Far East Journal of Mathematical Sciences (FJMS)
© 2015 Pushpa Publishing House, Allahabad, India
Published Online: February 2015
http://dx.doi.org/10.17654/FJMSJan2015_029_042
Volume 96, Number 1, 2015, Pages 29-42 **ISSN: 0972-0871**

WAVE GROUP EVOLUTION IN KdV-TYPE MODEL

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Abstract

An improvement of uni-directional wave model so called AB equation was proposed to study wave groups that appeared in the wave tanks of hydrodynamic laboratory [2]. The model was revised version of the KdV equation and could be interpreted as a higher order KdV equation for wave above finite depth and in certain approximation it became the KdV equation. In this paper, we will focus to investigate the evolution of the envelop of modulated wave which is described by the nonlinear Schrödinger equation (NLS). We derive the NLS-type equation from the AB equation. The asymptotic method is used to find the equation. The signs of the coefficients of the NLS-type equation that determine whether experimentally relevant wave groups are possible exist or not will be compared by the other NLS-type equations. The effect of the dispersion relation and the nonlinearity terms of the model to existence of experimentally relevant wave groups will be presented.

Received: July 25, 2014; Accepted: September 1, 2014
2010 Mathematics Subject Classification: 34E05, 74J30, 76B15.
Keywords and phrases: wave groups, AB equation, NLS-type equation.
Communicated by K. K. Azad

1. Introduction

Many theoretical investigations that deal with evolution of wave groups have been discussed. One of them was the theoretical investigation of wave groups evolution discussed by van Groesen in [1]; in the literature a KdV-type equation was used for uni-directional wave model and an NLS-type derived from the KdV equation as amplitude equation for modulated waves in water waves. In this paper, we will use a new KdV-type equation so called AB equation as a uni-directional surface wave model. The model was already proposed by van Groesen and Andonowati [2]. The model was improvement of the KdV equation and can be interpreted as a higher order KdV equation for wave above finite depth and in certain approximation it becomes the KdV equation. The nonlinear terms of the model were also improved to include the effects of short wave interactions. The model can be used for all wave lengths and correct for any depths (see [3]) and given by

$$\partial_t \eta = -\sqrt{g} A\left[\eta + \frac{1}{2} A(\eta A\eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2} B(\eta B\eta) + \frac{1}{4}(B\eta)^2 \right], \quad (1)$$

where η represents elevation, $A = \partial_x C/\sqrt{g}$ and $B = \sqrt{g}C^{-1}$ are pseudo differential operators with symbol $\hat{C}(k) = \Omega(k)/k$ (in the form of Fourier transform) with $\Omega(k) = c_0 k\sqrt{\frac{\tanh(kh)}{kh}}$, $c_0 = \sqrt{gh}$, g and h are gravity acceleration and water depth, respectively.

Equation (1) will be solved by using third order asymptotic expansions where the elevation $\eta(x, t)$ is expanded by power series in a small parameter ε. The expansions then obtain three linear partial differential equations up to the third order expansion and can be written as

$$\partial_t \eta^{(1)} + \sqrt{g} A\eta^{(1)} = 0, \quad (2)$$

$$\partial_t \eta^{(2)} + \sqrt{g} A\eta^{(2)} = -\sqrt{g} A\left[\frac{1}{2} A(\eta^{(1)} A\eta^{(1)}) - \frac{1}{4}(A\eta^{(1)})^2 + \frac{1}{2} B(\eta^{(1)} B\eta^{(1)}) + \frac{1}{4}(B\eta^{(1)})^2 \right], \quad (3)$$

$$\partial_t \eta^{(3)} + \sqrt{g} A\eta^{(3)} = -\sqrt{g} A\left[\frac{1}{2} A(\eta^{(1)} A\eta^{(2)} + \eta^{(2)} A\eta^{(1)}) - \frac{1}{4}(2A\eta^{(1)} A\eta^{(2)}) + \frac{1}{2} B(\eta^{(1)} B\eta^{(2)} + \eta^{(2)} B\eta^{(1)}) + \frac{1}{4}(2B\eta^{(1)} B\eta^{(2)})\right]. \quad (4)$$

Notation $\eta^{(i)}$ denotes the *i*th order approximation of η. Equations (2)-(4) relate to the first, second and third order equations, respectively.

In equation (2), the dispersion relation of AB equation can be obtained by taking $\eta^{(1)} = ae^{i(k_0 x - \omega t)}$ with amplitude constant *a*. Then we find dispersion relation $\omega = \Omega(k_0)$. The dispersion relation is the same as the dispersion relation for modified KdV equation in [1].

