# A POSTERIORI ERROR ESTIMATES OF THE STABILIZED CROUZEIX-RAVIART FINITE ELEMENT METHOD FOR THE LAMÉ-NAVIER EQUATIONS

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## Abstract

We obtain a posteriori error estimates for a variant of (non-locking) stabilized nonconforming methods based on the Crouzeix-Raviart

element introduced by Hansbo and Larson in [21]. We derive upper and lower a posteriori error bounds which are robust with respect to the nearly incompressible materials.

## 1. Introduction

The finite element methods are widely used for the numerical approximation of many problems occurring in engineering applications, like the Laplace equation, the Lamé system, etc. (see [10, 19]). In practice, adaptive techniques based on a posteriori error estimators have become indispensable tools for such methods. Hence, there now exist a large number of publications devoted to the analysis of some finite element approximations of problems from solid mechanics and obtaining locally defined a posteriori error estimates. We refer to the monographs [2, 7, 31] for a good overview on this topic.

For the elasticity system in the primal variables, several different approaches have been developed: residual type error estimators [5, 4, 32, 33, 17, 11, 22], methods based on the resolution of local subproblems by using higher order elements [4, 6, 8], averaging techniques (the so-called Zienkiewicz-Zhu estimators) [1, 2, 34, 35] and finally estimators based on equilibrated fluxes [3, 12, 25-27, 29]. For methods based on dual variables, like mixed methods, we refer to [13, 14, 16, 9, 24, 18]; note that such methods are usually locking-free and therefore the obtained estimators are usually locking-free.

Here we analyze two displacement methods based on the primal variables variant of the ones introduced by Hansbo and Larson in [21] which are nonconforming methods based on the Crouzeix-Raviart finite element. These methods are locking-free and very cheap. In [21], the authors derive a *priori* optimal error estimates uniform in the Lamé parameter $\lambda$ (see Theorem 3.1 of [21]). In this paper, we propose an a *posteriori* error analysis of these methods. Our analysis enters in the family of estimators of residual type since our error indicator is based on residuals on each triangle and jumps across the inter-element boundaries. We prove reliability of the indicator

uniformly in $\lambda$ (and $h$), in particular avoiding locking phenomena. The proof is based on a discrete divergence stability result (see [11] for another scheme), the use of an appropriate decomposition of the error into continuous and discontinuous parts [23]. Local efficiency of our indicator follows by using classical inverse estimates, the use of a strengthened norm allows to obtain its robustness.

Note that our methods enter in a similar framework than the ones in [17, 33, 11]. But our upper bound cannot be deduced from [17] because the assumption (2.12) from that paper is not satisfied by our scheme. On the other hand, the schemes in [17, 33, 11] are different from our proposed schemes, nevertheless the proof of the upper bound uses similar arguments than in [11].

The outline of the paper is as follows: We recall in Section 2 the boundary value problem and its numerical approximation. Section 3 is devoted to the proof of the discrete divergence stability result. The proof of the upper error bound is made in Section 4; and the lower bound is obtained in Section 5. Finally, some numerical tests that confirm our theoretical results are presented in the last section.

Let us finish this introduction with some notations used in the remainder of the paper: On $D$, the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. The usual norm and seminorm of $H^s(D)(s \geq 0)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. In the case $D = \Omega$, the index $\Omega$ will be omitted. Similar $H(div, \Omega)$ is defined by

$$H(div, \Omega) = \{v \in L^2(\Omega)^2 : div\, v \in L^2(\Omega)\},$$

equipped with its natural norm. Finally, the notation $a \lesssim b$ means the existence of a positive constant $C$, which is independent of the mesh size, of the considered quantities $a$ and $b$ and of the Lamé coefficient $\lambda$ such that $a \leq Cb$. In other words, the constant only depends on the aspect ratio of the mesh, on the domain $\Omega$ and on the shear modulus $\mu$, but not on Lamé coefficient $\lambda$ (see below).

## 2. A Discontinuous Galerkin Method for the Elasticity System

### 2.1. The boundary value problem of elasticity

Let $\Omega$ be a plane domain with a polygonal boundary. More precisely, we assume that $\Omega$ is a simply connected domain and that its boundary $\Gamma$ is the union of a finite number of linear segments $\overline{\Gamma}_j$, $1 \leq j \leq n_e$ ($\Gamma_j$ is assumed to be an open segment). In this domain, we consider an elastic isotropic material. Let $u = (u_1, u_2)$ be the displacement field and $f = (f_1, f_2) \in [L^2(\Omega)]^2$ the body force by unit of mass. For simplicity, we consider in $\Omega$ the pure homogeneous displacement problem. Thus, the displacement field $u = (u_1, u_2)$ satisfies the following equations and boundary condition:

$$\begin{cases} -div\,\sigma(u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma, \end{cases} \tag{2.1}$$

where the stress tensor is defined by

$$\sigma(u) := 2\mu\varepsilon(u) + \lambda\,tr\varepsilon(u)I. \tag{2.2}$$

The positive constants $\mu$ and $\lambda$ are called the *Lamé coefficients* where $\mu \equiv G$ is the shear modulus ([30, p. 207]) ([20, pp. 72-75, 57-58]). We assume that [10]

$$(\lambda, \mu) \in [\lambda_0, +\infty[ \times [\mu_1, \mu_2], \tag{2.3}$$

where $0 < \mu_1 < \mu_2$ and $\lambda_0 > 0$.

As usual, $\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the linearized strain tensor and $I$ is the identity tensor. For any tensor $\tau$ (of order 2), the trace of $\tau$ is $tr\tau := \tau_{11} + \tau_{22}$, and

$$div\tau := \left( \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} \right)$$

(in an orthogonal Cartesian frame). The classical variational formulation of

the boundary value problem (2.1) is the following: find $u \in [H_0^1(\Omega)]^2$ such that

$$\int_\Omega \sigma(u) : \varepsilon(v)\, dx = \int_\Omega f \cdot v \, dx, \quad \forall v \in [H_0^1(\Omega)]^2, \tag{2.4}$$

where $\sigma : \varepsilon = \sum_{i,\, j=1}^2 \sigma_{ij} \varepsilon_{ij}$ denotes the contraction of the two tensors. The existence and uniqueness of the solution in (2.4), for a given $f \in [H^{-1}(\Omega)]^2$, follows from the first Korn inequality (cf. [10, p. 286]). In the following, we suppose that $f \in [L^2(\Omega)]^2$.

