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THE SCRAMBLING INDEX OF PRIMITIVE TWO-COLORED TWO CYCLES WHOSE LENGTHS DIFFER BY 1

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Abstract

A two-colored digraph is a digraph each of whose arc is colored by red or blue. An (h, ℓ)-walk is a walk consisting of h red arcs and ℓ blue arcs. The scrambling index of a two-colored digraph is the smallest positive integer $h + \ell$ over all nonnegative integers h and ℓ such that for each pair of vertices u and v there is a vertex w such that there exist an (h, ℓ)-walk from u to w and an (h, ℓ)-walk from v to w. We study the scrambling index of primitive two-colored digraph consisting of two cycles whose lengths differ by 1. We present a lower bound and an upper bound for the scrambling index for such two-colored digraph. We then show that the lower and the upper bounds are sharp bounds.

1. Introduction

Let D be a digraph on n vertices $\{v_1, v_2, ..., v_n\}$. A walk of length ℓ

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from v_i to v_j is a sequence of arcs of the form $v_i = u_0 \to u_1, u_1 \to u_2, \ldots, u_{\ell-1} \to u_\ell = v_j$. A walk of length ℓ from v_i to v_j is denoted by $v_i \xrightarrow{\ell} v_j$ walk. A walk is open if $v_i \neq v_j$ and is closed otherwise. A $v_i \to v_j$ path is a walk with no repeated vertices, except possibly $v_i = v_j$. A cycle is a closed path. The distance from vertex v_i to vertex v_j, denoted by $d(v_i, v_j)$, is the length of the shortest $v_i \to v_j$ path. A digraph D is strongly connected provided for each ordered pair of vertices v_i and v_j there is a $v_i \to v_j$ walk. By a two cycles we mean a strongly connected digraph consisting of exactly two cycles.

A strongly connected digraph D is primitive provided there is a positive integer ℓ such that for each ordered pair of vertices v_i and v_j there exists a $v_i \xrightarrow{\ell} v_j$ walk. The smallest of such positive integer ℓ is the exponent of D. In 2009, Akelbek and Kirkland [1] introduced a new parameter on primitive digraph called *scrambling index*. The scrambling index of a primitive digraph is the smallest positive integer k such that for each pair of vertices v_i and v_j there exists a vertex v_t such that there are a $v_i \xrightarrow{k} v_t$ walk and a $v_j \xrightarrow{k} v_t$ walk in D. See [1, 2] for earliest discussion on the scrambling index of primitive digraphs.

A two-colored digraph $D^{(2)}$ is a digraph each of whose arcs is colored by either red or blue. An (h, ℓ)-walk in a two-colored digraph $D^{(2)}$ is a walk consisting of h red arcs and ℓ blue arcs. An (h, ℓ)-walk from v_i to v_j is also denoted by $v_i \xrightarrow{(h, \ell)} v_j$ walk. For a walk W in $D^{(2)}$ we denote $r(W)$ to be the number of red arcs in W and $b(W)$ to be the number of blue arcs in W. The length of W is $\ell(W) = r(W) + b(W)$. The vector $\begin{bmatrix} r(W) \\ b(W) \end{bmatrix}$ is the composition of the walk W.

The notions of primitivity and exponent of digraph have been generalized to that of two-colored digraph [3, 4]. A strongly connected two-colored digraph $D^{(2)}$ is primitive provided there exist nonnegative integers h and ℓ such that for each pair of ordered vertices v_i and v_j in $D^{(2)}$ there is a $v_i \overset{(h,\ell)}{\to} v_j$ walk in $D^{(2)}$. The smallest positive integer $h + \ell$ over all such nonnegative integers h and ℓ is called the *exponent* of $D^{(2)}$.

For a primitive two-colored digraph on n vertices $\{v_1, v_2, ..., v_n\}$, we define the *scrambling index* of $D^{(2)}$, denoted by $k(D^{(2)})$, to be the smallest positive integer $h + \ell$ over all nonnegative integers h and ℓ such that for each pair of vertices v_i and v_j in $D^{(2)}$ there is a vertex v_t with the property that there are a $v_i \overset{(h,\ell)}{\to} v_t$ walk and a $v_j \overset{(h,\ell)}{\to} v_t$ walk.

We discuss the scrambling index of primitive two-colored two cycles whose lengths $s + 1$ and s for some positive integer $s \geq 3$. In Section 2, we discuss primitivity of such two-colored two cycles. In Section 3, we present a way in setting up a lower bound and an upper bound for scrambling index. In Section 4, we present results on the scrambling index of two-colored two cycles whose lengths differ by 1.

2. Primitivity

Let $D^{(2)}$ be a strongly connected two-colored digraph and let the set of all cycles in $D^{(2)}$ be $C = \{C_1, C_2, ..., C_q\}$. We define a cycle matrix of $D^{(2)}$ to be a 2 by q matrix $M = \begin{bmatrix} r(C_1) & r(C_2) & \cdots & r(C_q) \\ b(C_1) & b(C_2) & \cdots & b(C_q) \end{bmatrix}$. If the rank of M is 1, the content of M is defined to be 0, and the content of M is defined to be the greatest common divisor of the determinants of 2 by 2 submatrices of M, otherwise. A two-colored digraph $D^{(2)}$ is primitive if and only if the content of M is 1 [3].

Let s and c be integers with $s \geq 3$ and $2 \leq c \leq s$. We discuss primitivity of two-colored two cycles on $n = 2s + 1 - c$ vertices $\{v_1, v_2, ..., v_n\}$ as shown in Figure 1. Let $C_1 : v_1 \to v_2 \to \cdots \to v_c \to v_{s+1} \to \cdots \to v_{n=2s+1-c} \to v_1$ be the cycle of length $s + 1$ and let $C_2 : v_1 \to v_2 \to \cdots \to v_c \to v_{c+1} \to \cdots \to v_s \to v_1$ be the cycle of length s. Notice that C_1 and C_2 have c vertices in common.

