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LEFT DERIVATIONS OF INCLINE ALGEBRAS AND RELATED SET

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Abstract

In this paper, we introduce the notion of left derivations of incline algebras, and prove some results on left derivations of an incline and integral incline. Also, we define the fixed set of left derivations on incline and show that it is a subincline and the fixed set of left derivations on incline is an ideal, principal ideal and model.

1. Introduction

Cao et al. [6] introduced the notion of incline algebras in their book, incline algebra and applications, and was studied by some authors (see [1, 2, 6, 7]). Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations. The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic system. Several authors (see [3, 4, 9, 13]) studied derivations in rings and near-rings. Jun and Xin [8]

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applied the notion of derivation in ring and near-ring theory to *BCI*-algebras. In 2010, Alshehri [3] introduced the notion of derivation in incline and discussed some properties of derivation in incline. In this paper, in Section 3, we introduce the concept of left derivations for an incline and investigate some of its properties. We show that if d is a monomorphic left derivation of an integral incline K, and $a \in K$ such that $a * dx = 0$ or $dx * a = 0$ for all $x \in K$, then either $a = 0$ or d is *zero*, and we prove that if $d^2 = 0$, then d is *zero*. Also, we show that if d_1, d_2 are two monomorphic left derivations of an incline, and $d_1 d_2 = 0$, then $d_2 d_1$ is a monomorphic left derivation of K. Also, we prove that if M is a nonzero ideal of an integral incline K, and d is a nonzero left derivation of K, then d is nonzero on M. In Section 4, we study the fixed set of a left derivation in incline. We denote it $\mathrm{Fix}_d(K)$. We show that $\mathrm{Fix}_d(K)$ is a subincline. Also, we prove that $\mathrm{Fix}_d(K)$ is an ideal, principal ideal and model by consider some conditions on the incline or the left derivation.

2. Preliminaries

An incline (algebra) is a set K with two binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in K$:

$(K1)$ $x + y = y + x$,

$(K2)$ $x + (y + z) = (x + y) + z$,

$(K3)$ $x * (y * z) = (x * y) * z$,

$(K4)$ $x * (y + z) = (x * y) + (x * z)$,

$(K5)$ $(y + z) * z = (y * x) + (z * x)$,

$(K6)$ $x + x = x$,

$(K7)$ $x + (x * y) = x$,

$(K8)$ $y + (x * y) = y$.

Furthermore, an incline algebra K is said to be *commutative* if $x * y + y * x$ for all $x, y \in K$. For convenience, we pronounce + (resp. $*$) as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice if and only if $x * x = x$ for all $x \in K$.

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in K$.

If $x \leq y$, then y is said to *dominate* x.

It is easy to see that ≤ is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$.

We say that ≤ is induced by operation +. It follows that:

(1) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$;

(2) $x + y \geq x$ and $x + y \geq y$ for all $x, y \in K$;

(3) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in K$;

(4) if $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

A subincline of an incline K is a non-empty subset M of K which is closed under addition and multiplication. A subincline M is said to be an *ideal* of an incline K if $x \in M$ and $y \leq x$, then $y \in M$. A proper ideal P of an incline K is said to be *prime* if for all $x, y \in K$, $x * y \in P$ implies either $x \in P$ or $y \in P$. An element 0 in an incline algebra K is a *zero* element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$, for any $x \in K$. An element e (≠ zero element) in an incline algebra K is called a *multiplicative identity* if for any $x \in K$, $x * e = e * x = x$. A *nonzero* element a in an incline algebra K with *zero* element is said to be a *left* (resp. *right*) *zero divisor* if there exists a *nonzero* $b \in K$ such that $a * b = 0$ (resp. $b * a = 0$). A *zero* divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline K with multiplicative identity e and *zero* element 0 is called an *integral incline* if it has no zero divisors. For $a \in K$, define a map $d_a : K \to K$ where $d_a(x) = x * a$ for all $x \in K$.

3. Left Derivations of Incline Algebras

Definition 3.1. Let K be an incline and $d : K \to K$ be a function. We call d a *left derivation* of K, if it satisfies the following condition for all $x, y \in K$:

$$d(x * y) = (x * dy) + (y * dx).$$

We often abbreviate $d(x)$ to dx.