## 2.2. The discontinuous Galerkin methods

In order to formulate the discrete problems introduced by Hansbo and Larson in [21], we first recall some notation.

Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations. Let us fix a triangulation $\mathcal{T}_h \in (\mathcal{T}_h)_{h>0}$ and let $\mathcal{E}_h$ denote the set of all edges of the triangulation $\mathcal{T}_h$. We split $\mathcal{E}_h$ into two disjoint subsets: $\mathcal{E}_h^I$ the subset of edges contained in $\Omega$ and $\mathcal{E}_h^\partial$ the subset of edges contained in the boundary of $\Omega$. Further, with each edge $E$, we associate a fixed unit normal $n_E$ such that for edges contained in $\partial\Omega$, $n_E$ is the exterior unit normal; we further fix a unit tangent $t_E$ so that $\{n_E, t_E\}$ forms an orthonormal basis of $\mathbb{R}^2$. For any edge $E$ and any triangle $T$, we denote by $|E|$ the length of $E$ and $|T|$ the area of $T$. As usual, for any triangle $T$, we denote by $\omega_T = \bigcup_{T' \in \mathcal{T}_h : T' \cap T \neq \varnothing} T'$, the patch associated with $T$. Similarly, for an edge $E$, we set $\omega_E = \bigcup_{T' \in \mathcal{T}_h : E \subset T'} T'$.

Let

$$DF(\mathcal{T}_h) := \{ v_h \in [L^2(\Omega)]^2; \ \forall T \in \mathcal{T}_h : v_{h|T} \in [P_1(T)]^2 \}$$

be the space of piecewise linear discontinuous functions (as usual $P_1(T)$ denotes the space of polynomials of degree less than or equal to 1 on $T$).

For a function $v_h \in DF(\mathcal{T}_h)$, we denote its jump at an edge $E$ by $[[v_h]]_E = v_h^+ - v_h^-$ for $E \in \mathcal{E}_h^I$ and $[[v_h]]_E = v_h^+$ for $E \in \mathcal{E}_h^\partial$, and the average $\langle v_h \rangle_E = (v_h^+ + v_h^-)/2$ for $E \in \mathcal{E}_h^I$ and $\langle v_h \rangle_E = v_h^+$ for $E \in \mathcal{E}_h^\partial$, where $v_h^\pm(x) = \lim_{h \to 0^+} v_h(x \mp hn_E)$ with $x \in E$. In the same spirit, we define its broken strain tensor $\varepsilon_h(v_h)$ by

$$\varepsilon_h(v_h)_T = \varepsilon(v_{h|T}), \quad \forall T \in \mathcal{T}_h,$$

its broken stress tensor $\sigma_h(v_h)$ and broken divergence $div_h v_h$ are defined similarly.

We further introduce the Crouzeix-Raviart space of vector fields $CR(\mathcal{T}_h)$:

$$CR(\mathcal{T}_h) := \{v_h \in [L^2(\Omega)]^2; \forall T \in \mathcal{T}_h : v_{h|T} \in [P_1(T)]^2,$$

$$[[P_0 v_h]]_E = 0, \forall E \in \mathcal{E}_h\}, \tag{2.5}$$

where for any edge $E$ and any $w \in L^2(E)$,

$$P_0 w := \frac{1}{|E|} \int_E w \, ds,$$

and therefore

$$[[P_0 v_h]]_E := P_0 [[v_h]]_E = \frac{1}{|E|} \int_E v_h^+ ds - \frac{1}{|E|} \int_E v_h^- ds.$$

The first discontinuous Galerkin method is a variant of the one introduced in [21] and reads as follows: find $u_h \in DF(\mathcal{T}_h)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in DF(\mathcal{T}_h). \tag{2.6}$$

The bilinear form is here defined by

$$a_h(u_h, v_h)$$

$$:= a_{0h}(u_h, v_h) + (2\mu + \lambda)\lambda\gamma_0 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} ([[P_0 u_h \cdot n_E]], [[P_0 v_h \cdot n_E]])_E$$

$$+ 2\mu\gamma_0' \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} ([[P_0 u_h \cdot t_E]], [[P_0 v_h \cdot t_E]])_E$$

$$- \sum_{E \in \mathcal{E}_h} ((\langle \sigma_h(u_h) \rangle_E n_E, [[v_h]]_E)_E + (\langle \sigma_h(v_h) \rangle_E n_E, [[u_h]]_E)_E), \qquad (2.7)$$

$$a_{0h}(u_h, v_h) := \sum_{T \in \mathcal{T}_h} (\sigma(u_h), \varepsilon(v_h))_T + 2\mu\gamma_1 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} ([[u_h]], [[v_h]])_E, \quad (2.8)$$

while the linear form is defined by

$$(f, v_h) = \int_\Omega f \cdot v_h \, dx.$$

Here the parameters $\gamma_0$, $\gamma_0'$ and $\gamma_1$ are the stabilizing parameters fixed below.

Here and below $(\sigma(u_h), \varepsilon(v_h))_T$ means the $L^2$-inner product $\int_T \sigma(u_h)$ : $\varepsilon(v_h) dx$. Similarly, for vector valued functions $v, w \in L^2(E)^2$, $(v, w)_E$ means the $L^2$-inner product $\int_E v \cdot w \, ds$.

