u)) = 2$ for $u = \chi(\Omega)$, where $\Omega \in \mathscr{R}_{\widetilde{u}}$ with $\Omega \cap \text{axis}(g) \neq \emptyset$. In particular, $i(\widetilde{u}, c) - d_{\mathcal{C}}(u, f(u))$ could be unbounded.

Note that it could be the case where both $i(\widetilde{u}, c)$ and $d_{\mathcal{C}}(u, f(u))$ are larger than two even if $\Omega \cap \text{axis}(g) \neq \emptyset$. However, if $i(\widetilde{u}, c) = 2$, then Lemma 4.1 tells us that $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) \geq 3$ for all $\Omega \subset \mathscr{R}_{\widetilde{u}}$ with $\Omega \cap \text{axis}(g) = \emptyset$. But if Ω is so chosen that $\Omega \cap \text{axis}(g) \neq \emptyset$, then by Lemma 4.2 and Lemma 3.3, $d_{\mathcal{C}}(u, f(u)) = 2$, where $u = \chi(\Omega)$.

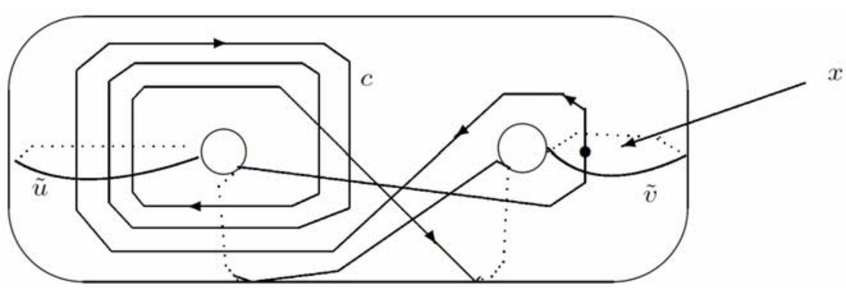


Figure 1

Recall that $(\tau, \Omega, \mathscr{U})$ is the configuration corresponding to u. For other vertices $v \in \hat{\mathcal{C}}_0(S)$, we let $(\tau_v, \Omega_v, \mathscr{U}_v)$ denote the configuration corresponding to v. It is easy to verify that $g(\Omega) = \Omega_w$ if $(\tau_w, \Omega_w, \mathscr{U}_w)$ is the configuration corresponding to $w = g^*(u)$.

Assume that $\Omega \bigcap \operatorname{axis}(g) \neq \emptyset$. By Lemma 2.1 of [14], there is a pair (Δ, Δ') of maximal elements of $\mathscr{U}_u$ such that Δ covers the repelling fixed point and Δ' covers the attracting fixed point of g. In addition, we have $\Omega \subset \mathbf{H} \backslash (\Delta \bigcup \Delta')$ and $\overline{\Omega} \bigcap g(\overline{\Omega}) = \emptyset$. Without loss of generality, we may assume that $g(\Delta) \bigcap \Delta' \neq \emptyset$.

The following lemma will be used frequently in the proof of Theorem 2.1 and Theorem 2.2.

Lemma 4.4. *With the above notations and conditions, we have* $d_{\mathcal{C}}(v, f(u)) \geq 2$ *provided that there is a maximal element* $\Delta_v \in \mathscr{U}_v$ *such that* $\mathbf{H} \backslash \Delta_v \subset g(\Delta)$. *Similarly,* $d_{\mathcal{C}}(v, u) \geq 2$ *if there is a maximal element* $\Delta_v \in \mathscr{U}_v$ *such that* $\mathbf{H} \backslash \Delta_v \subset \Delta'$. *In particular, if* v *is disjoint from both* u *and* $f(u)$, *then no maximal element of* $\mathscr{U}_v$ *exists whose boundary lies entirely in* $g(\Delta) \bigcap \Delta'$.

Proof. We outline the argument here for completeness. See [14] for more details. If $d_{\mathcal{C}}(v, w) = 1$, where $w = f(u)$, then $t_w(v) = v$. So for any maximal element $\Delta'_v \in \mathscr{U}_v$, $t_w^s(\Delta'_v) \in \mathscr{U}_v$ is also a maximal element for any integer $s \neq 0$. On the other hand, by the assumption, there is a maximal element $\Delta_v \in \mathscr{U}_v$ such that $\mathbf{H} \backslash \Delta_v \subset g(\Delta)$. By examining the action of τ_w on $\mathbf{H}$, we see that $\tau_w^s(\mathbf{H} \backslash \Delta_v)$ is disjoint from $\mathbf{H} \backslash \Delta_v$ for any $s \neq 0$. It follows that $\tau_w^s(\Delta_v) \bigcap \Delta_v \neq \emptyset$ and $\Delta_v \neq \tau_w^s(\Delta_v)$. We thus conclude that $\tau_w^s(\Delta_v)$ is not a maximal element of $\mathscr{U}_v$. This is a contradiction. Similarly, we can prove the other two statements. $\square$

Alternatively, the condition that there is a maximal element $\Delta_v \in \mathscr{U}_v$ such that $\mathbf{H} \backslash \Delta_v \subset g(\Delta)$ implies that $\overline{\Omega}_v \bigcap \overline{\Omega}_w = \emptyset$, which says that Ω_v and Ω_w are *disjoint* but not adjacent. The result is stated as follows for future references.