Example 3.2. If $K = \{a, b, c, d\}$, and we define the sum "+" and product "$*$" on K as:

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	b
d	d	b	b	d

Table 1

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

Table 2

Then $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by:

$$d(x) = \begin{cases} a & \text{if } x = a, b, d \\ c & \text{if } x = c. \end{cases}$$

Then we can see that d is a left derivation of K.

Example 3.3. If $K = \{a, b, c, d, f\}$, and we define the sum "+" and product "$*$" on K as:

$+$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
f	a	b	c	d	f

Table 1

$*$	a	b	c	d	f
a	a	b	c	d	f
b	b	b	d	d	f
c	c	f	c	f	f
d	d	f	d	f	f
f	f	f	f	f	f

Table 2

By computation, one can easily see that K is a noncommutative incline.

Define a map $d : K \to K$ by:

$$d(x) = \begin{cases} b & \text{if } x = a, b \\ f & \text{if otherwise.} \end{cases}$$

Then we can see that d is a left derivation of K.

Throughout our work, d will consider an additive map on K, i.e.,

$$d(x + y) = dx + dy \text{ for all } x, y \in K.$$

Proposition 3.4. *Let K be an incline and d be a left derivation of K. Then the following hold for all* $x, y \in K$:

(i) $d(x * y) \leq dx + dy.$

(ii) *If* $x \leq y$, *then* $d(x * y) \leq y.$

(iii) *If K is a distributive lattice, then* $dx \leq x.$

Proof. (i) $d(x * y) = (x * dy) + (y * dx)$, from (1), $x * dy \leq dy$ and $y * dx \leq dx$.

Then by (4): $d(x * y) = (x * dy) + (y * dx) \leq dx + dy.$

(ii) If $x \leq y$, from (1) and (3), we get $x * dy \leq y * dy \leq y$. Also, $y * dy \leq y$, then $d(x * y) = (x * dy) + (y * dx) \leq y + y = y.$

(iii) If K is a distributive lattice, then $dx = d(x * x) = (x * dx) + (x * dx) = x * dx$, from ($K6$) and so, $dx = x * dx \leq x$, from (1).

Proposition 3.5. *Let K be an incline with a zero element and d be a left derivation of K. Then* $d0 = 0.$

Proof. Let $x \in K$. Then

$$d0 = d(x * 0) = (x * d0) + (0 * dx) = x * d0.$$

Putting $x = 0$, we get $d0 = 0.$

Proposition 3.6. *Let K be an incline with multiplicative identity and d be a left derivation of K. Then the following hold for all* $x \in K$:

(i) $x * de \leq dx$.

(ii) *If* $de = e$, *then* $x \leq dx$.

(iii) *If K is a distributive lattice, then* $de = e$ *if and only if d is an identiy left derivation*.

Proof. (i) Let $x \in K$, $dx = d(x * e) = (x * de) + (e * dx) = (x * de) + dx$. Therefore, $x * de \leq dx$.

(ii) follows directly from (i).

(iii) We only need to prove the necessity.

Assume $de = e$. Using (ii), we get that $x \leq dx$ for all $x \in K$.

But *K* is a distributive lattice, so by Proposition 3.4(iii), we have $dx \leq x$. Thus, $dx = x$.

Definition 3.7. Let *K* be an incline and *d* be a left derivation of *K*.

(i) If *d* is one-to-one, then we call *d* a *monomorphic left derivation*.

(ii) If *d* is onto, then we call *d* an *epic left derivation*.

(iii) If $x \leq y$ implies $dx \leq dy$ for all $x, y \in K$, then we call *d* an *isotone left derivation*.

Proposition 3.8. *Let d be a left derivation of an integral incline K, and a be an element of K. If d a monomorphic, then the following hold for all* $x, y \in K$:

(i) *K is a commutative incline*.

(ii) *If* $a * dx = 0$ *for all* $x \in K$, *then either* $a = 0$ *or d is zero*.

(iii) *If* $dx * a = 0$ *for all* $x \in K$, *then either* $a = 0$ *or d is zero*.

Proof. (i) Let $x, y \in K$. Then

$$\begin{aligned} d(x * y) &= (x * dy) + (y * dx) \\ &= (y * dx) + (x * dy), \text{ from } (K1) \\ &= d(y * x). \end{aligned}$$

But d is monomorphic, then $x * y = y * x$, thus K is commutative.