Note that our bilinear form $a_h$ differs from the one in [21] by the second and third terms of the right-hand side of (2.7), indeed in that paper these two terms are replaced by

$$(2\mu + \lambda)\gamma_0 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} ([[P_0 u_h]], [[P_0 v_h]])_E.$$

This modification is made to have a discrete stability result and hence a robust a posteriori estimate that cannot be obtained if we keep the original term from [21]. Note further that the bilinear form $a_h$ is similar to the one in [11] except that the second and third terms are replaced by

$$\lambda^2 \gamma_0 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} ([[u_h \cdot n_E]], [[v_h \cdot n_E]])_E.$$

As a consequence, the well-posedness of the corresponding problem in [11] is obtained for $\gamma_1$ and $\gamma_0$ large enough (independently of $\lambda$), while our problem is well-posed only for $\gamma_0$ and $\gamma_0'$ large enough (independently of $\lambda$). Our approach allows then to use a penalization parameter $\gamma_1$ as small as we want (but bounded away from zero). This difference also implies the use of a different error norm.

The well-posedness of problem (2.6) follows from the strong coerciveness of $a_h$ in $DF(\mathcal{T}_h)$, that we now show (see Proposition 2.2 of [21] or Proposition 2.2 of [11]).

**Proposition 2.1.** *There exist three positive constants* $\gamma_{\min}$, $\gamma_{\min}'$ *and C independent of* $\lambda$, $\mu$ *and the mesh size h such that for all* $\gamma_0 > \gamma_{\min}$ *and all* $\gamma_0' > \gamma_{\min}'$ *and any* $v_h \in DF(\mathcal{T}_h)$, *we have*

$$a_h(v_h, v_h) \ge C \left( 2\mu \|\varepsilon_h(v_h)\|^2 + \lambda \|div_h v_h\|^2 + (2\mu + \lambda)\lambda \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[[P_0 v_h \cdot n_E]]\|_E^2 \right.$$

$$\left. + 2\mu \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[[P_0 v_h \cdot t_E]]\|_E^2 + 2\mu\gamma_1 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[[v_h]]\|_E^2 \right).$$

**Proof.** For a fixed edge $E$, as $\varepsilon_h(v_h)$ and $div_h v_h$ are piecewise constants, we have

$$(\langle \sigma_h(v_h) \rangle_E n_E, [[v_h]]_E)_E = 2\mu(\langle \varepsilon(v_h) \rangle_E n_E, [[P_0 v_h]]_E)_E$$

$$+ \lambda(\langle div_h v_h \rangle_E, [[P_0 v_h \cdot n_E]]_E)_E.$$

By Cauchy-Schwarz's and Young's inequalities, for any $\delta, \delta' > 0$, we obtain

$$|\left(\langle\sigma_h(v_h)\rangle_E n_E, [[v_h]]_E\right)_E| \leq \frac{\mu}{\delta'} h_E \|\langle\varepsilon(v_h)\rangle_E\|_E^2 + \mu\delta' h_E^{-1}\|[[P_0 v_h]]_E\|_E^2$$

$$+ \frac{\lambda}{2(2\mu+\lambda)\delta} h_E \|\langle div_h v_h\rangle_E\|_E^2$$

$$+ \frac{\lambda(2\mu+\lambda)\delta}{2} h_E^{-1}\|[[P_0 v_h \cdot n_E]]_E\|_E^2.$$

The regularity assumption on the meshes yields a positive constant $C_{tr}$ such that

$$h_E\|\langle div_h v_h\rangle_E\|_E^2 \leq C_{tr}\|div_h v_h\|_{\omega_E}^2, \quad h_E\|\langle\varepsilon(v_h)\rangle_E\|_E^2 \leq C_{tr}\|\varepsilon_h(v_h)\|_{\omega_E}^2.$$

These estimates in the previous one lead to

$$\left|\sum_{E\in\mathcal{E}_h}\left(\langle\sigma_h(v_h)\rangle_E n_E, [[v_h]]_E\right)_E\right|$$

$$\leq \frac{2\mu C_{tr}}{\delta'}\|\varepsilon_h(v_h)\|^2 + \frac{\lambda C_{tr}}{(2\mu+\lambda)\delta}\|div_h v_h\|^2$$

$$+ \sum_{E\in\mathcal{E}_h} h_E^{-1}\left(\mu\delta\|[[P_0 v_h]]_E\|_E^2 + \frac{\lambda(2\mu+\lambda)\delta}{2}\|[[P_0 v_h \cdot n_E]]_E\|_E^2\right).$$

By this estimate and the definition of $a_h$, we deduce that

$$a_h(v_h, v_h) \geq 2\mu\left(1 - \frac{2C_{tr}}{\delta'}\right)\|\varepsilon_h(v_h)\|^2 + \lambda\left(1 - \frac{2C_{tr}}{(2\mu+\lambda)\delta}\right)\|div_h v_h\|^2$$

$$+ 2\mu\gamma_1\sum_{E\in\mathcal{E}_h} h_E^{-1}\|[[v_h]]_E\|_E^2$$

$$+ \sum_{E\in\mathcal{E}_h} h_E^{-1}\left((2\mu+\lambda)\lambda\left(\gamma_0 - \delta - \frac{2\mu\delta'}{(2\mu+\lambda)\lambda}\right)\|[[P_0 v_h \cdot n_E]]\|_E^2\right.$$

$$\left.+ 2\mu(\gamma_0' - \delta')\|[[P_0 v_h \cdot t_E]]_E\|_E^2\right).$$

Now we fix $\delta$ and $\delta'$ such that $\left(1 - \frac{2C_{tr}}{\delta'}\right) = \left(1 - \frac{2C_{tr}}{(2\mu + \lambda)\delta}\right) = \frac{1}{2}$, and the

conclusion follows with the choice $\gamma_{\min} = \max\left\{1, \frac{4C_{tr}}{(2\mu + \lambda)} + \frac{8\mu C_{tr}}{(2\mu + \lambda)\lambda}\right\}$

and $\gamma'_{\min} = 4C_{tr}$. □

By the previous result, we deduce that for $\gamma_0 > \gamma_{\min}$ and $\gamma'_0 > \gamma'_{\min}$, problem (2.6) is well posed. As shown in Proposition 2.1 of [21], the method is consistent, namely,

$$a_h(u - u_h, v_h) = 0, \quad \forall v_h \in DF(\mathcal{T}_h). \tag{2.9}$$

Restricting the discontinuous Galerkin method to the space of Crouzeix-Raviart functions, we obtain the following simplified scheme: find $u_{0h} \in CR(\mathcal{T}_h)$ such that

$$a_{0h}(u_{0h}, v_h) = (f, v_h), \quad \forall v_h \in CR(\mathcal{T}_h). \tag{2.10}$$