The paper is organized as follows. In Section 2, we will derive the NLS-type equation for the AB equation. The NLS-type is well-known as amplitude equation for modulated wave in water waves, see Dingemans [4]. By choosing the solution of AB equation as monochromatic with modulated amplitude and applying third order asymptotic method, we found the NLS-type equation. The coefficient of the NLS-type will be compared with other NLS-type.

In Section 3, the characteristic of wave groups for AB equation will be discussed. The characteristics of wave groups are determined by looking coefficients of NLS-type equation that are found in Section 2. Section 4 will be focussed on comparison of the coefficients of NLS-type of AB and the other NLS-type. Conclusions are written in Section 5.

2. Standard NLS-type Equation

Here we look for an envelop equation derived from the AB equation. The equation is based on investigating wave groups propagation. To the end, the solution of (2) will be chosen in the form of a monochromatic wave with modulated amplitude $a(x, t)$,

$$\eta^{(1)}(x, t) = a(x, t)e^{i(k_0 x - \omega t)} + c.c. \quad (5)$$

Here k_0 and ω represent wave number and frequency of the monochromatic,

respectively; *c.c* means complex conjugates. Expression (5) is called *first order harmonic mode*.

Substitution of (5) in (2) yields

$$(\partial_t a)e^{i(k_0x-\omega t)} - i\omega a e^{i(k_0x-\omega t)} + \sqrt{g}\sum_{n=0}\frac{1}{n!}(-i)^n \hat{A}^n(k_0)(\partial_x^n a)e^{i(k_0x-\omega t)} = 0, \quad (6)$$

where $\hat{A}$ is symbol of pseudo differential operator A in the form of Fourier transform and $\hat{A}^n$ represents nth derivative of $\hat{A}$ with respect to k.

For an envelop which has a large spatial extension, we introduce a scaling in the spatial variable. Along spatial variable ξ is introduced a frame moving with velocity V_g. Thus we write $a = \mathcal{A}(\xi, \tau)$, $\xi = \varepsilon(x - V_g t)$ and $\tau = \varepsilon^2 t$ with group velocity V_g. Applying these expressions into (6), we obtain

$$(-\varepsilon V_g \partial_\xi \mathcal{A} + \varepsilon^2 \partial_\tau \mathcal{A})e^{i\theta} - i\omega \mathcal{A} e^{i\theta} + \sqrt{g}\sum_{n=0}\frac{1}{n!}(-i)^n \hat{A}^n(k_0)\varepsilon^n(\partial_\xi^n \mathcal{A})e^{i\theta} = 0, \quad (7)$$

with $\theta = k_0x - \omega t$.

Since $\omega = \Omega(k_0)$ and $V_g = \Omega'(k_0)$, from (7) we obtain

$$\partial_\tau \mathcal{A} - \frac{i}{2}\Omega''(k_0)\partial_\xi^2 \mathcal{A} = O(\varepsilon). \quad (8)$$

For the second order solution, the right hand side (RHS) of equation (3) is obtained as interaction between solution of the first order with itself,

$$\text{RHS} = -\sqrt{g}\hat{A}(2k_0)\left[\frac{1}{4}(\hat{B}^2(k_0) - \hat{A}^2(k_0)) + \frac{1}{2}(\hat{A}(k_0)\hat{A}(2k_0) + \hat{B}(k_0)\hat{B}(2k_0))\right]\mathcal{A}e^{2i\theta} + c.c.$$

The solution of the second order equation can be chosen as

$$\eta^{(2)}(x,\, t) = (b(x,\, t)e^{2i\theta} + c(x,\, t)) + c.c. \tag{9}$$

This solution is superposition of the second order double harmonic mode and second order nonharmonic long wave.

Substituting (9) into the second order equation, we find

$$(\partial_t b)e^{2i\theta} - 2i\omega b e^{2i\theta} + \partial_t c + \sqrt{g}\sum_{n=0}\frac{1}{n!}(-i)^n \hat{A}^n(2k_0)(\partial_x^n b)e^{2i\theta}$$

$$+ \sqrt{g}\sum_{n=0}\frac{1}{n!}(-i)^n \hat{A}^n(0)(\partial_x^n c)e^{2i\theta} = \text{RHS}. \tag{10}$$

For solution of b and c, we will follow the same line as in solution for a. We then introduce $b = \mathcal{B}(\xi,\, \tau),\ c = \mathcal{C}(\xi,\, \tau),\ \xi = \varepsilon(x - V_g t)$ and $\tau = \varepsilon^2 t$ for equation (9). Then we get in the second order equation

$$-2i\omega\mathcal{B}e^{2i\theta} + \sqrt{g}\hat{A}(2k_0)\mathcal{B}e^{2i\theta} = \text{RHS}. \tag{11}$$