(ii) Let $a * dx = 0$ for all $x \in K$ and let $y \in K$, replace x by $x * y$. Then

$$\begin{aligned} 0 &= a * dx \\ &= a * d(x * y) \\ &= a * (x * dy) + a * (y * dx) \\ &= a * (x * dy) + a * (dx * y), \text{ from (i)} \\ &= a * (x * dy). \end{aligned}$$

Putting $x = e$, we get that $a * dy = 0$, but K has no zero divisors, so $a = 0$ or $dy = 0$ for all $y \in K$. Thus, we have that $a = 0$ or d is zero.

(iii) Similar to (ii).

Proposition 3.9. *Let K be an incline and d be a left derivation of K. Then the following hold for all* $x, y \in K$:

(i) $d(x * y) \leq dx$;

(ii) $d(x * y) \leq dy$;

(iii) *d is an isotone derivation.*

Proof. (i) Let $x, y \in K$. Then by ($K7$), we have:

$$dx = d(x + (x * y)) = dx + d(x * y).$$

Hence, $d(x * y) \leq dx$.

(ii) Similar to (i).

(iii) Let $x \leq y$. Then $x + y = y$, and so

$$dy = d(x + y) = dx + dy.$$

Hence, $dx \leq dy$.

Proposition 3.10. *Let K be an integral incline and d be a left derivation of K. Denote* $d^{-1}(0) = \{x \in K \mid dx = 0\}$. *Then* $d^{-1}(0)$ *is an ideal of K.*

Proof. Let $x, y \in d^{-1}(0)$, thus $dx = dy = 0$.

From the hypothesis, we get $x + y \in d^{-1}(0)$, also $d(x * y) = 0$, and so $x * y \in d^{-1}(0)$. Then $d^{-1}(0)$ is a subincline of *K*.

Now, let $x \in K$, $y \in d^{-1}(0)$ such that $x \leq y$, thus $dy = 0$, and $(K8)$ gives us that

$$0 = dy = d(y + (x * y)) = dy + d(x * y),$$

hence $d(x * y) = 0$. Then

$$0 = d(x * y) = x * dy + y * dx = y * dx.$$

Since *K* has no zero divisors, either $y = 0$ or $dx = 0$.

If $dx = 0$, then $x \in d^{-1}(0)$ and if $y = 0$, then $x = 0$, by Proposition 3.4, $x \in d^{-1}(0)$.

Theorem 3.11. *Let K be an integral incline and d be a monomorphic left derivation of K. Define* $d^2(x) = d(x)$ *for all* $x \in K$. *If* $d^2 = 0$, *then d is zero.*

Proof. Let $x, y \in K$. Then

$$0 = d^2(x * y)$$

$$= d(d(x * y))$$

$$= d(x * dy + y * dx)$$

$$= d(x * dy) + d(y * dx)$$

$$= x * d^2 y + dy * dx + y * d^2 x + dx * dy$$

$$= dy * dx + dx * dy$$

$$= dx * dy + dx * dy, \text{ from Proposition 3.8(i),}$$

then from ($K6$), we get that $dx * dy = 0$. Since K has no zero divisors, we have that $dx = 0$ for all $x \in K$ or $dy = 0$ for all $y \in K$.

In two cases, we have $d = 0$.

Theorem 3.12. *Let K be an incline and d_1, d_2 are monomorphic left derivations of K. Define $d_1 d_2(x) = d_1(d_2 x)$ for all $x \in K$. If $d_1 d_2 = 0$, then $d_2 d_1$ is a monomorphic left derivation of K.*

Proof. First, let $x, y \in K$, since d_1, d_2 monomorphic that is mean, if $d_1(x) = d_1(y)$ implies $x = y$, also if $d_2(x) = d_2(y)$ implies $x = y$.

Now if $d_1 d_2(x) = d_1 d_2(y)$ implies $d_1(d_2 x) = d_1(d_2 y)$ implies $d_2 x = d_2 y$ implies $x = y$, i.e., $d_1 d_2$ is monomorphic of K and also $d_2 d_1$.