As before, this second method is consistent in the following sense:

$$a_{0h}(u - u_{0h}, v_h) = 0, \quad \forall v_h \in CR(\mathcal{T}_h) \cap H_0^1(\Omega)^2. \tag{2.11}$$

Observe that Theorem 2.3 of [28] guarantees the displacement vector field $u \in [H^{3/2+\varepsilon}(\Omega)]^2$ for some $\varepsilon \in \left]0, \frac{1}{2}\right[$. Consequently, $\sigma(u) \cdot n_E$ has sense on each edge $E \in \mathcal{E}_h$ as an element of $[H^\varepsilon(E)]^2 \subset [L^2(E)]^2$ and Green's formula can be applied element by element. Hence, the consistency always holds.

We finally refer to [21] for robust a priori error estimates.

Our goal is to bound the "triple norm" $\||e\|| := \sqrt{a_{0h}(e, e)}$ of the error $e := u - u_h$. But in order to get a robust estimate, we need to use the parameter depending norm:

$$\||e\||_\lambda^2 = \||e\||^2 + \lambda^2 \|\Pi_0 div_h e\|^2 + \lambda^2 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[[P_0 e \cdot n_E]]\|_E^2,$$

where $\Pi_0$ is the $L^2$-projection on $L_0^2(\Omega)^2$ defined by

$$\Pi_0 f = f - \frac{1}{|\Omega|}\int_\Omega f(x)dx, \ \forall f \in L^2(\Omega)^2.$$

Before going on notice that if $u_{0h} \in CR(\mathcal{T}_h)$, then

$$\int_\Omega div_h\, u_{0h} dx = 0.$$

## 3. Discrete Stability

The stability of the continuous problem (2.4) was proved in [11, Prop. 3.2] and means that

$$\lambda \| div\, u \| \lesssim \| f \|.$$

A discrete version of this estimate was also proved in Corollary 3.7 of [11] for their own scheme. Using similar arguments, we also obtain a discrete stability.

**Proposition 3.1.** *If $u_h \in DF(\mathcal{T}_h)$ is the solution of* (2.6), *then*

$$\| u_h \|_\lambda \lesssim \| f \|.$$

**Proof.** Lemma 3.5 of [11] yields $w \in DF(\mathcal{T}_h) \cap H(div, \Omega)$ such that

$$div\, w = -\lambda \Pi_0(div_h\, u_h),$$

$$\| \nabla_h w \|^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1}\| [[w]] \|_E^2 \lesssim \lambda^2 \| \Pi_0(div_h u_h) \|^2.$$

With such a choice, we notice that in $a_h(u_h, w)$, the term

$$(2\mu + \lambda)\lambda\gamma_0 \sum_{E \in \mathcal{E}_h} h_E^{-1}([[P_0 u_h \cdot n_E]], [[P_0 w \cdot n_E]])_E$$

is zero and therefore the arguments of Proposition 3.6 of [11] can be applied.

□

## 4. Robust Upper Bound of the Error

For all $T \in \mathcal{T}_h$, the local estimator $\eta_T$ is defined by

$$\eta_T^2 = h_T^2 \| f_T \|_T^2 + \sum_{E \in \mathcal{E}_h : E \subset T} h_E \| [[\sigma(u_h) \cdot n_E]]_E \|_E^2$$

$$+ \sum_{E \in \mathcal{E}_h : E \subset T} h_E^{-1} (\| [[u_h]] \|_E^2 + \lambda^2 \| [[P_0 u_h \cdot n_E]] \|_E^2),$$

where $f_T = \frac{1}{|T|} \int_T f(x) dx$. The global estimator is given by

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2.$$

Finally, let $osc^2(f) = \sum_{T \in \mathcal{T}_h} h_T^2 \| f - f_T \|_T^2$ be the oscillating term.

Note that the difference between our estimator and the one from [11] relies on the replacement of the term $\lambda^2 h_E^{-1} \| [[u_h \cdot n_E]] \|_E^2$ by the smaller one $\lambda^2 h_E^{-1} \| [[P_0 u_h \cdot n_E]] \|_E^2$ (that is even zero for the Crouzeix-Raviart element, see below).

Now we are able to state our locking-free a posteriori error estimate:

**Theorem 4.1.** *With the previous notation, we have the following a posteriori estimate*:

$$\| e \|_\lambda \lesssim \eta + osc(f). \tag{4.1}$$

**Proof.** We proceed as in the proof of Theorem 4.2 of [11], using in particular Lemma 4.1 in [11], namely, we take $\omega \in H_0^1(\Omega)^2$ such that

$$div \, \omega = \lambda \Pi_0 (div_h e),$$

$$\| \omega \|_{1, \Omega} \lesssim \lambda \| \Pi_0 (div_h u_h) \|,$$

and such that there exist $\beta > 0$ and $\kappa > 0$ for which

$$4\mu\| \varepsilon(e^c) \|^2 + \lambda^2 \| \Pi_0(div_h e) \|^2$$

$$\leq \kappa(2\mu(\varepsilon(e^c), \varepsilon(e^c + \beta\omega)) + \beta\lambda(div_h e, div\omega)), \tag{4.2}$$

where $e^c$ is the continuous part of the error (see for instance, [23, 11]), namely,

$$e^c = u - u_h^c \quad \text{and} \quad u_h = u_h^c + u_h^\perp,$$

with $u_h^c \in V_h^c := DF(\mathcal{T}_h) \cap H_0^1(\Omega)^2$ and $u_h^\perp \in DF(\mathcal{T}_h)^\perp$, where $DF(\mathcal{T}_h)^\perp$ is an appropriate orthogonal complement of $V_h^c$ into $DF(\mathcal{T}_h)$.