Lemma 4.5. *Let* $u, v \in \hat{\mathcal{C}}_0(S)$ *be such that* $d_{\mathcal{C}}(u, v) = 1.$ *Let* $(\tau_u, \Omega_u, \mathscr{U}_u)$ *and* $(\tau_v, \Omega_v, \mathscr{U}_v)$ *be the configurations corresponding to u and v, respectively. Then* $\overline{\Omega}_u$ *and* $\overline{\Omega}_v$ *cannot be disjoint.*

Let $u' \in \mathcal{C}'_0(S)$. Then $\widetilde{u}'$ is trivial. By Lemma 5.1 and Lemma 5.2 of [15], u' corresponds to a parabolic fixed point z' of G; that is, there is a primitive parabolic element $T \in G$ such that $T(z') = z'$. If $d_{\mathcal{C}}(u, u') = 1$, then $t_u t_{u'} = t_{u'} t_u$. Thus, $\tau_u T = T\tau_u$, which in turn implies that τ_u fixes z', and thus that $z' \in \Omega_u \cap \mathbf{S}^1$. The converse remains valid and the result can be stated in the following lemma (see [15] for more detailed argument):

Lemma 4.6. *Let* $u \in \hat{\mathcal{C}}_0(S)$ *and* $u' \in \mathcal{C}'_0(S)$. *Let* $z' \in \mathbf{S}^1$ *be the parabolic fixed point of G corresponding to* u'. *Then* $d_{\mathcal{C}}(u, u') = 1$ *if and only if* $z' \in \Omega_u \cap \mathbf{S}^1$.

Return to the case in which $\Omega = \Omega_u$ and $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) = 2$ but $D(\Omega, g(\Omega))$ is arbitrary (where we recall that $u = \chi(\Omega)$, and $g^* = f$).

Lemma 4.7. *Let* $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$ *be a non-trivial vertex, and let* $g \in G$ *be an essential hyperbolic element so that* $i(\widetilde{u}, \varrho(axis(g))) \geq 2$. *Suppose that* $\Omega \in \mathscr{R}_{\widetilde{u}}$ *is such that* $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) = 2$. *Then* $V(\chi(\Omega), g^*) \cap \mathcal{C}'_0(S) = \emptyset$. *Furthermore, we have* $\varepsilon(V(\chi(\Omega), g^*)) \subset W(\widetilde{u}, \varrho(axis(g)))$, *where* ε *is the projection given in* (1.1).

Proof. If $\Omega \cap \text{axis}(g) = \emptyset$, then by Lemma 4.1, we see that $d_{\mathcal{C}}(\chi(\Omega), \chi(g(\Omega))) \geq 3$. This contradicts the hypothesis. So it must be the case that $\Omega \cap \text{axis}(g) \neq \emptyset$. Write $u = \chi(\Omega)$. By Lemma 3.1, $\chi(g(\Omega)) = g^*(\chi(\Omega)) = g^*(u) = f(u)$. Since $d_{\mathcal{C}}(u, f(u)) = 2$, $(u, f(u))$ does not fill S. Choose $v \in V(u, f)$, which means that $d_{\mathcal{C}}(v, u) = d_{\mathcal{C}}(v, f(u)) = 1$.

If $\tilde{v} = \varepsilon(v)$ is trivial, i.e., $v \in \mathcal{C}'_0(S)$, then by Lemma 4.6, $z \in (\Omega \cap g(\Omega)) \cap \mathbf{S}^1$. This is impossible since $\overline{\Omega} \cap g(\overline{\Omega}) = \varnothing$. This proves $V(\chi(\Omega), g^*) \cap \mathcal{C}'_0(S) = \varnothing$.

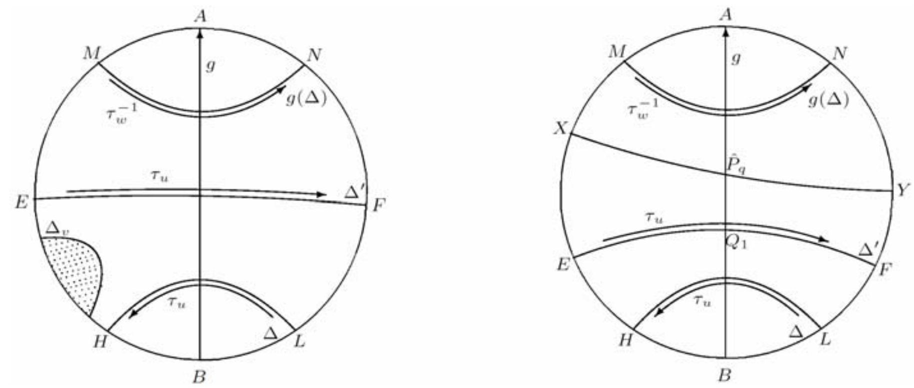
We thus assume that $v \in \hat{\mathcal{C}}_0(S)$. Then $\tilde{v} \in \mathcal{C}_0(\tilde{S})$. Suppose that $\tilde{v}$ is not in $W(\tilde{u}, \varrho(\text{axis}(g)))$. By Lemma 4.4, we need to find a maximal element $\Delta_v \in \mathscr{U}_v$ such that $\mathbf{H} \backslash \Delta_v \subset \Delta'$ or $\mathbf{H} \backslash \Delta_v \subset g(\Delta)$. To do so, we first notice that $g \in G$ is an essential hyperbolic element. As a result, $\text{axis}(g)$ cannot lie in Ω_v. There remain two cases:

Case 1. There is a maximal element $\Delta_v \in \mathscr{U}_v$ such that $\text{axis}(g) \subset \Delta_v$. Since $\tilde{u}$ is disjoint from $\tilde{v}$ and since $\partial\Delta_v \in \{\varrho^{-1}(\tilde{v})\}$, we see that $\partial\Delta_v$ is disjoint from all geodesics drawn in Figure 2(a).

Notice that $\partial\Delta', \partial\Delta, \partial g(\Delta) \in \{\varrho^{-1}(\tilde{u})\}$, so they are all mutually disjoint. As it turns out, $(\mathbf{H} \backslash \Delta_v) \cap \mathbf{S}^1$ lies in one of the eight subarcs of $\mathbf{S}^1 \backslash \{A, B, E, F, H, L, M, N\}$. A possible Δ_v is drawn in Figure 2(a). As we see, $\mathbf{H} \backslash \Delta_v \subset g(\Delta)$ or $\mathbf{H} \backslash \Delta_v \subset \Delta'_v$, and we are done.

Case 2. $\text{axis}(g) \cap \Omega_v \neq \varnothing$. There is a pair (Δ_v, Δ'_v) of maximal elements of $\mathscr{U}_v$, where Δ_v covers the repelling fixed point of g and Δ'_v covers the attracting fixed point of g, so that $\Omega_v \subset \mathbf{H} \backslash (\Delta_v \cup \Delta'_v)$.