Second, let $x, y \in K$. Then

$$0 = d_1 d_2(x * y)$$

$$= d_1(x * d_2 y + y * d_2 x)$$

$$= d_1(x * d_2 y) + d_1(y * d_2 x)$$

$$= x * d_1 d_2 y + d_2 y * d_1 x + y * d_1 d_2 x + d_2 x * d_1 y$$

$$= d_2 y * d_1 x + d_2 x * d_1 y.$$

Then

$$\begin{aligned} d_2d_1(x * y) &= d_2(d_1(x * y)) \\ &= d_2(x * d_1y + y * d_1x) \\ &= d_2(x * d_1y) + d_2(y * d_1x) \\ &= x * d_2d_1y + d_1y * d_2x + y * d_2d_1x + d_1x * d_2y \\ &= x * d_2d_1y + y * d_2d_1x. \end{aligned}$$

This implies that d_2d_1 is a left derivation of *K*.

From first and second, d_2d_1 is a monomorphic left derivation of *K*.

Theorem 3.13. *Let M be a nonzero ideal of an integral incline K. If d is a nonzero left derivation of K, then d is a nonzero left derivation on M.*

Proof. Assume that $d = 0$ on *M* and $x \in M$. Then $dx = 0$.

Let $y \in K$, since $x * y \leq x$ and *M* is an ideal of *K*, thus we have $x * y \in M$. Therefore, $d(x * y) = 0$, then we get that

$$0 = d(x * y) = x * dy + y * dx = x * dy.$$

But *K* has no zero divisors, so $x = 0$ for all $x \in M$ or $dy = 0$ for all $y \in K$. Since $M \neq 0$, we get that $dy = 0$ for all $y \in K$. This contradicts $d \neq 0$ on *K*.

4. The Fixed Set of a Left Derivation in Incline

Definition 4.1. Let *K* be an incline and *d* be a left derivation on *K*. Denote

$$\mathrm{Fix}_d(K) = \{x \in K : dx = x\}.$$

Proposition 4.2. *Let K be a commutative incline and d be a left derivation of K. Then* $\mathrm{Fix}_d(K)$ *is a subincline of K.*

Proof. Let $x, y \in \mathrm{Fix}_d(K)$. Then

$$d(x+y) + d(x) + d(y) = x + y,$$

thus $x + y \in \mathrm{Fix}_d(K)$. Also,

$$\begin{aligned} d(x * y) &= (x * dy) + (y * dx) \\ &= (x * y) + (x * y) \\ &= x * y, \end{aligned}$$

hence $x * y \in \mathrm{Fix}_d(K)$, and so $\mathrm{Fix}_d(K)$ is a subincline of *K*.

Proposition 4.3. *Let K be an incline and d be a left derivation on K. Then the following hold*:

(i) *If K with a zero then,* $0 \in \mathrm{Fix}_d(K)$.

(ii) *If K with multiplicative identity such that* $e \in \mathrm{Fix}_d(K)$, *then* $x \leq dx$.

Proof. (i) Clear from Proposition 3.4.

(ii) Let $x \in K$ such that $e \in \mathrm{Fix}_d(K)$, from Proposition 3.6(ii), $x \leq dx$.

Theorem 4.4. *Let K be an incline with multiplicative identity and d be a left derivation on K where x is dominate dx for all* $x \in K$. *If* $e \in \mathrm{Fix}_d(K)$, *then d is an identity*.

Proof. From Proposition 4.3(ii), we have $x \leq dx$, but from the hypothesis $dx \leq x$.

Thus, $d(x) = x$ for all $x \in K$ and so *d* is an identity.

Theorem 4.5. *Let K be a commutative incline with multiplicative identity*. *Then the following hold*:

(i) d_a *is a left derivation of K.*

(ii) *If* $e \in \mathrm{Fix}_{d_a}(K)$, *then* $x \leq d_a(x)$ *and* d_a *is an identity map, moreover* $\mathrm{Fix}_{d_a}(K)$ *is an ideal of K.*

Proof. (i) Let $x, y \in K$. Then $d_a(x * y) = (x * y) * a$.

But

$$x * d_a(y) + y * d_a(x) = x * (y * a) + y * (x * a)$$
$$= (x * y) * a + (y * x) * a$$
$$= (x * y) * a + (x * y) * a$$
$$= (x * y) * a.$$

Therefore,

$$d_a(x * y) = (x * d_a y) + (y * d_a x),$$

i.e., d_a is a left derivation of *K*.