In a first step using this splitting and the standard estimate

$$\| \nabla_h u_h^\perp \|^2 \lesssim \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \| [[u_h]] \|_E^2, \tag{4.3}$$

we have (compare with the estimate (29) from [11])

$$\| e \|_\lambda^2 \lesssim \| \varepsilon(e^c) \|^2 + \lambda^2 \| \Pi_0(div_h e) \|^2$$

$$+ \sum_{E \in \mathcal{E}_h} h_E^{-1}(\| [[u_h]] \|_E^2 + \lambda^2 \| [[P_0 u_h \cdot n_E]] \|_E^2).$$

Hence, it remains to estimate the first two terms of this right-hand side. For that purpose, we use (4.2) and transform its right-hand side as in [11] to get

$$4\mu\| \varepsilon(e^c) \|^2 + \lambda^2 \| \Pi_0(div_h e) \|^2$$

$$\lesssim (\sigma(u), \varepsilon(w)) - (\sigma_h(u_h), \varepsilon(w)) + 2\mu(\varepsilon(u_h^\perp), \varepsilon(w)) - \lambda(div_h e, div\, e^c), \tag{4.4}$$

where $w = e^c + \beta\omega$ that belongs to $H_0^1(\Omega)^2$ and satisfies (see [11, p. 1550])

$$\| w \|_{1, \Omega} \lesssim \| e \|_\lambda. \tag{4.5}$$

For the two first terms of this right-hand side, we use the continuous and discrete formulation to get successively (with $w_h = I_{Cl}(w)$ the Clément interpolant of $w$)

$$(\sigma(u), \varepsilon(w)) - (\sigma_h(u_h), \varepsilon(w)) = (\sigma(u), \varepsilon(w - w_h)) - (\sigma_h(u_h), \varepsilon(w - w_h))$$

$$+ (\sigma(u), \varepsilon(w_h)) - (\sigma_h(u_h), \varepsilon(w_h))$$

$$= (\sigma(u), \varepsilon(w - w_h)) - (\sigma_h(u_h), \varepsilon(w - w_h))$$

$$- \sum_{E \in \mathcal{E}_h} (\langle \sigma(w_h) \rangle_E n_E, [\![u_h]\!]_E)_E.$$

Using again the continuous formulation and piecewise Green's formula, we arrive at

$$(\sigma(u), \varepsilon(w)) - (\sigma_h(u_h), \varepsilon(w))$$

$$= (f, w - w_h) - \sum_{E \in \mathcal{E}_h} (([\![\sigma_h(u_h) \cdot n_E]\!], w - w_h)_E$$

$$+ (\langle \sigma(w_h) \rangle_E n_E, [\![u_h]\!]_E)_E).$$

The first term is estimated in a standard way. The second term is also estimated in a classical way, namely, by classical local error estimates on the Clément interpolant ([31, Lemma 1.4, p. 11]) using Lemma 3.4 of [11], we get

$$\left| \sum_{E \in \mathcal{E}_h} ([\![\sigma_h(u_h) \cdot n_E]\!], w - w_h)_E \right| \lesssim \eta \|w\|_{1,\Omega}.$$

For the last term, we can write (see above)

$$(\langle \sigma(w_h) \rangle_E n_E, [\![u_h]\!]_E)_E = 2\mu(\langle \varepsilon(w_h) \rangle_E n_E, [\![u_h]\!]_E)_E$$

$$+ \lambda(\langle div_h w_h \rangle_E, [\![P_0 u_h \cdot n_E]\!]_E)_E.$$

Hence, using scaling arguments, we obtain

$$\left| \sum_{E \in \mathcal{E}_h} (\langle \sigma(w_h) \rangle_E n_E, [[u_h]]_E)_E \right|$$

$$\lesssim \| w_h \|_{1, \Omega} \left( \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} (\| [[u_h]] \|_E^2 + \lambda^2 \| [[P_0 u_h \cdot n_E]] \|_E^2) \right)^{\frac{1}{2}}.$$

Hence, using again Lemma 1.4 of [31], we get

$$\left| \sum_{E \in \mathcal{E}_h} (\langle \sigma(w_h) \rangle_E n_E, [[u_h]]_E)_E \right| \lesssim \eta \| w \|_{1, \Omega}.$$

The third term of (4.4) is estimated by using (4.3), namely,

$$| 2\mu(\varepsilon(u_h^\perp), \varepsilon(w)) | \lesssim \| \nabla_h u_h^\perp \| \| \nabla w \| \lesssim \eta \| \nabla w \|.$$

For the last term of (4.4) since $e^c$ belongs to $H_0^1(\Omega)^2$ its divergence is of mean zero and therefore as in step 2b in the proof of Theorem 4.2 of [11, p. 1550], we find

$$-\lambda(div_h e, div\, e^c) \leq -\lambda(\Pi_0 div_h e, div_h u_h^\perp).$$

Hence, by Cauchy-Schwarz's inequality and the estimate (4.3), we again arrive at

$$-\lambda(div_h e, div\, e^c) \lesssim \eta \lambda \| \Pi_0 div_h e \| \lesssim \eta \| e \|_\lambda.$$

All the previous estimates lead to the conclusion.    □

## 5. Lower Bound of the Error

For a subset $\omega$ of $\overline{\Omega}$, we denote by

$$\| e \|_{\lambda, \omega}^2 := (\sigma_h(e), \varepsilon_h(e))_\omega + \lambda^2 \| \Pi_0 div_h e \|_\omega^2$$

$$+ \sum_{E \in \mathcal{E}_h : E \subset \omega} \frac{1}{h_E} (2\mu\gamma_1 \| [[u_h]] \|_E^2 + \lambda^2 \| [[P_0 u_h \cdot n_E]] \|_E^2),$$

$$osc_\omega^2(f) := \sum_{T \in \mathcal{T}_h : T \subset \omega} h_T^2 \| f - f_T \|_T^2.$$

We start with the element residual.