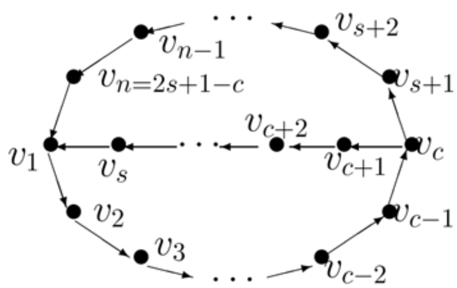

Figure 1. Two cycles whose lengths differ by 1.

Proposition 2.1. *Let $D^{(2)}$ be a strongly connected primitive two-colored two cycles of lengths $s + 1$ and s, respectively. The cycle matrix of $D^{(2)}$ is either of the form* $M = \begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix}$ *or* $M = \begin{bmatrix} 1 & 1 \\ s & s-1 \end{bmatrix}$.

Proof. The cycle matrix of $D^{(2)}$ is of the form $M = \begin{bmatrix} s+1-a & s-b \\ a & b \end{bmatrix}$ for some $0 \leq a \leq s + 1$ and $0 \leq b \leq s$. Since $D^{(2)}$ is primitive, $\det(M) = s(b - a) + b = \pm 1$. This implies $(b - a) \leq 0$. If $\det(M) = 1$, then $s(b - a) + b = 1$. Since $b \leq s$, we conclude that $b - a = 0$. Therefore in this case we have $b = a = 1$. If $\det(M) = -1$, then $s(b - a) + b = -1$. Since $b \leq s$, we conclude that $b - a = -1$. This implies $b = s - 1$ and $a = s$. Hence we now conclude that either $M = \begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix}$ or $M = \begin{bmatrix} 1 & 1 \\ s & s-1 \end{bmatrix}$. □

We assume without loss of generality that the cycle matrix of $D^{(2)}$ is the matrix $M = \begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix}$. Hence either $D^{(2)}$ has two blue arcs or $D^{(2)}$ has only one blue arc.

3. Lower and Upper Bounds

In this section, we discuss a way in setting up bounds for the scrambling index of two-colored two cycles.

Proposition 3.1. *Let* $D^{(2)}$ *be a primitive two-colored two cycles and let* P_{v_i, v_t} *be a* $v_i \to v_t$ *path that contains a vertex of both cycles. If for some nonnegative integers* h *and* ℓ *the system* $M\mathbf{z} + \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix} = \begin{bmatrix} h \\ \ell \end{bmatrix}$ *has nonnegative integer solution, then there is a* $v_i \overset{(h,\ell)}{\to} v_t$ *walk.*

Proof. Let $\mathbf{z} = (z_1, z_2)^T$ be the solution to the system and let v_q be a vertex in the path P_{v_i, v_t} that lies on both cycles. The walk that starts at v_i, moves to v_q along the path P_{v_i, v_q}, and moves z_1 and z_2 times around the cycles C_1 and C_2, respectively, and back at v_q and finally follows the path P_{v_q, v_t} to v_t is an (h, ℓ)-walk from v_i to v_t. □

For a vertex v_t in $D^{(2)}$, the local scrambling index of v_i and v_j at the vertex v_t, denoted by $k_{v_i, v_j}(v_t)$, is the smallest positive integer $h + \ell$ over all pairs of nonnegative integers h and ℓ such that there are a $v_i \overset{(h,\ell)}{\to} v_t$ walk and a $v_j \overset{(h,\ell)}{\to} v_t$ walk. The *local scrambling index* of vertices v_i and v_j in $D^{(2)}$, denoted $k_{v_i, v_j}(D^{(2)})$, is defined to be $k_{v_i, v_j}(D^{(2)}) = \min_{v_t \in V(D^{(2)})}\{k_{v_i, v_j}(v_t)\}$. Hence $\max_{v_i, v_j \in V(D^{(2)})}\{k_{v_i, v_j}(D^{(2)})\} \leq k(D^{(2)})$.

The following lemma is a basis for finding a lower bound for the scrambling index of two-colored two cycles.

Lemma 3.2. *Let* $D^{(2)}$ *be a primitive two-colored two cycles with cycle matrix M and let* $\det(M) = 1$. *Let* v_i *and* v_j *be any two distinct vertices in* $D^{(2)}$. *If* $k_{v_i, v_j}(v_t)$ *is obtained by an* (h, ℓ)*-walk, then*

$$\begin{bmatrix} h \\ \ell \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t}) \\ r(C_1)b(P_{v_j, v_t}) - b(C_1)r(P_{v_j, v_t}) \end{bmatrix},$$

and hence

$$k_{v_i, v_j}(v_t) \geq \ell(C_1)[b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t})]$$

$$+ \ell(C_2)[r(C_1)b(P_{v_j, v_t}) - b(C_1)r(P_{v_j, v_t})]$$

for some paths P_{v_i, v_t} *and* P_{v_j, v_t}.