By hypothesis, $\tilde{v}$ is non-trivial and does not lie in $W(\tilde{u}, \varrho(\text{axis}(g)))$. Write $\#\{c(\tau), \tilde{v}\} = \{P_1, ..., P_r\}$. There is a parametrization $c = c(\tau)$ and an intersection point $P_q \in \{P_1, ..., P_r\}$, $P_q = c(\tau_q^1)$, such that $\tau_1^2 < \tau_q^1$, where $c(\tau_1^2)$ is the first intersection point in $\#\{c(\tau), \tilde{u}\}$. P_q corresponds to the intersection point $\hat{P}_q$ shown in Figure 2(b).



(a) (b)

Figure 2

Since u, v are disjoint, $\widetilde{u}, \widetilde{v}$ are also disjoint, which implies that $\{\varrho^{-1}(\widetilde{u})\}$ and $\{\varrho^{-1}(\widetilde{v})\}$ are disjoint. But $\partial\Delta, \partial\Delta', \partial g(\Delta) \in \{\varrho^{-1}(\widetilde{u})\}$. We conclude that there is one geodesic $[XY] \in \{\varrho^{-1}(\widetilde{v})\}$ that passes through $\hat{P}_q$ and lies entirely in the region $g(\Delta) \cap \Delta'$.

If $\partial\Delta_v \subset \mathbf{H}\backslash g(\Delta)$, then $\mathbf{H}\backslash\Delta_v \subset \Delta'$. If $\partial\Delta_v \subset g(\Delta) \cap \Delta'$ (note that $\partial\Delta_v$ may or may not be equal to $[XY]$), then also $\mathbf{H}\backslash\Delta_v \subset \Delta'$. We are done. If $\partial\Delta_v \subset \mathbf{H}\backslash(\Delta \cup \Delta')$, then since $[XY] \subset g(\Delta) \cap \Delta'$, Δ'_v must contain $[XY]$, which tells us that $\mathbf{H}\backslash\Delta'_v \subset g(\Delta)$. Finally if $\partial\Delta_v \subset \Delta$, then Δ'_v contains $[XY]$. It follows that $\mathbf{H}\backslash\Delta'_v \subset g(\Delta)$.

It remains to consider the case where $\partial\Delta_v \in \{\partial\Delta, \partial\Delta', \partial g(\Delta)\}$. If $\partial\Delta_v = \partial\Delta$, then $t_v = t_u$. Thus, $u = v$. If $\partial\Delta_v = \partial g(\Delta)$, then $t_w = t_v$. So $w = v$. If $\partial\Delta_v = \partial\Delta'$, then Δ'_v contains $[XY]$. It follows that $\mathbf{H}\backslash\Delta'_v \subset g(\Delta)$. Geometrically, in this case, $v = u_0$, where u_0 together with u forms the boundary of an x-punctured cylinder. $\square$

Proof of Theorem 2.1. By Lemma 4.7, we see that the restriction $\varepsilon|_{V(u, f)}$ sends $V(u, f)$ to $W(\widetilde{u}, c)$. We first claim that $\varepsilon|_{V(u, f)}$ is onto.

Indeed, pick $\widetilde{v} \in W(\widetilde{u}, c)$, and let $c = c(\tau)$ be a parametrization for c such that

$$\tau_i^1 < \tau_j^2 \text{ for all } 1 \leq i \leq r \text{ and } 1 \leq j \leq k, \tag{4.1}$$

where $\#\{\widetilde{v}, c\} = \{c(\tau_1^1), \ldots, c(\tau_r^1)\}$ and $\#\{\widetilde{u}, c\} = \{c(\tau_1^2), \ldots, c(\tau_k^2)\}$. Take $x = c(0)$ and move x along $c(\tau)$. After meeting $\widetilde{v}$ at $P_1, \ldots, P_r$ in the order, c starts meeting $\widetilde{u}$ at order $Q_1, \ldots, Q_k$ and never meets $\widetilde{v}$ again before arriving at x because of (4.1). It turns out that v is disjoint from $g^*(u)$ as well (where v is obtained from $\widetilde{v}$ by deleting x). We see that v is disjoint from u and $f(u)$. By the assumption, $d_{\mathcal{C}}(u, f(u)) = 2$ and $d_{\mathcal{C}}(u, v) = d_{\mathcal{C}}(f(u), v) = 1$. By the triangle inequality, we obtain

$$2 = d_{\mathcal{C}}(u, f(u)) \leq d_{\mathcal{C}}(u, v) + d_{\mathcal{C}}(v, f(u)) = 2.$$

It follows that $d_{\mathcal{C}}(u, f(u)) = d_{\mathcal{C}}(u, v) + d_{\mathcal{C}}(v, f(u))$. Hence $v \in V(u, f)$ and $\varepsilon(v) = \widetilde{v}$. This proves that $\varepsilon|_{V(u, f)}$ is onto.

We next claim that $\varepsilon|_{V(u, f)}$ is one-to-one. Suppose there are $v_1, v_2 \in V(u, f)$ so that $\varepsilon(v_1) = \varepsilon(v_2)$. Let $\widetilde{v} = \varepsilon(v_i)$, $i = 1, 2$. Let $(\tau_i, \Omega_i, \mathscr{U}_i)$ be the configurations corresponding to v_i.