**Lemma 5.1.** *For all $T \in \mathcal{T}_h$, we have*

$$h_T \| f_T \|_T \lesssim \| e \|_{\lambda, T} + osc_T(f).$$

**Proof.** If $b_T$ is the standard bubble function associated with $T$, then we set $w_T = f_T b_T$. Hence, by a standard inverse inequality, we have

$$\| f_T \|_T^2 \lesssim \int_T f_T \cdot w_T dx = \int_T (f_T - f) \cdot w_T dx - \int_T div \, \sigma(u - u_h) \cdot w_T dx,$$

reminding that $div \, \sigma(u) = -f$. For the second term of this right-hand side applying Green's formula, we obtain

$$\| f_T \|_T^2 \lesssim \int_T (f_T - f) \cdot w_T dx + \int_T \sigma(u - u_h) : \varepsilon(w_T) dx.$$

Now we can write

$$\int_T \sigma(u - u_h) : \varepsilon(w_T) dx = 2\mu \int_T \varepsilon(u - u_h) : \varepsilon(w_T) dx$$

$$+ \lambda \int_T div(u - u_h) div(w_T) dx$$

$$= 2\mu \int_T \varepsilon(u - u_h) : \varepsilon(w_T) dx$$

$$+ \lambda \int_\Omega div_h(u - u_h) div(w_T) dx.$$

Since $w_T$ can be seen as a function in $H_0^1(\Omega)^2$, we deduce that

$$\int_T \sigma(u - u_h) : \varepsilon(w_T) dx = 2\mu \int_T \varepsilon(u - u_h) : \varepsilon(w_T) dx$$

$$+ \lambda \int_\Omega \Pi_0 div_h(u - u_h) div(w_T) dx$$

$$= 2\mu \int_T \varepsilon(u - u_h) : \varepsilon(w_T) dx$$

$$+ \lambda \int_T \Pi_0 div_h(u - u_h) div(w_T) dx.$$

By Cauchy-Schwarz's inequality and the inverse inequality

$$\| \varepsilon(w_T) \|_T \lesssim h_T^{-1} \| f_T \|_T,$$

we obtain

$$h_T \| f_T \|_T \lesssim osc_T(f) + \| \varepsilon(u - u_h) \|_T + \lambda \| \Pi_0 div_h(u - u_h) \|_T.$$

The conclusion follows from the direct estimate

$$\| \varepsilon(e) \|_T^2 \leq \frac{1}{2\mu} \int_T \sigma(e) : \varepsilon(e) dx. \tag{5.1}$$

$\square$

We argue in a similar manner for the normal jump, namely, we have the

**Lemma 5.2.** *For all* $E \in \mathcal{E}_h$, *we have*

$$h_E^{1/2} \| [[\sigma(u_h) \cdot n_E]]_E \|_E \lesssim \| e \|_{\lambda, \omega_E} + osc_{\omega_E}(f).$$

**Proof.** Denoting by $b_E$ the edge bubble, we set $w_E = [[\sigma(u_h) \cdot n_E]]_E b_E$ that belongs to $[H_0^1(\omega_E)]^2$. Hence, by a standard inverse inequality, we have

$$\| [[\sigma(u_h) \cdot n_E]]_E \|_E^2 \sim \int_E [[\sigma(u_h) \cdot n_E]]_E \cdot w_E ds$$

$$= \int_E [[\sigma(u_h - u) \cdot n_E]]_E \cdot w_E ds$$

$$= \sum_{T \in \mathcal{T}_h : T \subset \omega_E} \int_{\partial T} \sigma(u_h - u) \cdot n \cdot w_E ds.$$

Hence, an integration by parts yields

$$\| \, [[\sigma(u_h) \cdot n_E]]_E \, \|_E^2 \sim \sum_{T \in \mathcal{T}_h : T \subset \omega_E} \left( \int_T \sigma(u_h - u) : \nabla(w_E) dx + \int_T f \cdot w_E dx \right).$$

Using the same arguments as before, we obtain

$$h_E^{1/2} \| \, [[\sigma(u_h) \cdot n_E]]_E \, \|_E \lesssim \sum_{T \in \mathcal{T}_h : T \subset \omega_E} ( \| \, \varepsilon(u_h - u) \, \|_T + h_T \| \, f_T \, \|_T )$$

$$+ \lambda \| \Pi_0 div_h(u - u_h) \|_{\omega_E} + osc_{\omega_E}(f).$$

The conclusion follows from Lemma 5.1 and the estimate (5.1).      □

These two lemmas directly yield the

**Theorem 5.3.** *For all* $T \in \mathcal{T}_h$, *we have*

$$\eta_T \lesssim \| \, e \, \|_{\lambda, \omega_T} + osc_{\omega_T}(f).$$

## 6. The Nonconforming Galerkin Method Based on the Crouzeix-Raviart Element

The arguments of the three previous sections can be directly applied to the discontinuous method (2.10). They are even simpler because $[[P_0 u_h \cdot n]]_E = 0$ on any edge $E$. Hence, we have not to use the extra term

$$\lambda^2 \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \| \, [[P_0 e \cdot n]] \, \|_E^2$$

in the norm of the error (as it is zero). More precisely, we can prove the next results:

**Theorem 6.1.** *We have the following a posteriori estimate*:

$$\| \, e \, \| + \lambda \| \Pi_0 div_h e \, \| \lesssim \eta_0 + osc(f), \tag{6.1}$$

*where* $\eta_0^2 = \sum_{T \in \mathcal{T}_h} \eta_{0T}^2$, *and for all* $T \in \mathcal{T}_h$, *the local estimator* $\eta_{0T}$ *is*