Proof. Since $\det(M) = 1$, there are integers q_1 and q_2 such that $\begin{bmatrix} h \\ \ell \end{bmatrix} = M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. Since every walk can be decomposed into a path and some cycles, $\begin{bmatrix} h \\ \ell \end{bmatrix} = M\mathbf{z} + \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix}$ for some path P_{v_i, v_t} from v_i to v_t and some nonnegative integer vector $\mathbf{z}$. Comparing these equations we have $\mathbf{z} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - M^{-1} \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix} \geq 0$. Hence

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq M^{-1} \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix} = \begin{bmatrix} b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t}) \\ r(C_1)b(P_{v_i, v_t}) - b(C_1)r(P_{v_i, v_t}) \end{bmatrix}.$$

Thus $q_1 \geq b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t})$ for some path P_{v_i, v_t} from v_i to

v_t. Similarly, we have $q_2 \geq r(C_1)b(P_{v_j, v_t}) - b(C_1)r(P_{v_j, v_t})$ for some path P_{v_j, v_t} from v_j to v_t. If $k_{v_i, v_j}(v_t)$ is obtained by an (h, ℓ)-walk, then

$$\begin{bmatrix} h \\ \ell \end{bmatrix} = M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t}) \\ r(C_1)b(P_{v_j, v_t}) - b(C_1)r(P_{v_j, v_t}) \end{bmatrix}.$$

Thus

$$k_{v_i, v_j}(v_t) \geq \ell(C_1)[b(C_2)r(P_{v_i, v_t}) - r(C_2)b(P_{v_i, v_t})]$$
$$+ \ell(C_2)[r(C_1)b(P_{v_j, v_t}) - b(C_1)r(P_{v_j, v_t})]$$

for some paths P_{v_i, v_t} and P_{v_j, v_t}. □

4. Results

We begin with the case where $D^{(2)}$ has only one blue arc. Notice that the blue arc of $D^{(2)}$ must be of the form $v_a \to v_{a+1}$ for some $1 \leq a \leq c - 1$.

Lemma 4.1. *Let $D^{(2)}$ be a primitive two-colored two cycles with cycles of length $s + 1$ and s as shown in Figure 1. If $D^{(2)}$ has a unique blue arc $v_a \to v_{a+1}$, for some $1 \leq a \leq c - 1$, then $k(D^{(2)}) \geq s^2 - c + 1$.*

Proof. We assume that there are a $v_{a+1} \overset{(h, \ell)}{\to} v_t$ walk and a $v_a \overset{(h, \ell)}{\to} v_t$ walk for some vertex v_t in $D^{(2)}$. Let $e_1 = b(C_2)r(P_{v_{a+1}, v_t}) - r(C_2)b(P_{v_{a+1}, v_t})$ and $e_2 = r(C_1)b(P_{v_a, v_t}) - b(C_1)r(P_{v_a, v_t})$. We consider two cases depending on the position of the vertex v_t.

Case 1. The vertex v_t lies on the $v_1 \to v_a$ path. There are two paths P_{v_{a+1}, v_t} from v_{a+1} to v_t. They are an $(s - a + d(v_1, v_t), 0)$-path and an $(s - a + d(v_1, v_t) + 1, 0)$-path. Considering the $(s - a + d(v_1, v_t), 0)$-path we have

$$e_1 = b(C_2)r(P_{v_{a+1},v_t}) - r(C_2)b(P_{v_{a+1},v_t})$$
$$= (1)(s - a + d(v_1, v_t)) - (s-1)(0) = s - a + d(v_1, v_t).$$

Considering the $(s - a + d(v_1, v_t) + 1, 0)$-path we have $e_1 = s - a + d(v_1, v_t) + 1$. So we choose $e_1 = s - a + d(v_1, v_t)$.

There are two paths P_{v_a,v_t} from v_a to v_t. They are an $(s - a + d(v_1, v_t), 1)$-path and an $(s - a + d(v_1, v_t) + 1, 1)$-path. Considering the $(s - a + d(v_1, v_t), 1)$-path we have $e_2 = a - d(v_1, v_t)$. Considering the $(s - a + d(v_1, v_t) + 1, 1)$-path we have $e_2 = a - d(v_1, v_t) - 1$. So we choose $e_2 = a - d(v_1, v_t) - 1$.

By Lemma 3.2 we conclude that

$$k_{v_a,v_{a+1}}(v_t) \geq \ell(C_1)e_1 + \ell(C_2)e_2 = (s+1)e_1 + se_2 = s^2 - a + d(v_1, v_t) \quad (1)$$

for each vertex v_t that lies on the $v_1 \to v_a$ path.

Case 2. The vertex v_t lies on the $v_{a+1} \to v_n$ path or $v_{a+1} \to v_s$ path. There is a unique path P_{v_{a+1},v_t} from v_{a+1} to v_t which is a $(d(v_{a+1}, v_t), 0)$-path. Using this path we have $e_1 = d(v_{a+1}, v_t)$. There is a unique path P_{v_a,v_t} from v_a to v_t which is a $(d(v_{a+1}, v_t), 1)$-path. Using this path we have $e_2 = s - d(v_{a+1}, v_t)$. Since $d(v_{a+1}, v_t) = d(v_1, v_t) - a$, by Lemma 3.2 we conclude that

$$k_{v_a,v_{a+1}}(v_t) \geq (s+1)e_1 + se_2 = s^2 - a + d(v_1, v_t) \qquad (2)$$

for each vertex v_t that lies on the $v_{a+1} \to v_n$ path or $v_{a+1} \to v_s$ path.

From (1) and (2), we conclude that $k_{v_a,v_{a+1}}(D^{(2)}) \geq s^2 - a$ and hence $k(D^{(2)}) \geq s^2 - a$. Since $1 \leq a \leq c - 1$, $k(D^{(2)}) \geq s^2 - c + 1$. $\square$

We next discuss the case where $D^{(2)}$ has two blue arcs. We first consider the case where the blue arcs have the same initial vertex and then discuss the case where the blue arcs have different initial vertices.