If, say, for v_1 we have $\Omega_1 \cap \text{axis}(g) = \varnothing$, then by Lemma 4.1, $d_{\mathcal{C}}(v_1, f(v_1)) \geq 3$. Since $v_1 \in V(u, f)$, we have $d_{\mathcal{C}}(u, v_1) = d_{\mathcal{C}}(f(u), v_1) = 1$. Hence the triangle inequality yields that

$$\begin{aligned} 2 &= d_{\mathcal{C}}(u, f(u)) = d_{\mathcal{C}}(u, v_1) + d_{\mathcal{C}}(v_1, f(u)) \\ &= d_{\mathcal{C}}(f(u), f(v_1)) + d_{\mathcal{C}}(v_1, f(u)) \geq d_{\mathcal{C}}(v_1, f(v_1)) \geq 3. \end{aligned}$$

This is a contradiction. Similarly, we assert that $\Omega_2 \cap \text{axis}(g) \neq \varnothing$. So we only need to handle the case in which $\Omega_i \cap \text{axis}(g) \neq \varnothing$. Since $\varepsilon(v_1) = \varepsilon(v_2)$, $v_1, v_2 \in F_{\widetilde{v}}$. From the bijection (3.1), we see that $\Omega_1, \Omega_2 \in \mathscr{R}_{\widetilde{v}}$. As an immediate consequence, either Ω_1, Ω_2 are disjoint, or Ω_1, Ω_2 are adjacent, or $\Omega_1 = \Omega_2$.

The condition $v_1 \in V(u, f)$ says that v_1 is disjoint from u and w, where $w = f(u)$. By the argument of Lemma 4.4, $\tau_u^r \tau_w^{-s}$ sends every maximal element of $\mathscr{U}_1$ to a maximal element. In particular, for the maximal element $\Delta_{v_1} \in \mathscr{U}_1$ that contains the repelling fixed point of g and $\text{axis}(g) \cap \partial\Delta_{v_1} \neq \emptyset$, it cannot occur that $\mathbf{H} \backslash \Delta_{v_1} \subset g(\Delta)$ or $\mathbf{H} \backslash \Delta_{v_1} \subset \Delta'$. This further implies that $\overline{\Omega}_1$ contains $\text{axis}(g) \cap (g(\Delta) \cap \Delta')$ (see Figure 2(b)). Similarly, the condition $v_2 \in V(u, f)$ leads to that $\overline{\Omega}_2$ also contains $\text{axis}(g) \cap (g(\Delta) \cap \Delta')$. Hence $\overline{\Omega}_1 \cap \overline{\Omega}_2$ contains $\text{axis}(g) \cap (g(\Delta) \cap \Delta')$. In particular, $\overline{\Omega}_1 \cap \overline{\Omega}_2 \neq \emptyset$. This tells us that $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are not disjoint. If they are adjacent, $\overline{\Omega}_1 \cap \overline{\Omega}_2$ is the axis of a simple hyperbolic element of G, which coincides with $\text{axis}(g)$. This implies that g is simple, which is clearly a contradiction. So we must have $\Omega_1 = \Omega_2$, which means $\chi(\Omega_1) = \chi(\Omega_2)$. That is, $v_1 = v_2$. This shows that $\varepsilon|_{V(u, f)}$ is one-to-one, and hence the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Suppose $d_{\mathcal{C}}(u, f(u)) \leq 2$ for some $u \in F_{\tilde{u}}$. If $d_{\mathcal{C}}(u, f(u)) = 1$, then $\chi(\Omega) = u$ and Ω satisfies $\Omega \cap \text{axis}(g) \neq \emptyset$. By Lemma 4.3, $i(\tilde{u}, c) = 1$. In particular, c cannot be expressed as a concatenation of two filling curves c_1 and c_2. Assume that $d_{\mathcal{C}}(u, f(u)) = 2$. Then by Lemma 4.3 and Lemma 4.1, $i(\tilde{u}, c) \geq 2$ (as discussed in Section 3, the converse is not true, but $i(\tilde{u}, c) = 2$ does imply that $d_{\mathcal{C}}(u, f(u)) = 2$ for $u = \chi(\Omega) \in F_{\tilde{u}}$, where $\Omega \cap \text{axis}(g) \neq \emptyset$).

From Theorem 2.1, we know that $V(u, f) \cap \mathcal{C}_0'(S) = \emptyset$ and $\varepsilon(V(u, f)) = W(\tilde{u}, c)$. Choose $v \in V(u, f)$ and write $\tilde{v} = \varepsilon(v)$. Since u, v are disjoint, $\tilde{u}$ and $\tilde{v}$ are also disjoint and by Theorem 2.1 again, they are separated by c, which means that with the aid of a parametrization $c = c(\tau)$, $\tau \in [0, 1]$, all points Q_j in $\#\{c, \tilde{u}\}$ can be expressed as $Q_j = c(\tau_j^2)$ and all points P_i in $\#\{c, \tilde{v}\}$ can be expressed as $P_i = c(\tau_j^1)$, where $\tau_i^1 < \tau_j^2$ for all $1 \leq i \leq r$ and $1 \leq j \leq k$. Without loss of generality, we assume $\tau_1^2 = \min\{\tau_1^2, ..., \tau_k^2\}$ and $\tau_k^2 = \max\{\tau_1^2, ..., \tau_k^2\}$. Then $c(\tau)$, $\tau_1^2 < \tau < \tau_k^2$, is a path in c connecting $Q_1 = c(\tau_1^2)$ and $Q_k = c(\tau_k^2)$, which is denoted by c_0. We see that $c_1 = c_0 \cdot \beta$ is homotopic to δ_1.

We claim that $\tilde{v}$ is disjoint from δ_1. Since $\tilde{v} \in W(\tilde{u}, c)$, it is clear that c_0 does not meet $\tilde{v}$. But $\tilde{v}$ is disjoint from $\tilde{u}$ and β is a part of $\tilde{u}$. We see that $\tilde{v}$ does not meet β either, which says $\tilde{v}$ does not meet c_1 (because c_1 is the union of β and c_0). This further implies that $\tilde{v}$ is disjoint from δ_1, contradicting that $\delta_1 \in \mathscr{S}$. This proves Theorem 2.2. □

5. Proof of Theorem 2.3

As usual, we let $c \in \mathscr{S}$ be determined by an element $f \in \mathscr{F}$. Let $\tilde{u} \in \mathcal{C}_0(\tilde{S})$ be given an orientation. With respect to $\tilde{u}$, the geodesic c can be expressed as a concatenation $c_1 \cdot c_2$, where c_1 is freely homotopic to a geodesic δ_1. Moreover, c_1 can further be written as the concatenation $c_{11}, ..., c_{1,k-1}$ of curves, where each c_{1i} is freely homotopic to nontrivial geodesics δ_{1i}. Set $\Sigma = \{\delta_{11}, ..., \delta_{1,k-1}\}$.