Collecting term $e^{2i\theta}$ in (11), we get

$$\mathcal{B}(\xi,\, \tau) = \frac{C_0}{2\Omega(k_0) - \Omega(2k_0)}\mathcal{A}^2 \tag{12}$$

with

$$C_0 = \Omega(2k_0)\left[\frac{\Omega(k_0)}{2g}\left(\frac{1}{2}\Omega(k_0) - \Omega(2k_0)\right) + \frac{gk_0^2}{\Omega(k_0)}\left(\frac{1}{\Omega(2k_0)} + \frac{1}{4\Omega(k_0)}\right)\right].$$

In the third order equation, a resonance term and a constant term will appear due to interactions between first and second order solution in the nonlinear terms of AB equation. The constant term can be written as

$$(V - \Omega'(0))\partial_\xi \mathcal{C} - \frac{\Omega'(0)}{2}\left[\frac{g}{\hat{C}^2(k_0)} - \frac{k_0\hat{C}^2(k_0)}{g} + \frac{2g}{\hat{C}(k_0)\hat{C}(0)}\right]\partial_\xi|\,\mathcal{A}\,|^2, \tag{13}$$

while the resonance term can be written by

$$
\begin{aligned}
&-\frac{\sqrt{g}}{2}[[2\hat{A}^2(k_0)\hat{A}(2k_0)+2\hat{B}(k_0)\hat{B}(2k_0)\hat{A}(k_0)\\
&+\hat{A}(k_0)(\hat{B}^2(k_0)-\hat{A}^2(k_0))]\overline{\mathcal{A}}\mathcal{B}\\
&+[2\hat{B}(k_0)\hat{A}(k_0)\hat{B}(0)+(\hat{B}^2(k_0)+\hat{A}^2(k_0))\hat{A}(k_0)]\mathcal{A}\mathcal{C}]e^{i\theta}\\
&-\left(\partial_\tau\mathcal{A}-\frac{1}{2}i\Omega''(k_0)\partial_\tau^2\mathcal{A}\right)e^{i\theta}. \qquad (14)
\end{aligned}
$$

These terms have to be vanished to satisfy solvability condition for asymptotic valid solution. Therefore from (13) we get

$$
\mathcal{C}=\frac{\frac{\Omega'(0)}{2}\left[\frac{g}{\hat{C}^2(k_0)}-\frac{k_0\hat{C}^2(k_0)}{g}+\frac{2g}{\hat{C}(k_0)\hat{C}(0)}\right]}{\Omega'(k_0)-\Omega'(0)}|\mathcal{A}|^2. \qquad (15)
$$

By substituting (15) into resonance term (14) and using solvability condition, we obtain an equation for the amplitude $\mathcal{A}$ given by

$$
\partial_\tau\mathcal{A}+i\beta\partial_\xi^2\mathcal{A}+i\gamma|\mathcal{A}|^2\mathcal{A}=0, \qquad (16)
$$

where the coefficients of equation (16) are given by

$$
\beta=-\frac{1}{2}\Omega''(k_0) \text{ and } \gamma=\gamma_1+\gamma_2,
$$

with $\gamma_1=D_1F_1$ and $\gamma_2=D_2F_2$, where

$$
D_1=\frac{2k_0^3}{g}\hat{C}^2(k_0)\hat{C}(2k_0)+\frac{k_0g}{\hat{C}(2k_0)}+\frac{1}{2}\left(\frac{g}{\hat{C}^2(k_0)}+\frac{k_0^2\hat{C}^2(k_0)}{g}\right)k_0\hat{C}(k_0),
$$

$$
D_2=\frac{k_0g}{\hat{C}(0)}+\frac{1}{2}\left(\frac{g}{\hat{C}^2(k_0)}-\frac{k_0^2\hat{C}^2(k_0)}{g}\right)k_0\hat{C}(k_0),
$$

$$
\begin{aligned}
F_1=\frac{\Omega(2k_0)}{2\Omega(k_0)-\Omega(2k_0)}\Bigg[&\frac{\Omega(k_0)}{2g}\left(\frac{1}{2}\Omega(k_0)-\Omega(2k_0)\right)\\
&+\frac{gk_0^2}{\Omega(k_0)}\left(\frac{1}{\Omega(2k_0)}+\frac{1}{4\Omega(k_0)}\right)\Bigg],
\end{aligned}
$$

$$F_2 = \frac{\Omega'(0)}{2(\Omega'(k_0) - \Omega'(0))}\left[\frac{g}{\hat{C}^2(k_0)} - \frac{k_0^2\hat{C}^2(k_0)}{g} + \frac{2g}{\hat{C}(k_0)\hat{C}(0)}\right].$$

Equation (16) was known as standard NLS-type equation and we denote this as NLS_{AB}.