*defined by*

$$\eta_{0T}^2 = h_T^2 \| f_T \|_T^2 + \sum_{E \in \mathcal{E}_h : E \subset T} h_E \, \| [[\sigma(u_h) \cdot n_E]]_E \|_E^2$$

$$+ \sum_{E \in \mathcal{E}_h : E \subset T} \frac{1}{h_E} \| [[u_h]] \|_E^2.$$

**Theorem 6.2.** *For all* $T \in \mathcal{T}_h$, *we have*

$$\eta_{0T} \lesssim \| e \|_{\omega_T} + \lambda \| \Pi_0 div_h(e) \|_{\omega_T} + osc_{\omega_T}(f),$$

*where*

$$\| e \|_{\omega}^2 := \sum_{T \in \mathcal{T}_h : T \subset \omega} (\sigma(e), \varepsilon(e))_T + 2\mu\gamma_1 \sum_{E \in \mathcal{E}_h : E \subset \omega} \frac{1}{h_E} \| [[u_h]] \|_E^2.$$

Theorem 6.1 is also a consequence of Theorem 2.1 of [15] since $u_{0h}$ satisfies the assumptions of this theorem. We give here an alternative proof.

## 7. Numerical Experiments

In this section, some numerical experiments are performed to illustrate the above theoretical results, similarly to the ones proposed in [11]. In all the following cases, Crouzeix-Raviart finite elements $CR(\mathcal{T}_h)$ defined by (2.5) are used on a regular mesh composed of triangles.

### 7.1. Divergence-free smooth solutions

The first test consists in solving equation (2.1) on the unit square $\Omega = (0, 1)^2$, for the exact solution given by:

$$u(x, y) = \frac{1}{\pi^2} (\sin(\pi x) \sin(\pi y))^{n-1} (\sin(\pi x) \cos(\pi y), -\sin(\pi y) \cos(\pi x)),$$

where $n \geq 2$ is a given integer and where $f$ is chosen accordingly. It should be noted that $u$ is smooth, divergence-free, and that $u = 0$ on $\partial\Omega$.

For the first example, we set $n = 2$. We plot in Figure 7.1 the convergence rates of the error (Figure 7.1(a)) and of the estimator (Figure 7.1(b)) for several values of $\lambda$, using a uniform mesh refinement strategy. It can be observed a good convergence towards zero for both of them corresponding to the one theoretically expected (namely, order 1 in $h$ corresponding to order $-1/2$ in the total number of degrees of freedom $ndof$).
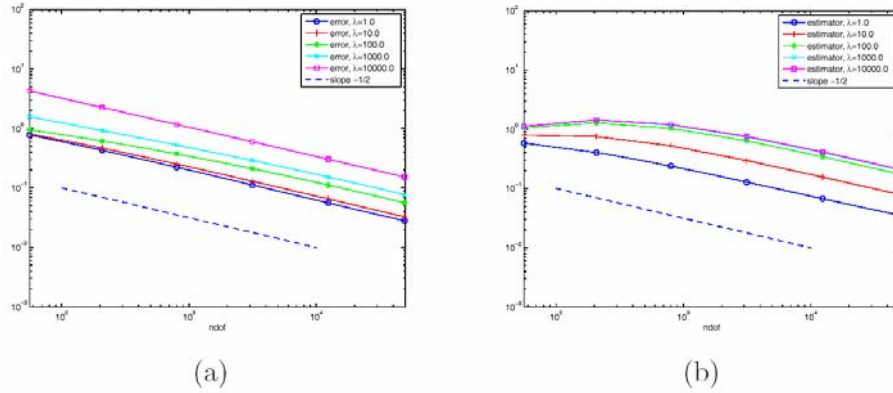


(a)                                      (b)

**Figure 7.1.** Error (a) and estimator (b) convergence rates in $ndof$ for the regular solution, $n = 2$, $\lambda$ from 1 to $10^4$.
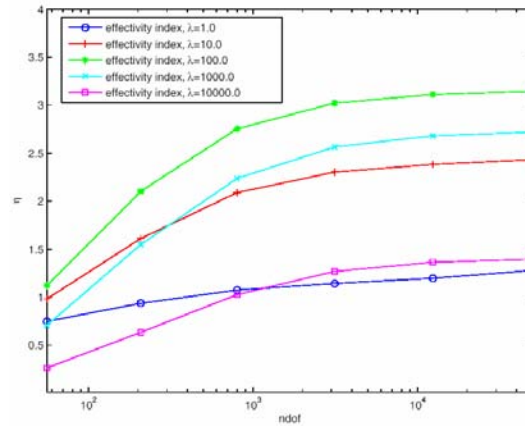


**Figure 7.2.** Effectivity index $I_{eff}$ for the regular solution, $n = 2$, $\lambda$ from 1 to $10^4$.

Then the effectivity index defined by:

$$I_{eff} = \frac{\eta_0}{\| | e \| | + \lambda \| \Pi_0 div_h e \|}$$

is plotted in Figure 7.2. First, it can be observed that for a given value of $\lambda$, the reliability of the estimator is ensured, as underlined by Theorem 6.1. Moreover, even if the value of $I_{eff}$ is not totally independent of $\lambda$, it remains in a reasonable range (in the order of 1).