We prove Theorem 2.3 by establishing the following result:

Theorem 5.1. *If* Σ *contains two filling close geodesics* δ_{1i}, δ_{1j} *for* $1 \leq i, j \leq k-1$ *and* $|i - j| \geq 2$, *then* $d_{\mathcal{C}}(u, f(u)) \geq 4$ *for those* $u = \chi(\Omega) \in F_{\tilde{u}}$ *for which* $\Omega \cap axis(g) \neq \emptyset$.

Proof. Let (Δ, Δ') be the pair of maximal elements of $\mathscr{U}_u$ such that Δ

and Δ' cover the repelling and attracting fixed points of g, respectively. We know that $\Omega \subset \mathbf{H} \backslash (\Delta \cup \Delta')$. If $d_{\mathcal{C}}(u, f(u)) = 1$, then by Lemma 4.3, $i(\tilde{u}, c) = 1$, contradicting that $c = \delta_1 \in \mathscr{S}^*$. If $d_{\mathcal{C}}(u, f(u)) = 2$, then by Theorem 2.1, for any $v \in V(u, f)$, we have $v \notin \mathcal{C}'_0(S)$ and $\varepsilon(v) \in W(\tilde{u}, c)$. This implies that $\varepsilon(v)$ does not intersect δ_1, which says that $\delta_1 \notin \mathscr{S}$. So $\delta_1 \notin \mathscr{S}^*$.

Assume that $d_{\mathcal{C}}(u, f(u)) = 3$, where we write $u = \chi(\Omega)$ for an $\Omega \in \mathscr{R}_{\tilde{u}}$ satisfying $\Omega \cap \text{axis}(g) \neq \emptyset.\Omega$ Let $[u, v_1, v_2, f(u)]$ be a geodesic path in $\mathcal{C}_0(S)$ connecting u and $f(u)$. Note that $v_1, v_2 \in \mathcal{C}'_0(S) \cup \hat{\mathcal{C}}_0(S)$.

Consider the case in which $v_i \in \hat{\mathcal{C}}_0(S)$, $i = 1, 2$. Denote by $(\tau_i, \Omega_i, \mathscr{U}_i)$ the configuration corresponding to v_i. Let $\Delta_i \in \mathscr{U}_i$ be the maximal element that contains $\text{axis}(g)$ if $\Omega_i \cap \text{axis}(g) = \emptyset$ (Figure 2(a) with $\Delta_v = \Delta_i$); and let $\Delta_i, \Delta'_i \in \mathscr{U}_i$ be the pair of maximal elements so that Δ_i covers the repelling fixed point B and Δ'_i covers the attracting fixed point A of g if $\Omega_i \cap \text{axis}(g) \neq \emptyset$ (Figure 2(b) with $\Delta = \Delta_i$ and $\Delta' = \Delta'_i$).

Since $\tilde{v}_i$ and $\tilde{u}$ are simple closed geodesics, all geodesics in $\{\varrho^{-1}(\tilde{v}_i)\}$ are mutually disjoint, and so are the geodesics in $\{\varrho^{-1}(\tilde{u})\}$. In particular, since $\{\partial\Delta, \partial\Delta', \partial g(\Delta)\} \subset \{\varrho^{-1}(\tilde{u})\}$, $\partial\Delta$, $\partial\Delta'$ and $\partial g(\Delta)$ are mutually disjoint. If, in addition, v_i and u are disjoint, then $\{\varrho^{-1}(\tilde{v}_i)\}$ is also disjoint from $\{\varrho^{-1}(\tilde{u})\}$ which particularly implies that $\{\varrho^{-1}(\tilde{v}_i)\}$ is disjoint from $\{\partial\Delta, \partial\Delta', \partial g(\Delta)\}$. We will use these properties in each case below.

There are several possibilities: (1) $v_1, v_2 \in \mathcal{C}'_0(S)$, (2) $v_1 \in \mathcal{C}'_0(S)$ and $v_2 \in \hat{\mathcal{C}}_0(S)$, (3) $v_1 \in \hat{\mathcal{C}}_0(S)$ and $v_2 \in \mathcal{C}'_0(S)$, and (4) $v_1, v_2 \in \hat{\mathcal{C}}_0(S)$.

Case 1. $v_1, v_2 \in \mathcal{C}_0'(S)$. We claim that this case cannot occur, as any two twice punctured disks enclosing x have an overlap near x, as it turns out, v_1, v_2 must intersect, contradicting that $[v_1, v_2]$ is an edge in $\mathcal{C}_1(S)$.

Case 2. $v_1 \in \mathcal{C}_0'(S)$ and $v_2 \in \hat{\mathcal{C}}_0(S)$. By Lemmas 5.1 and 5.2 of [12], v_1 corresponds to a parabolic fixed point z_1 of G, where $z_1 \in \mathbf{S}^1$.

If $\Omega_2 \cap \text{axis}(g) = \emptyset$, then since $d_{\mathcal{C}}(v_2, f(u)) = 1$, either $\Delta_2 \subset g(\Delta)$ or $g(\Delta) \subset \Delta_2$. The former cannot occur, as Δ_2 covers the point A while $g(\Delta)$ does not. Hence we have $g(\Delta) \subset \Delta_2$. Notice that v_1 is disjoint from v_2. By Lemma 4.6, $z_1 \in \Omega_2 \cap \mathbf{S}^1$. This implies that $z_1 \in (\mathbf{H} \backslash \Delta_2) \cap \mathbf{S}^1$. Hence $z_1 \in (\mathbf{H} \backslash g(\Delta)) \cap \mathbf{S}^1$. But $\mathbf{H} \backslash g(\Delta) \subset \Delta'$. Therefore, z_1 is covered by Δ'. From Lemma 4.6, $d_{\mathcal{C}}(v_1, u) \geq 2$, contradicting that $d_{\mathcal{C}}(v_1, u) = 1$.