(ii) Let $x \in K$. Then

$$d_a(x) = d_a(x * e)$$
$$= x * d_a(e) + e * d_a(x)$$
$$= x + d_a(x).$$

Therefore,

$$x \leq d_a(x).$$

But

$$d_a(x) = x * a \leq x,$$

hence

$$d_a(x) = x,$$

so d_a is an identity map.

Now, $\mathrm{Fix}_{d_a}(K)$ is a subincline from Proposition 4.2 and if $x \in K$, $y \in \mathrm{Fix}_{d_a}(K)$ such that $x \leq y$

$$\Rightarrow x + y = y \text{ and } d_a(x + y) = d_a(y)$$

$$\Rightarrow d_a(x) + d_a(y) = d_a(y) = y = x + y.$$

But d_a is an identity, then $d_a(x) = x$.

Therefore, $x \in \mathrm{Fix}_{d_a}(K)$ and so $\mathrm{Fix}_{d_a}(K)$ is an ideal of K.

Clearly, if K is a commutative incline and d_a is a left derivation, then d_a is a left isotone derivation, from Proposition 3.8(iii).

Proposition 4.6. *Let K be a commutative incline and d_a, d_b be two isotone left derivations where $d_a(x)$, $d_b(x)$ are dominate x for all $x \in K$. Then*

$$d_a = d_b \textit{ if and only if } \mathrm{Fix}_{d_a}(K) = \mathrm{Fix}_{d_b}(K).$$

Proof. It is clear that $d_a = d_b$ implies $\mathrm{Fix}_{d_a}(K) = \mathrm{Fix}_{d_b}(K)$.

Conversely, let $\mathrm{Fix}_{d_a}(K) = \mathrm{Fix}_{d_b}(K)$ and $x \in K$. Then

$$\begin{aligned}
d_a^2(x) &= d_a(d_a(x)) \\
&= d_a(x * a) \\
&= (x * d_a(a) + a * d_a(x)) \\
&= x * (a * a) + a * (x * a) \\
&= (x * a) * a + (a * x) * a \\
&= (x * a) * a + (x * a) * a \\
&= (x * a) * a \\
&= d_a(x) * a \\
&\leq d_a(x).
\end{aligned}$$

Hence, $d_a^2(x) \leq d_a(x)$.

From the hypothesis, we get $d_a(x) \leq d_a^2(x)$. That is,

$$d_a^2(x) = d_a(x).$$

Similarly, we can prove that

$$d_b^2(x) = d_b(x).$$

Thus, $d_a(x) \in \mathrm{Fix}_{d_a}(K) = \mathrm{Fix}_{d_b}(K)$, i.e., $d_b(d_a x) = d_a(x)$.

Also, $d_b(x) \in \mathrm{Fix}_{d_b}(K) = \mathrm{Fix}_{d_a}(K)$, i.e., $d_a(d_b x) = d_b(x)$.

Since d_a, d_b are isotone and they are dominate x,

$$d_b(d_a(x)) \leq d_b x = d_a(d_b(x)).$$

By a similar way, we can get

$$d_a(d_b(x)) \leq d_a x = d_b(d_a(x)).$$

Hence, $d_a(x) = d_b(d_a(x)) = d_a(d_b(x)) = d_b x$, that is, $d_a = d_b$.

Proposition 4.7. *Let K be an incline and* $a \in K$. *Then*

$$\langle a \rangle = \{x \in K : x \leq a\} \text{ is an ideal of } K.$$

Proof. Let $x, y \in \langle a \rangle$, that is, $x \leq a$, $y \leq a$.

Thus,

$$x + y \leq a + a = a \text{ and } x * y \leq a * a \leq a,$$

so

$$x + y,\ x * y \in \langle a \rangle, \text{ that is, } \langle a \rangle \text{ is a subincline of } K.$$

Now, let $x \in K$ and $y \in \langle a \rangle$ such that $x \leq y$. Then

$$x \leq y \leq a, \text{ and so } x \leq a.$$

That is, $x \in \langle a \rangle$ and $\langle a \rangle$ is an ideal of K.

We call $\langle a \rangle$ a *principal ideal* and denote it by I_a.