For the second example, we set $n = 25$ in order to generate high gradients in the vicinity of the mid-point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consequently, instead of a uniform one, a local refinement mesh strategy is performed based on the local error estimator efficiency proven in Theorem 6.2. Here, this local refinement ensures that the mesh remains conform (there is no hanging-nodes creation), as well as regular (the minimal angle allowed in the triangulation is specified, in order to preserve the shape regularity of the mesh). We refer to [31] for further details and for the algorithm description (red, green and blue refinements). The resulting meshes obtained for iteration steps 1, 4 and 7 corresponding to $\lambda = 10^4$ are displayed in Figure 7.3, so that we can observe that the high gradients area is well tracked by the remeshing process. The convergence rates of the error and the estimator are plotted in Figure 7.4 and the effectivity index in Figure 7.5. Once again, the conclusions are the same than for the first example: experimental convergence rates of order 1 and robustness with respect to large values of $\lambda$, as well as only a small variation in $\lambda$ of the effectivity index (around 1) for all the considered values of $\lambda$.
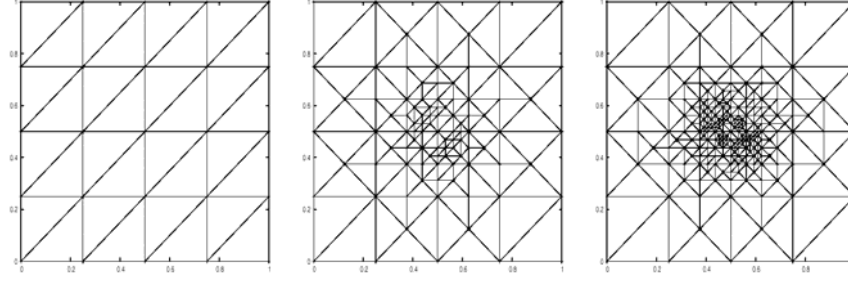
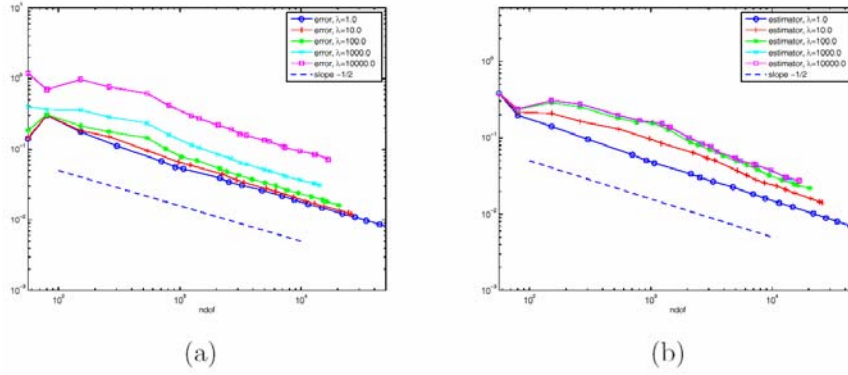**Figure 7.3.** Mesh refinements 1, 4 and 7 for the regular solution, $n = 25$, $\lambda = 10^4$.



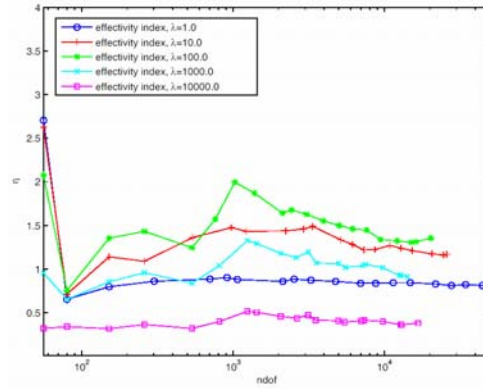**Figure 7.4.** Error (a) and estimator (b) convergence rates in *ndof* for the regular solution, $n = 25$, $\lambda$ from 1 to $10^4$.



**Figure 7.5.** Effectivity index $I_{eff}$ for the regular solution, $n = 25$, $\lambda$ from 1 to $10^4$.

## 7.2. Singular solution

This second test consists now in solving equation (2.1) on the domain $\Omega$ defined by:

$$\Omega = (-1, 1)^2 \backslash \{(x, y); -0.1x \leq y \leq 0.1x, 0 \leq x\},$$

with the external force $f$ chosen to be:

$$f(x, y) = -r^{-\frac{3}{4}}(1, 0), \quad r = \sqrt{x^2 + y^2}.$$

In that case, the exact solution exhibits a singular behaviour around the origin. We want to check if the mesh is correctly refined around the singularity when the estimator is used to carry out the refinement process, and in the same time if the global estimator decreases consequently towards zero. The resulting meshes obtained for iteration steps 1, 7 and 10 corresponding to $\lambda = 10^4$ are displayed in Figure 7.6, and we can see that the mesh is automatically refined around the singularity. Moreover, Figure 7.7 shows that the estimator converges towards zero, with experimental convergence rates of order 1.
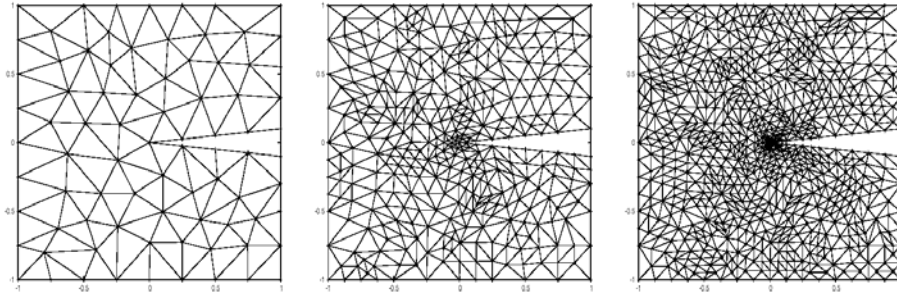


**Figure 7.6.** Mesh refinements 1, 7 and 10 for the singular solution, $\lambda = 10^4$.
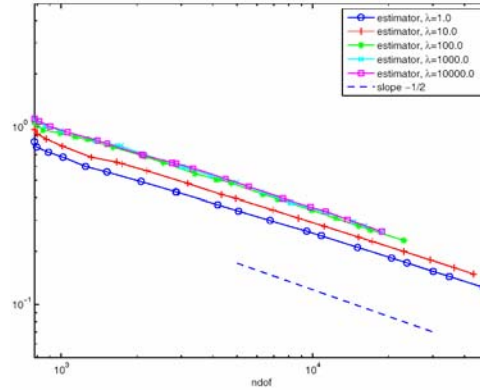
**Figure 7.7.** Estimator convergence rate in *ndof* for the singular solution, λ from 1 to $10^4$.

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