If $\Omega_2 \cap \text{axis}(g) \neq \emptyset$, then $\Omega_2 \subset \mathbf{H} \backslash (\Delta_2 \cup \Delta_2')$. By the assumption, $d_{\mathcal{C}}(v_2, f(u)) = 1$. Hence either $g(\Delta) \subset \Delta_2$ or $\Delta_2 \subset g(\Delta)$. Suppose that the former occurs. Then $d_{\mathcal{C}}(v_1, v_2) = 1$ implies (from Lemma 4.6) that $z_1 \in \Omega_2 \cap \mathbf{S}^1$. We deduce that z_1 is covered by Δ'. By Lemma 4.6 again, $d_{\mathcal{C}}(v_1, u) \geq 2$. This leads to a contradiction.

Suppose now $\Delta_2 \subset g(\Delta)$. By the assumption, $\delta_1 \in \mathscr{S}^*$. This means that $\{\varrho^{-1}(\tilde{v}_2)\}$ intersects $\hat{c}_0$ at least twice (where and below $\hat{c}_0$ denotes the geodesic segment $\text{axis}(g) \cap g(\Delta) \cap \Delta'$); that is to say, there are geodesics $[X_2 Y_2], [X_2' Y_2'] \in \{\varrho^{-1}(\tilde{v}_2)\}$ that are entirely contained in $g(\Delta) \cap \Delta'$, where $[X_2' Y_2']$ is the one closest to $[MN] = \partial g(\Delta)$. If $\partial \Delta_2 = [MN]$ or $[X_2', Y_2']$, then since $\Omega_2 \subset \mathbf{H} \backslash (\Delta_2 \cup \Delta_2')$ and since $d_{\mathcal{C}}(v_1, v_2) = 1$, by Lemma 4.6, $z_1 \in \Omega_2 \cap \mathbf{S}^1$, and hence $z_1 \in \Delta'$. By Lemma 4.6 again, $d_{\mathcal{C}}(u, v_1) \geq 2$, contradicting the hypothesis. If $\partial \Delta_2 = [X_2 Y_2]$ or any other geodesics of $\{\varrho^{-1}(\tilde{v}_2)\}$ intersecting $\hat{c}_0$, then by Lemma 4.6, Δ_2' must contain $[X_2' Y_2']$

and so $\Delta_2' \cap g(\Delta) \neq \varnothing$. It follows that $d_{\mathcal{C}}(f(u), v_2) \geq 2$. This is again a contradiction.

Case 3. $v_1 \in \hat{\mathcal{C}}_0(S)$ and $v_2 \in \mathcal{C}_0'(S)$. This time, v_2 corresponds to a parabolic fixed point z_2 of G. By the assumption, $\tilde{v}_1$ is nontrivial and u, v_1 are disjoint.

Suppose that $\Omega_1 \cap \text{axis}(g) = \varnothing$. By the assumption, $d_{\mathcal{C}}(u, v_1) = 1$. Hence either $\Delta_1 \subset \Delta$ or $\Delta \subset \Delta_1$. The former case does not occur, as Δ_1 covers the point A while Δ does not. So we must have $\Delta \subset \Delta_1$; or $\mathbf{H}\backslash\Delta_1 \subset \mathbf{H}\backslash\Delta$. The assumption says $d_{\mathcal{C}}(v_1, v_2) = 1$. By Lemma 4.6, $z_2 \in \Omega_1 \cap \mathbf{S}^1$. But $\Omega_1 \subset \mathbf{H}\backslash(\Delta_1 \cup \Delta_1')$. So z_2 is covered by $g(\Delta)$. It follows that $d_{\mathcal{C}}(f(u), v_2) \geq 2$.

Suppose that $\Omega_1 \cap \text{axis}(g) \neq \varnothing$. In this case, $\Omega_1 \subset \mathbf{H}\backslash(\Delta_1 \cup \Delta_1')$. By the assumption, $d_{\mathcal{C}}(u, v_1) = 1$. This implies that $\Delta \subset \Delta_1$ or $\Delta_1 \subset \Delta$. Assume that $\Delta_1 \subset \Delta$. Since $\{\varrho^{-1}(\tilde{v}_1)\}$ intersects $\hat{c}_0$ at least twice, let $[X_1Y_1]$, $[X_1'Y_1']$ be such geodesics, where $[X_1'Y_1']$ is the one in $\{\varrho^{-1}(\tilde{v}_1)\}$ closest to $[MN]$ $= \partial g(\Delta)$. Note that $[X_1Y_1]$, $[X_1'Y_1']$ are contained in $\Delta' \cap g(\Delta)$. It follows that Δ_1' contains $[X_1'Y_1']$. But $z_2 \in \Omega_1 \cap \mathbf{S}^1$ and $\Omega_1 \subset \mathbf{H}\backslash(\Delta_1 \cup \Delta_1')$. We conclude that z_2 is covered by $g(\Delta)$. So $d_{\mathcal{C}}(v_2, f(u)) \geq 2$.

Assume now that $\Delta \subset \Delta_1$. The assumption leads to that $\{\varrho^{-1}(\tilde{v}_1)\}$ intersects $\hat{c}_0$ at least twice. Let $[X_1Y_1]$, $[X_1'Y_1']$ be as above. Then since $d_{\mathcal{C}}(u, v_1) = 1$, $\Delta_1 \subset \mathbf{H}\backslash\Delta'$. Hence by Lemma 4.6, Δ_1' contains $[X_1Y_1]$. It follows that $z_2 \in \Omega_1 \cap \mathbf{S}^1$ must be covered by $g(\Delta)$. This contradicts that $d_{\mathcal{C}}(v_2, f(u)) = 1$.

Case 4. $v_1, v_2 \in \hat{\mathcal{C}}_0(S)$. Then for $i = 1, 2$, v_i corresponds to the configurations $(\tau_i, \Omega_i, \mathscr{U}_i)$. By the assumption, v_1 and v_2 are disjoint.