Definition 4.8. Let K be an incline and I_a be a principal ideal of K. Then we call I_a a *model ideal* if it satisfies the following condition:

$$x * a = y * a \text{ implies } (x + b) * a = (y + b) * a \text{ for all } b \in I_a.$$

Proposition 4.9. *Let K be an incline. Then every principal ideal I_a is a model.*

Proof. Let I_a be a principal ideal of K, and let

$$x * y = y * a \text{ for some } x,\ y \in K.$$

Then for all $b \in I_a$, $b \leq a$, hence

$$\begin{aligned}(x + b) * a &= (x * a) + (b * a)\\ &= (y * a) + (b * a)\\ &= (y + b) * a.\end{aligned}$$

That is, I_a is a model.

Corollary 4.10. *Let K be a commutative incline with multiplicative identity and $e \in \mathrm{Fix}_{d_a}(K)$. Then $\mathrm{Fix}_{d_a}(K)$ is a model ideal.*

Proof. For all $x \in \mathrm{Fix}_{d_a}(K)$, that is,

$$x = d_a(x) = x * a \leq a.$$

So, $\mathrm{Fix}_{d_a}(K)$ is a principal ideal, then it is a model from Proposition 4.9.

Theorem 4.11. *Let K be a commutative incline with multiplicative identity and $e \in \mathrm{Fix}_{d_a}(K)$. If I be a principle ideal of K, then there exists a unique isotone left derivation d_a such that $\mathrm{Fix}_{d_a}(K) = I$.*

Proof. Let $I = \langle a \rangle$ be a principle ideal of K.

Since for all $x \in \mathrm{Fix}_{d_a}(K)$,

$$d_a(x) = x = x * a \leq a$$

$$\Rightarrow \mathrm{Fix}_{d_a}(K) = \langle a \rangle.$$

Now, in order to prove the uniqueness, we assume that there exist two isotone left derivations d_a and d_b such that $\mathrm{Fix}_{d_a}(K) = I$ and $\mathrm{Fix}_{d_b}(K) = I$. So, $\mathrm{Fix}_{d_a}(K) = \mathrm{Fix}_{d_b}(K)$, then $d_a = d_b$ by Proposition 4.6.

Definition 4.12. Let K be an incline and d_1, d_2 are two left derivations on K. Define a map $d_1 + d_2 : K \to K$ as follows:

$$(d_1 + d_2)(x) = d_1(x) + d_2(x) \text{ for all } x \in K.$$

Proposition 4.13. *Let K be an incline and d_1, d_2 are two left derivations on K. Then $d_1 + d_2$ is a left derivation on K.*

Proof. Let $x, y \in K$. Then

$$\begin{aligned}(d_1 + d_2)(x * y) &= d_1(x * y) + d_2(x * y) \\ &= x * d_1 y + y * d_1 x + x * d_2 y + y * d_2 x \\ &= x * (d_1 y + d_2 y) + y * (d_1 x + d_2 x) \\ &= x * (d_1 + d_2)(y) + y * (d_1 + d_2)(x).\end{aligned}$$

Thus, $d_1 + d_2$ is a left derivation on K.

Proposition 4.14. *Let K be an incline and d_1, d_2 are two left derivations on K where $d_2(x)$ is a dominate $d_1(x)$ for all $x \in K$. Then $x * x \in \mathrm{Fix}_{d_1+d_2}(K)$ for all $x \in \mathrm{Fix}_{d_1+d_2}(K)$.*

Proof. Let $x \in \mathrm{Fix}_{d_1+d_2}(K)$, since $d_1(x) \leq d_2(x)$, hence $d_1(x) + d_2(x) = d_2(x)$, i.e., $(d_1 + d_2)(x) = d_2 x$.

But $x \in \mathrm{Fix}_{d_1+d_2}(K)$, that is, $(d_1 + d_2)(x) = x$.

Hence, we get $d_2(x) = x$.

Also, $x = d_1(x) + d_2(x) = d_1(x) + x$, then $d_1(x) \le x$.

Now,

$$\begin{aligned}(d_1 + d_2)(x * x) &= d_1(x * x) + d_2(x * x)\\ &= (x * d_1x) + (x * d_1x) + (x * d_2x) + (x * d_2x)\\ &= (x * d_1x) + (x * d_1x) + (x * x) + (x * x)\\ &= x * d_1x + x * x\\ &= x * (d_1x + x)\\ &= x * x.\end{aligned}$$

Therefore, $x * x \in \mathrm{Fix}_{d_1+d_2}(K)$.