Lemma 4.2. *Let $D^{(2)}$ be a primitive two-colored two cycles with cycles of lengths $s+1$ and s as shown in Figure 1. If $D^{(2)}$ has two blue arcs $v_c \to v_{c+1}$ and $v_c \to v_{s+1}$, then $k(D^{(2)}) \geq s^2 + (s+1-c)$.*

Proof. Suppose there are a $v_c \overset{(h,\ell)}{\to} v_t$ walk and a $v_{s+1} \overset{(h,\ell)}{\to} v_t$ walk. Define $e_1 = b(C_2)r(P_{v_{s+1},v_t}) - r(C_2)b(P_{v_{s+1},v_t})$ and $e_2 = r(C_1)b(P_{v_c,v_t}) - b(C_1)r(P_{v_c,v_t})$. We consider three cases.

Case 1. The vertex v_t lies on the $v_1 \to v_c$ path. There is a unique P_{v_{s+1},v_t} path from v_{s+1} to v_t which is an $(s+1-c+d(v_1, v_t), 0)$-path. Using this path we find that $e_1 = s+1-c+d(v_1, v_t)$. There are two paths P_{v_c,v_t} from v_c to v_t. They are an $(s+1-c+d(v_1, v_t), 1)$-path and an $(s-c+d(v_1, v_t), 1)$-path. Considering the $(s+1-c+d(v_1, v_t), 1)$-path we have $e_2 = c-1-d(v_1, v_t)$. Considering the $(s-c+d(v_1, v_t), 1)$-path we have $e_2 = c-d(v_1, v_t)$. Thus we conclude that $e_2 = c-1-d(v_1, v_t)$. Lemma 3.2 implies that

$$k_{v_{s+1},v_c}(v_t) \geq (s+1)e_1 + se_2 = s^2 + s + 1 - c + d(v_1, v_t) \tag{3}$$

for each vertex v_t that lies on the $v_1 \to v_c$ path.

Case 2. The vertex v_t lies on $v_{c+1} \to v_s$ path. There is a unique P_{v_{s+1},v_t} path from v_{s+1} to v_t which is an $(s+d(v_{c+1}, v_t), 1)$-path. Using this path we find that $e_1 = d(v_{c+1}, v_t)+1$. There is a unique path P_{v_c,v_t} from v_c to v_t which is a $(d(v_{c+1}, v_t), 1)$-path. Using this path we have that $e_2 = s - d(v_{c+1}, v_t)$. Since $d(v_{c+1}, v_t) = d(v_1, v_t) - c$, Lemma 3.2 implies that

$$k_{v_{s+1}, v_c}(v_t) \geq (s+1)e_1 + se_2 = s^2 + s + 1 - c + d(v_1, v_t) \tag{4}$$

for each vertex v_t that lies on the $v_{c+1} \to v_s$ path.

Case 3. The vertex v_t lies on $v_{s+1} \to v_n$ path. There is a unique path P_{v_{s+1}, v_t} from v_{s+1} to v_t which is an $(s+1-c-d(v_t, v_1), 0)$-path. Using this path we find that $e_1 = s+1-c-d(v_t, v_1)$. There is a unique path from v_c to v_t which is an $(s+1-c-d(v_t, v_1), 1)$-path. Using this path we find that $e_2 = c-1+d(v_t, v_1)$. By Lemma 3.2 we have

$$\begin{bmatrix} h \\ \ell \end{bmatrix} \geq M \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} s^2 + 1 - c - d(v_t, v_1) \\ s \end{bmatrix}.$$

We consider $v_{s+1} \to v_t$ walk. Since the path P_{v_{s+1}, v_t} is an $(s+1-c-d(v_t, v_1), 0)$-path, the solution to the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_{s+1}, v_t}) \\ b(P_{v_{s+1}, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 + 1 - c - d(v_t, v_1) \\ s \end{bmatrix}$$

is $z_1 = 0$ and $z_2 = s$. This implies there is no $(s^2 + 1 - c - d(v_t, v_1), s)$-walk from v_{s+1} to v_t. We note that the shortest $v_{s+1} \to v_t$ walk that contains at least $s^2 + 1 - c - d(v_t, v_1)$ red arcs and at least s blue arcs is an $(s^2 + s + 1 - c - d(v_t, v_1), s+1)$-walk. This implies $k_{v_{s+1}, v_c}(v_t) \geq s^2 + s + 1 - c + (s+1-d(v_t, v_1))$. Since v_t lies on C_1, $d(v_1, v_t) = s+1-d(v_t, v_1)$. Therefore

$$k_{v_{s+1}, v_c}(v_t) \geq s^2 + s + 1 - c + d(v_1, v_t) \tag{5}$$

for each vertex v_t that lies on $v_{s+1} \to v_n$ path.

From (3), (4) and (5) we conclude that $k_{v_{s+1}, v_c}(D^{(2)}) \geq s^2 + s + 1 - c$. Hence $k(D^{(2)}) \geq s^2 + s + 1 - c$. □

We now discuss the case where $D^{(2)}$ has two blue arcs with different initial vertices.

Lemma 4.3. *Let* $D^{(2)}$ *be a primitive two-colored two cycles with cycles of lengths* $s+1$ *and* s *as shown in Figure* 1. *If* $D^{(2)}$ *has two blue arcs* $v_y \to v_{y+1}$ *and* $v_x \to v_{x+1}$ *for some* $c+1 \le x \le s$ *and* $s+1 \le y \le n$, *then*

$$k(D^{(2)}) \ge s^2 + |\, d(v_{y+1}, v_1) - d(v_{x+1}, v_1)\,|s + \max\{d(v_{y+1}, v_1), d(v_{x+1}, v_1)\}.$$

Proof. For simplicity, we define $d_1 = d(v_{y+1}, v_1)$ and $d_2 = d(v_{x+1}, v_1)$. We consider two cases where $d(v_{y+1}, v_1) > d(v_{x+1}, v_1)$ and $d(v_{y+1}, v_1) \le d(v_{x+1}, v_1)$.