Suppose that $\Omega_1 \cap \text{axis}(g) = \varnothing$ and $\Omega_2 \cap \text{axis}(g) = \varnothing$. By the same argument as in Case 3, $\Delta \subset \Delta_1$. By Lemma 4.5, $d_{\mathcal{C}}(u, v_1) = 1$ implies that $\Omega_1 \cap \Omega \neq \varnothing$. Hence $\mathbf{H} \backslash \Delta_1 \subset \mathbf{H} \backslash (\Delta \cup \Delta')$. If $\Delta_1 \subset \Delta_2$, i.e., $\mathbf{H} \backslash \Delta_2 \subset \mathbf{H} \backslash \Delta_1$, then $\mathbf{H} \backslash \Delta_2 \subset g(\Delta)$. By Lemma 4.5, $d_{\mathcal{C}}(f(u), v_2) \geq 2$, contradicting that $d_{\mathcal{C}}(f(u), v_2) = 1$. Suppose $\Delta_2 \subset \Delta_1$, i.e., $\mathbf{H} \backslash \Delta_1 \subset \mathbf{H} \backslash \Delta_2$. See Figure 3(a). Note that $\{\varrho^{-1}(\tilde{v}_2)\}$ intersects $\hat{c}_0$ more than once, and all geodesics in $\{\varrho^{-1}(\tilde{v}_2)\}$ are disjoint, as a member of $\{\varrho^{-1}(\tilde{v}_2)\}$, $\partial \Delta_2$ must be disjoint from any geodesic in $\{\varrho^{-1}(\tilde{v}_2)\}$ intersecting $\hat{c}_0$. It follows that $\mathbf{H} \backslash \Delta_2 \subset g(\Delta)$. By Lemma 4.5, $d_{\mathcal{C}}(f(u), v_2) \geq 2$, contradicting that $d_{\mathcal{C}}(f(u), v_2) = 1$.

If $\Omega_1 \cap \text{axis}(g) = \varnothing$ and $\Omega_2 \cap \text{axis}(g) \neq \varnothing$, then by the discussion above, we know that $\Delta \subset \Delta_1$ and $\mathbf{H} \backslash \Delta_1 \subset \mathbf{H} \backslash (\Delta \cup \Delta')$. By the assumption, there are at least two geodesics in $\{\varrho^{-1}(\tilde{v}_2)\}$ intersecting $\hat{c}_0$. Let $[X_2 Y_2]$ be the one which is closest to $[EF] = \partial \Delta'$. See Figure 3(b).

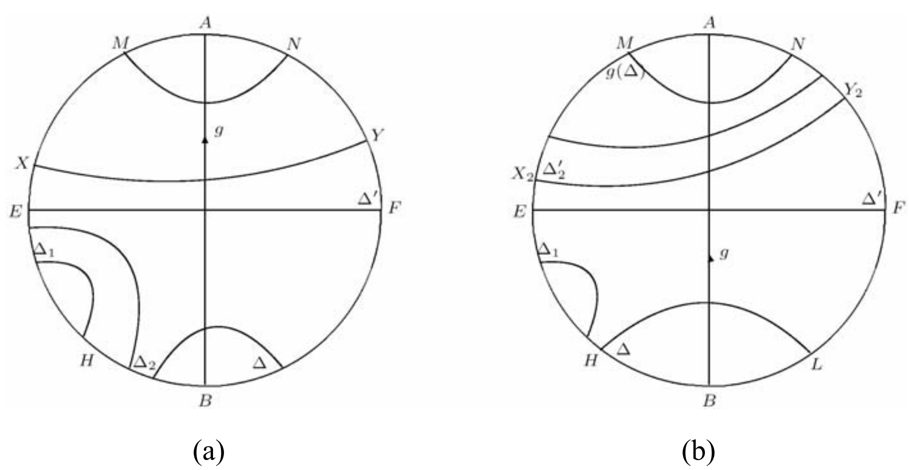


Figure 3

Recall that (Δ_2, Δ_2') is the pair of the maximal elements of $\mathscr{U}_2$ that crosses axis(g). We claim that Δ_2' contains $[X_2 Y_2]$. Otherwise, by Lemma 2.1 of [14], Δ_2 contains $[X_2 Y_2]$. Since $\Omega_2 \subset \mathbf{H} \backslash (\Delta_2 \cup \Delta_2')$ and $\Omega_1 \subset \mathbf{H} \backslash \Delta_1$, we see that $\overline{\Omega}_2 \cap \overline{\Omega}_1 = \varnothing$. By Lemma 4.4, $d_{\mathcal{C}}(v_1, v_2) \geq 2$, contradicting that $d_{\mathcal{C}}(v_1, v_2) = 1$. We conclude that Δ'_2 contains $[X_2Y_2]$. But then $\Delta'_2 \cap g(\Delta) \neq \varnothing$. So again, by Lemma 4.5, $d_{\mathcal{C}}(v_1, v_2) \geq 2$, which also leads to a contradiction.

The case where $\Omega_1 \cap \text{axis}(g) \neq \varnothing$ and $\Omega_1 \cap \text{axis}(g) = \varnothing$ can be handled similarly.