Proposition 4.15. *Let K be an incline and* d_1, d_2 *are two left derivations on K where* $d_2(x)$ *is a dominate* $d_1(x)$ *and* $d_1(x)$ *is a dominate x for all* $x \in K$. *Then* $\mathrm{Fix}_{d_1+d_2}(K) = \mathrm{Fix}_{d_1}(K) \cap \mathrm{Fix}_{d_2}(K)$.

Proof. Let $x \in \mathrm{Fix}_{d_1+d_2}(K)$, hence $(d_1 + d_2)(x) = x$, thus

$$d_1(x) + d_2(x) = x.$$

Since $d_2(x)$ is a dominate $d_1(x)$, we get

$$d_1(x) + d_2(x) = d_2(x),$$

that is, $d_2(x) = x$ and so $x \in \mathrm{Fix}_{d_2}(K)$. Also, we have

$$d_1(x) + x = x,$$

that is, $d_1(x) \le x$.

But $d_1(x)$ dominate x, hence $d_1(x) = x$ and thus $x \in \text{Fix}_{d_1}(K)$.

Therefore, $x \in \text{Fix}_{d_1}(K) \cap \text{Fix}_{d_2}(K)$.

Now, let $x \in \text{Fix}_{d_1}(K) \cap \text{Fix}_{d_2}(K)$, thus

$$d_1(x) = d_2(x) = x,$$

and so

$$(d_1 + d_2)(x) = d_1(x) + d_2(x) = x + x = x.$$

Hence, $x \in \text{Fix}_{d_1+d_2}(K)$. This completes the proof.

Definition 4.16. Let K be an incline and d_1, d_2 are two left derivations on K. Define a map $d_1 * d_2 : K \to K$ as follows:

$$(d_1 * d_2)(x) = \begin{cases} d_1(x) & \text{if } d_2(x) \le d_1(x) \\ d_2(x) & \text{if } d_1(x) \le d_2(x), \text{ for all } x \in K \\ d_1(x) * d_2(x) & \text{otherwise.} \end{cases}$$

Definition 4.17. Let K be an incline. We call K a *dominante incline* if x dominate y or y dominate x for all $x, y \in K$.

During the previous definition of easy conclusion, if K is a dominante incline and d_1, d_2 are two left derivations on K, then $d_1 * d_2$ is a left derivation on K.

Proposition 4.18. *Let K be an incline and* d_1, d_2 *be two left derivations on K where* $d_2(x)$ *is a dominate* $d_1(x)$ *and x dominate* $d_2(x)$ *for all* $x \in K$. *Then* $d_1 * d_2$ *is a left derivation on K. Moreover,*

$$\text{Fix}_{d_1 * d_2}(K) = \text{Fix}_{d_1}(K) \cup \text{Fix}_{d_2}(K).$$

Proof. Since $d_1(x) \le d_2(x)$, thus $(d_1 * d_2)(x) = d_2(x)$ is a left derivation on K.

Let $x \in \mathrm{Fix}_{d_1 * d_2}(K)$, i.e., $x = (d_1 * d_2)(x) = d_2(x)$, that is, $x \in \mathrm{Fix}_{d_2}(K)$ and so $x \in \mathrm{Fix}_{d_1}(K) \cup \mathrm{Fix}_{d_2}(K)$.

Now, let $x \in \mathrm{Fix}_{d_1}(K) \cup \mathrm{Fix}_{d_2}(K)$, thus $d_1(x) = x$ or $d_2(x) = x$. If $d_2(x) = x$, then $d_1 * d_2(x) = d_2(x) = x$.

Hence, $x \in \mathrm{Fix}_{d_1 * d_2}(K)$.

If $d_1(x) = x$, then we get $x \leq d_2(x)$ from $d_1(x) \leq d_2(x)$ and by hypothesis, $d_2(x) \leq x$.

Then $d_2(x) = x$, and $(d_1 * d_2)(x) = d_2(x) = x$.

Therefore, $x \in \mathrm{Fix}_{d_1 * d_2}(K)$.

Corollary 4.19. *If K is a dominante incline and* d_1, d_2 *are two left derivations on K, then* $\mathrm{Fix}_{d_1 + d_2}(K) = \mathrm{Fix}_{d_1 * d_2}(K)$.

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