Case 1. $d(v_{y+1}, v_1) > d(v_{x+1}, v_1)$. Suppose there are $v_{y+1} \overset{(h,\ell)}{\to} v_t$ walk and $v_x \overset{(h,\ell)}{\to} v_t$ walk for some vertex v_t in $D^{(2)}$. Let $e_1 = b(C_2)r(P_{v_{y+1}, v_t}) - r(C_2)b(P_{v_{y+1}, v_t})$ and $e_2 = r(C_1)b(P_{v_x, v_t}) - b(C_1)r(P_{v_x, v_t})$. We shall show that $k(D^{(2)}) \ge s^2 + (d_1 - d_2)s + d_1$. We consider three subcases depending on the position of the vertex v_t.

Subcase 1a. The vertex v_t lies on the $v_1 \to v_y$ path or $v_1 \to v_x$ path. There is a unique P_{v_{y+1}, v_t} path from v_{y+1} to v_t which is a $(d_1 + d(v_1, v_t), 0)$-path. Using this path we have $e_1 = d_1 + d(v_1, v_t)$. There is a unique path P_{v_x, v_t} from v_x to v_t which is a $(d_2 + d(v_1, v_t), 1)$-path. Using this path we have $e_2 = s - d_2 - d(v_1, v_t)$. Lemma 3.2 implies

$$k_{v_{y+1}, v_x}(v_t) \ge (s+1)e_1 + se_2 = s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t) \quad (6)$$

for each vertex v_t that lies on the $v_1 \to v_y$ path or $v_1 \to v_x$ path.

Subcase 1b. The vertex v_t lies on $v_{y+1} \to v_n$ path. There is a unique path P_{v_{y+1}, v_t} from v_{y+1} to v_t which is a $(d_1 - d(v_t, v_1), 0)$-path. Using this path we have $e_1 = d_1 - d(v_t, v_1)$. There is a unique path P_{v_x, v_t} from v_x to v_t which is a $(d_2 + s - d(v_t, v_1), 2)$-path. Using this path, we have $e_2 = s - d_2 + d(v_t, v_1)$. From Lemma 3.2, we have

$$\begin{bmatrix} h \\ \ell \end{bmatrix} \geq M \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1) \\ s + d_1 - d_2 \end{bmatrix}.$$

We consider walk from the vertex v_{y+1} to v_t. We note that the path P_{v_{y+1}, v_t} from v_{y+1} to v_t is a $(d_1 - d(v_t, v_1), 0)$-path. This implies the solution to the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_{y+1}, v_t}) \\ b(P_{v_{y+1}, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1) \\ s + d_1 - d_2 \end{bmatrix}$$

is $z_1 = 0$ and $z_2 = s + d_1 - d_2$. Since the path P_{v_{y+1}, v_t} lies entirely on C_1, there is no walk from v_{y+1} to v_t that consists of $s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1)$ red arcs and $s + d_1 - d_2$ blue arcs. Notice that the shortest walk from v_{y+1} to v_t that contains at least $s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1)$ red arcs and at least $s + d_1 - d_2$ blue arcs is the walk with $s^2 + (d_1 - d_2)s + d_2 - d(v_t, v_1)$ red arcs and $s + d_1 - d_2 + 1$ blue arcs. This implies $k_{v_{y+1}, v_x}(v_t) \geq s^2 + (d_1 - d_2)s + d_1 + s + 1 - d(v_t, v_1)$. Since v_t lies on C_1, we have $d(v_t, v_1) = s + 1 - d(v_1, v_t)$. Therefore

$$k_{v_{y+1}, v_x}(v_t) \geq s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t) \tag{7}$$

for each vertex v_t that lies on the $v_{y+1} \to v_n$ path.

Subcase 1c. The vertex v_t lies on $v_{x+1} \to v_s$ path. There is a unique path from v_{y+1} to v_t which is a $(d_1 + s - 1 - d(v_t, v_1), 1)$-path. Considering this path, we have $e_1 = d_1 - d(v_t, v_1)$. There is a unique path from v_x to v_t which is a $(d_2 - d(v_t, v_1), 1)$-path. Considering this path we have $e_2 = s - d_2 + d(v_t, v_1)$. From Lemma 3.2, we have

$$\begin{bmatrix} h \\ \ell \end{bmatrix} \geq M \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1) \\ s + d_1 - d_2 \end{bmatrix}.$$

We consider walks from v_x to v_t. We note that the path P_{v_x, v_t} from v_x to v_t is a $(d_2 - d(v_t, v_1), 1)$-path. This implies the solution to the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_x, v_t}) \\ b(P_{v_x, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1) \\ s + d_1 - d_2 \end{bmatrix}$$

is $z_1 = s + d_1 - d_2 - 1$ and $z_2 = 0$. Since the path P_{v_x, v_t} lies entirely on the cycle C_2, there is no walk from v_x to v_t that consists of $s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1)$ red arcs and $s + d_1 - d_2$ blue arcs. Notice that the shortest walk from v_x to v_t that contains at least $s^2 + (d_1 - d_2 - 1)s + d_2 - d(v_t, v_1)$ red arcs and at least $s + d_1 - d_2$ blue arcs is the walk with $s^2 + (d_1 - d_2)s + d_2 - d(v_t, v_1) - 1$ red arcs and $s + d_1 - d_2 + 1$ blue arcs. This implies $k_{v_{y+1}, v_x}(v_t) \geq s^2 + (d_1 - d_2)s + d_1 + s - d(v_t, v_1)$. Since v_t lies on C_2, we have $d(v_t, v_1) = s - d(v_1, v_t)$. Therefore

$$k_{v_{y+1}, v_x}(v_t) \geq s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t) \tag{8}$$

for each vertex v_t that lies on the $v_{x+1} \to v_s$ path.