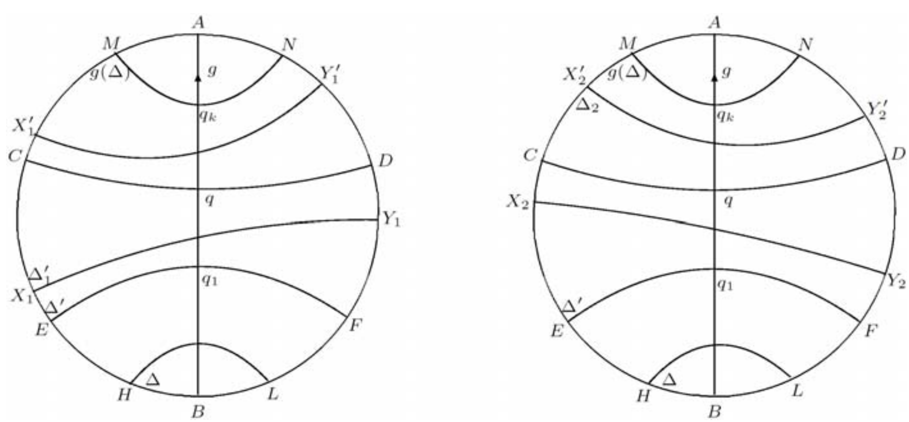
If $\Omega_1 \cap \text{axis}(g) \neq \varnothing$ and $\Omega_2 \cap \text{axis}(g) \neq \varnothing$, we note that $\{\varrho^{-1}(\widetilde{v}_1)\}$, $\{\varrho^{-1}(\widetilde{v}_2)\}$ and $\{\varrho^{-1}(\widetilde{u})\}$ are mutually disjoint. From the assumption, $\{\varrho^{-1}(\widetilde{u})\}$ intersects $\hat{c}_0$ at points $q_1, \ldots, q_k$, where $q_1 = [EF] \cap \hat{c}_0$ and $q_k = [MN] \cap \hat{c}_0$. See Figure 4.

Note that c_{1i} and c_{1j} correspond to two intervals E_i and E_j on $\hat{c}_0$. Since $|i - j| \geq 2$, there is a geodesic $[CD]$ that separates E_i and E_j. Denote $q = [CD] \cap \hat{c}_0$. Then the geodesic segment $\hat{c}_0$, which is a portion of $\text{axis}(g)$, can be written as $[q_1q] \cup [qq_k]$. Thus, $E_i \subset [q_1q]$ and $E_j \subset [qq_k]$. By the assumption, δ_{1i} and $\delta_{1j} \in \mathscr{S}$ for some i, j with $j - i \geq 2$. This means that $\widetilde{v}_1$ and $\widetilde{v}_2$ intersect δ_{1i} and δ_{1j}. Therefore, the segments $[q_1q]$ and $[qq_k]$ both intersect geodesics in $\sum_1 = \{\hat{v} \in \{\varrho^{-1}(\widetilde{v}_1)\} : \hat{v} \subset g(\Delta) \cap \Delta'\}$. Similarly, $[q_1q]$ and $[qq_k]$ also intersect geodesics in $\sum_2 = \{\hat{v} \in \{\varrho^{-1}(\widetilde{v}_2)\} : \hat{v} \subset g(\Delta) \cap \Delta'\}$.

For simplicity, let $[X'_1Y'_1] \in \sum_1$ denote the geodesic that is closest to the geodesic $[MN]$, and let $[X_1Y_1] \in \sum_1$ denote the geodesic that is closest to the geodesic $[EF]$. Then $[X'_1Y'_1]$ lies in the component of $\mathbf{H}$ bounded by $[MN]$ and $[CD]$, and $[X_1Y_1]$ lies in the component of $\mathbf{H}$ bounded by $[EF]$ and $[CD]$. See Figure 4(a). Likewise, let $[X'_2Y'_2] \in \sum_2$ be the geodesic closest to $[MN]$, and $[X_2Y_2] \in \sum_2$ be the geodesic closest to $[EF]$. It is also obvious that $[X_2'Y_2']$ lies in the component of $\mathbf{H}$ bounded by $[MN]$ and $[CD]$, and $[X_2Y_2]$ lies in the component of $\mathbf{H}$ bounded by $[EF]$ and $[CD]$. See Figure 4(b). Notice that geodesics in Σ_1 and Σ_2 are boundaries of elements of $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively. Consider the pairs (Δ_1, Δ_1') and (Δ_2, Δ_2') of maximal elements of $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively. Recall that Δ_1' and Δ_2' cover the attracting fixed point A of g, while Δ_1 and Δ_2 cover the repelling fixed point B of g.

We claim that $\partial\Delta_1' = [X_1Y_1]$ or $\partial\Delta_1'$ is disjoint from Δ'. If not, then by Lemma 2.1 of [14], either $\partial\Delta_1 = [X_1Y_1]$ or $[X_1Y_1]$ is contained in Δ_1. Since $\Omega_1 \subset \mathbf{H}\backslash(\Delta_1 \cup \Delta_1')$, we see that Ω_1 is disjoint from Ω. From Lemma 4.5, $d_{\mathcal{C}}(u, v_1) \geq 2$, contradicting that $d_{\mathcal{C}}(u, v_1) = 1$. So Δ_1' contains $[CD]$.

We also claim that $\partial\Delta_2 = [X_2'Y_2']$ or $\partial\Delta_2$ is disjoint from $g(\Delta)$. Suppose not. By Lemma 2.1 of [14] again, either $\partial\Delta_2' = [X_2'Y_2']$ or $[X_2'Y_2']$ is contained in Δ_2'. But $\Omega_2 \subset \mathbf{H}\backslash(\Delta_2 \cup \Delta_2')$. Hence Ω_2 is disjoint from $g(\Omega)$. Clearly, $\chi(g(\Omega)) = g^*(\chi(\Omega)) = f(u)$. From Lemma 4.5, $d_{\mathcal{C}}(f(u), v_2) \geq 2$, contradicting that $d_{\mathcal{C}}(f(u), v_2) = 1$. So Δ_2 contains $[CD]$.



(a) (b)

Figure 4

In particular, $\Delta_1' \cap \Delta_2 \neq \varnothing$. As members of $\varrho^{-1}(\widetilde{v}_1)$ and $\varrho^{-1}(\widetilde{v}_1)$, $\partial\Delta_1'$ is disjoint from $\partial\Delta_2$. Since $\Omega_1 \subset \mathbf{H}\backslash(\Delta_1 \cup \Delta_1')$ and $\Omega_2 \subset \mathbf{H}\backslash(\Delta_2 \cup \Delta_2')$, we conclude that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \varnothing$. By Lemma 4.5, we see that $d_{\mathcal{C}}(v_1, v_2) \geq 2$, contradicting that $d_{\mathcal{C}}(v_1, v_2) = 1$. □

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