From (6), (7) and (8) we conclude that $k_{v_{y+1}, v_x}(D^{(2)}) \geq s^2 + (d_1 - d_2)s + d_1$ and hence

$$k(D^{(2)}) \geq s^2 + (d_1 - d_2)s + d_1 \tag{9}$$

whenever $d_1 > d_2$.

Case 2. $d(v_{y+1}, v_1) \le d(v_{x+1}, v_1)$. Suppose there are a $v_y \overset{(h,\ell)}{\to} v_t$ walk and a $v_{x+1} \overset{(h,\ell)}{\to} v_t$ walk for some vertex v_t in $D^{(2)}$. Let $e_1 = b(C_2)r(P_{v_{x+1}, v_t}) - r(C_2)b(P_{v_{x+1}, v_t})$ and $e_2 = r(C_1)b(P_{v_y, v_t}) - b(C_1)r(P_{v_y, v_t})$. We shall show that $k(D^{(2)}) \ge s^2 + (d_2 - d_1)s + d_2$. We consider three subcases depending on the position of the vertex v_t.

Subcase 2a. The vertex v_t lies on the $v_1 \to v_y$ path or $v_1 \to v_x$ path. There is a unique path P_{v_{x+1}, v_t} from v_{x+1} to v_t which is a $(d_2 + d(v_1, v_t), 0)$-path. Using this path we find that $e_1 = d_2 + d(v_1, v_t)$. There is a unique path P_{v_y, v_t} from v_y to v_t which is a $(d_1 + d(v_1, v_t), 1)$-path. Using this path we find that $e_2 = s - d_1 - d(v_1, v_t)$. From Lemma 3.2 we have

$$k_{v_y, v_{x+1}}(v_t) \ge (s+1)e_1 + se_2 = s^2 + (d_2 - d_1)s + d_2 + d(v_1, v_t) \quad (10)$$

for each vertex v_t that lies on the $v_1 \to v_y$ path or $v_1 \to v_x$ path.

Subcase 2b. The vertex v_t lies on $v_{y+1} \to v_n$ path. There is a unique path P_{v_{x+1}, v_t} from v_{x+1} to v_t which is a $(d_2 + s - d(v_t, v_1), 1)$-path. Using this path we find that $e_1 = d_2 + 1 - d(v_t, v_1)$. There is a unique path P_{v_y, v_t} from v_y to v_t which is a $(d_1 - d(v_t, v_1), 1)$-path. Using this path we find that $e_2 = s - d_1 + d(v_t, v_1)$. Since v_t lies on C_1, we have $s + 1 - d(v_t, v_1) = d(v_1, v_t)$. From Lemma 3.2, we have

$$\begin{aligned} k_{v_y, v_{x+1}}(v_t) &\ge (s+1)e_1 + se_2 = s^2 + (d_2 - d_1)s + d_2 + (s+1) - d(v_t, v_1) \\ &= s^2 + (d_2 - d_1)s + d_2 + d(v_1, v_t) \qquad (11) \end{aligned}$$

for each vertex v_t that lies on the $v_{y+1} \to v_n$ path.

Subcase 2c. The vertex v_t lies on $v_{x+1} \to v_s$ path. There is a unique path P_{v_{x+1}, v_t} from v_{x+1} to v_t which is a $(d_2 - d(v_t, v_1), 0)$-path. Using this path we find that $e_1 = d_2 - d(v_t, v_1)$. There is a unique path P_{v_y, v_t} from v_y to v_t which is a $(d_1 + s - 1 - d(v_t, v_1), 2)$-path. Using this path we find that $e_2 = s - d_1 + 1 + d(v_t, v_1)$. Since v_t lies on C_2, we have $s - d(v_t, v_1) = d(v_1, v_t)$. From Lemma 3.2 we have

$$k_{v_y, v_{x+1}}(v_t) \geq (s+1)e_1 + se_2 = s^2 + (d_2 - d_1)s + d_2 + s - d(v_t, v_1)$$

$$= s^2 + (d_2 - d_1)s + d_2 + d(v_1, v_t) \tag{12}$$

for each vertex v_t that lies on the $v_{x+1} \to v_s$ path.

From (10), (11) and (12), we conclude that $k_{v_{y+1}, v_x}(D^{(2)}) \geq s^2 + (d_2 - d_1)s + d_2$ and hence

$$k(D^{(2)}) \geq s^2 + (d_2 - d_1)s + d_2 \tag{13}$$

whenever $d_2 \geq d_1$.

Finally, from (9) and (13), we conclude that $k(D^{(2)}) \geq s^2 + |d_1 - d_2|s + \max\{d_1, d_2\}$. □

Theorem 4.4. *Let $D^{(2)}$ be a primitive two-colored two cycles with cycles of lengths $s+1$ and s as shown in Figure 1. If $c < s$, then $s^2 - c + 1 \leq k(D^{(2)}) \leq s^2 + (s+1-c)s + s + 1 - c$.*

Proof. From Lemma 4.1, Lemma 4.2 and Lemma 4.3, we conclude that $k(D^{(2)}) \geq s^2 - c + 1$. We next show that $k(D^{(2)}) \leq s^2 + (s+1-c)s + s + 1 - c$. We shall show that there exists a vertex v_t in $D^{(2)}$ such that for each vertex v_i, $i = 1, 2, ..., n$, there is a $(2s^2 - cs, 2s + 1 - c)$-walk from v_i to

v_t. We shall show the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix} = \begin{bmatrix} 2s^2 - cs \\ 2s + 1 - c \end{bmatrix} \tag{14}$$

has a nonnegative integer solution for some path P_{v_i, v_t} from v_i to v_t. The solution to the system (14) is $z_1 = s + 1 - c - r(P_{v_i, v_t}) + (s-1)b(P_{v_i, v_t})$ and $z_2 = r(P_{v_i, v_t}) + (1 - b(P_{v_i, v_t}))s$.

We first consider the case where $D^{(2)}$ has a unique blue arc. Let v_t to be the terminal vertex of the blue arc. Then for any path P_{v_i, v_t} we have $b(P_{v_i, v_t}) = 1$. Using this path we have $z_1 = 2s - c - r(P_{v_i, v_t})$ and $z_2 = r(P_{v_i, v_t}) \geq 0$. Since $r(P_{v_i, v_t}) \leq s$ and $c < s$ we have $z_1 \geq 0$.

We next consider the case where $D^{(2)}$ has two blue arcs and let $v_t = v_1$. If $b(P_{v_i, v_1}) = 0$, then v_i lies on the $v_{x+1} \to v_1$ path or $v_{y+1} \to v_1$ path. This implies $r(P_{v_i, v_1}) \leq s + 1 - c$ and hence $z_1 \geq 0$ and $z_2 \geq r(P_{v_i, v_1}) + s \geq s$. If $b(P_{v_i, v_1}) = 1$, v_i lies on the $v_2 \to v_x$ path or $v_2 \to v_y$ path, then $z_2 \geq r(P_{v_i, v_1}) \geq 1$ and $z_1 = 2s - c - r(P_{v_i, v_1})$. We note in this case that $r(P_{v_i, v_1}) \leq s$ and $c < s$, thus $z_1 \geq 0$.

For each vertex v_i, $i = 1, 2, \ldots, n$, there is a vertex v_t such that the system (14) has a nonnegative integer solution for some path P_{v_i, v_t}. Proposition 3.1 guarantees that for each vertex v_i, $i = 1, 2, \ldots, n$, there is a $(2s^2 - cs, 2s + 1 - c)$-walk from v_i to v_t. Thus $k(D^{(2)}) \leq s^2 + (s + 1 - c)s + s + 1 - c$. □

We note that the bounds given in Theorem 4.4 are sharp bounds as shown in the following corollaries.

Corollary 4.5. *Let* $D^{(2)}$ *be a primitive two-colored two cycles with cycles of lengths* $s+1$ *and s as shown in Figure* 1. *If* $D^{(2)}$ *has a unique blue arc* $v_{c-1} \to v_c$, *then* $k(D^{(2)}) = s^2 - c + 1$.

Proof. By Lemma 4.1, $k(D^{(2)}) \geq s^2 - c + 1$. It remains to show that $k(D^{(2)}) \leq s^2 - c + 1$. We show that for each $i = 1, 2, \ldots, n$, the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_i, v_1}) \\ b(P_{v_i, v_1}) \end{bmatrix} = \begin{bmatrix} s^2 - s - c + 2 \\ s - 1 \end{bmatrix} \tag{15}$$

has nonnegative integer solution for some path P_{v_i, v_1} from v_i to v_1. The solution to the system (15) is $z_1 = s - c + 1 + sb(P_{v_i, v_1}) - r(P_{v_i, v_1}) - b(P_{v_i, v_1})$ and $z_2 = c - 2 + r(P_{v_i, v_1}) - sb(P_{v_i, v_1})$.

If the vertex v_i lies on the $v_1 \to v_{c-1}$ path, then there is an $(s - d(v_1, v_i), 1)$-path from v_i to v_1. Using this path we have $z_1 = s - c + d(v_1, v_i)$ and $z_2 = c - 2 - d(v_1, v_i)$. Since $s \geq c$ we have $z_1 \geq 0$. Since $d(v_1, v_i) \leq c - 2$ we have $z_2 \geq 0$.

If the vertex v_i lies on the $v_c \to v_s$ path, then there is an $(s - d(v_1, v_i), 0)$-path from v_i to v_1. Using this path we have $z_1 = d(v_1, v_i) - c + 1$ and $z_2 = s + c - 2 - d(v_1, v_i)$. Since $d(v_1, v_i) \geq c - 1$ we have $z_1 \geq 0$. Since $d(v_1, v_i) \leq s - 1$ we have $z_2 \geq c - 1$.

If v_i lies on the $v_{s+1} \to v_n$ path, then there is an $(s + 1 - d(v_1, v_i), 0)$-path from v_i to v_1. Using this path we have $z_1 = d(v_1, v_i) - c$ and $z_2 = s + c - 1 - d(v_1, v_i)$. Since v_i lies on the $v_{s+1} \to v_n$ path, $d(v_1, v_i) \geq c$. Hence $z_1 \geq 0$. Since $s \geq d(v_1, v_i)$ we have $z_2 \geq c - 1$.

Therefore for each vertex v_i, $i = 1, 2, \ldots, n$, there is a path P_{v_i, v_1} from v_i to v_1 such that the system (15) has a nonnegative integer solution. Proposition

3.1 guarantees that for each v_i, $i = 1, 2, ..., n$, there is a $v_i \overset{(h,\ell)}{\to} v_1$ walk with $h = s^2 - s - c + 2$ and $\ell = s - 1$. Thus $k(D^{(2)}) \leq s^2 - c + 1$.

We now conclude that $k(D^{(2)}) = s^2 - c + 1$. □

Corollary 4.6. *Let $D^{(2)}$ be a primitive two-colored two cycles with cycles of lengths $s + 1$ and s as shown in Figure* 1. *If $D^{(2)}$ has two blue arcs $v_c \to v_{s+1}$ and $v_s \to v_1$ and $c < s$, then $k(D^{(2)}) = s^2 + (s + 1 - c)s + s + 1 - c$.*

Proof. By Theorem 4.4, $k(D^{(2)}) \leq s^2 + (s + 1 - c)s + s + 1 - c$. Note that this is a special case of Lemma 4.3 with $d_1 = s + 1 - c$ and $d_2 = 0$. Case 1 of the proof of Lemma 4.3 guarantees that $k(D^{(2)}) \geq s^2 + (s + 1 - c)s + s + 1 - c$. Therefore, $k(D^{(2)}) = s^2 + (s + 1 - c)s + s + 1 - c$. □

Theorem 4.7. *Let $D^{(2)}$ be a primitive two-colored two cycles with cycles of lengths $s + 1$ and s as shown in Figure* 1. *If $c = s$, then $s^2 - s + 1 \leq k(D^{(2)}) \leq s^2 + 1$.*

Proof. From Lemma 4.1, Lemma 4.2 and Lemma 4.3, we conclude that $k(D^{(2)}) \geq s^2 - c + 1 = s^2 - s + 1$. It remains to show that $k(D^{(2)}) \leq s^2 + 1$. We shall show that there exists a vertex v_t in $D^{(2)}$ such that for each vertex v_i, $i = 1, 2, ..., n$, there is an $(s^2 + s - 1, s)$-walk from v_i to v_t.

It suffices to show that there is a vertex v_t in $D^{(2)}$ such that for each $i = 1, 2, ..., n$, the system

$$M\mathbf{z} + \begin{bmatrix} r(P_{v_i, v_t}) \\ b(P_{v_i, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 - s + 1 \\ s \end{bmatrix} \tag{16}$$

has a nonnegative integer solution for some path P_{v_i, v_t} from v_i to v_t. The solution to the system (16) is $z_1 = (1 - r(P_{v_i, v_t})) + b(P_{v_i, v_i})(s - 1)$ and $z_2 = s(1 - b(P_{v_i, v_t})) + r(P_{v_i, v_i}) - 1$.

Assume $D^{(2)}$ has only one blue arc and let v_t to be the vertex v_{a+2} for some $1 \leq a \leq c - 1$. Then for each $i = 1, 2, ..., n$, $i \neq a + 1$, there is an $(r(P_{v_i, v_t}), 1)$-path from v_i to v_t with $r(P_{v_i, v_t}) \geq 1$. Using this path we find that $z_1 = s - r(P_{v_i, v_t}) \geq 0$ and $z_2 = r(P_{v_i, v_t}) - 1 \geq 0$. If $i = a + 1$, then there is a $(1, 0)$-path from v_i to v_t. Using this path we find that $z_1 = 0$ and $z_2 = s$.

We now assume $D^{(2)}$ has two blue arcs. Since $c = s$, we have $n = s + 1$. Therefore, there are two possibilities for the two blue arcs of $D^{(2)}$. They are either the arcs $v_s \to v_1$ and $v_s \to v_{s+1}$ or the arcs $v_s \to v_1$ and $v_{s+1} \to v_1$.

If $v_s \to v_{s+1}$ is blue, we let v_t to be v_1. For each $i = 1, 2, ..., n - 1$, there is an $(r(P_{v_i, v_1}), 1)$-path from v_i to v_1 with $1 \leq r(P_{v_i, v_1}) \leq s$. Using this path we find that $z_1 = s - r(P_{v_i, v_1}) \geq 0$ and $z_2 = r(P_{v_i, v_1}) - 1 \geq 0$. We note that there is a $(1, 0)$-path from v_{s+1} to v_1. Using this path we find that $z_1 = 0$ and $z_2 = s$.

If $v_{s+1} \to v_1$ is a blue arc, we let v_t to be the vertex v_2. For each $i = 2, ..., n$, there is an $(r(P_{v_i, v_2}), 1)$-path from v_i to v_2 with $1 \leq r(P_{v_i, v_2}) \leq s - 1$. Using this path we find that $z_1 = s - r(P_{v_i, v_2}) \geq 1$ and $z_2 = r(P_{v_i, v_1}) - 1 \geq 0$. We note that there is a $(1, 0)$-path from v_1 to v_2. Using this path, we find that $z_1 = 0$ and $z_2 = s$.

Therefore, there is a vertex v_t in $D^{(2)}$ such that for each $i = 1, 2, ..., n$, the system (16) has a nonnegative integer solution for some path P_{v_i, v_t}. This implies for each $i = 1, 2, ..., n$ there exists a vertex v_t in $D^{(2)}$ such that there is an $(s^2 - s + 1, s)$-walk from v_i to v_t. Hence $k(D^{(2)}) \leq s^2 + 1$. $\square$

We note that the bounds given in Theorem 4.7 are sharp bounds. From Corollary 4.5 the lower bound is achieved if the two-colored two cycles $D^{(2)}$ has a unique blue arc $v_{s-1} \to v_s$. The upper bound is achieved if the two-colored two cycles $D^{(2)}$ has two blue arcs with the same initial vertex $v_s \to v_1$ and $v_s \to v_{s+1=n}$. Lemma 4.2 guarantees that $k(D^{(2)}) \geq s^2 + s + 1 - c = s^2 + 1$. Combining this and Theorem 4.7, we have $k(D^{(2)}) = s^2 + 1$.

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