3. Characteristic of Wave Groups

The parameters β and γ are the important parameters in the NLS equation. They determine the kind of wave groups that described by the model [4, 1], so this will be looked first. We will demonstrate this by looking for 'steady' solution. Next, we try to compare with other NLS-type equations. One of them is NLS-type that derived from the KdV equation, we called it by NLS_{KdV}. The solution of equation (16) will be solved by choosing the ansatz as

$$\mathcal{A}(\xi, \tau) = f(\xi)e^{-(\pm i\tau)}. \tag{17}$$

Substitution of (17) in (16) yields

$$-i(\pm) f(\xi) + i\beta f''(\xi) + i\gamma f^3(\xi) = 0. \tag{18}$$

Here f' and f'' denote the derivative of f with respect to ξ. Equation (18) can be rewritten as

$$\beta f''(\xi) = \pm f(\xi) - \gamma f^3(\xi). \tag{19}$$

Considering (19), we observe that the solution very much depends on coefficients β and γ. Multiplying each side by $f'(\xi)$ then integrating them with respect to ξ, we find

$$\beta\frac{1}{2}(f')^2 = \pm\frac{1}{2}f^2 - \gamma\frac{1}{4}f^4 + E = -V(f) + E, \tag{20}$$

where

$$V(f) = -\left(\pm\frac{1}{2}f^2 - \gamma\frac{1}{4}f^4\right), \tag{21}$$

for some constant E. The constant E can be regarded as energy when we consider this by interpreting the equation f as a mechanical system (i.e.

Newton's equation of motion for a particle with mass β under the influence of a conservative force with potential energy $V(f)$).

In the sequel, we denote $V_{\pm}$ corresponding to the sign $\pm$ in (21) and $\pm\gamma$, with $\gamma > 0$. The graph of this function can be seen in Figure 1. The continuous line shows the V_{+}, while the dot line shows the graph of V_{-}.

The function V has minimum and maximum values at $f = \pm 1/\sqrt{\gamma}$ with the maximum value $\frac{1}{4\gamma}$ and the minimum value $-\frac{1}{4\gamma}$.

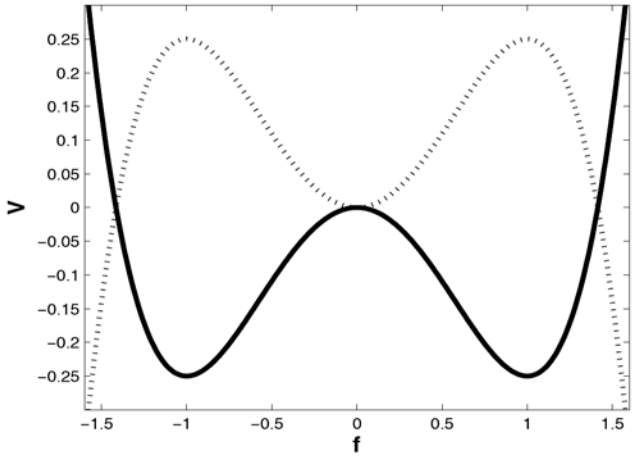


Figure 1. Graph of the function of V_{+} (solid line) and V_{-} (dashed line) with $\gamma = 1$.

Let us consider the case for V_{-}, by using the initial condition of V, where the maximum of V is $\frac{1}{4\gamma}$, we get

$$\beta \frac{1}{2}(f')^2 = -\frac{1}{2} f^2 - \gamma \frac{1}{4} f^4 + E = \left(-\frac{1}{2} f^2 + \gamma^* \frac{1}{4} f^4 \right) + \frac{1}{4\gamma^*}, \quad (22)$$

where $\gamma^* = |\gamma|$. Equation (22) can be rewritten as

$$f' = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\gamma^* \beta}} - \frac{\sqrt{\gamma^*}}{\sqrt{\beta}} f^2 \right). \quad (23)$$

The solution of (23) is given by

$$f(\xi) = \frac{1}{\sqrt{\gamma^*}} \tanh\left(\frac{1}{\sqrt{2\beta}}\xi\right). \tag{24}$$

Thus, the solution of (16) is given by

$$\mathcal{A}(\xi, \tau) = \frac{1}{\sqrt{\gamma^*}} \tanh\left(\frac{1}{\sqrt{2\beta}}\xi\right) e^{i\tau}. \tag{25}$$

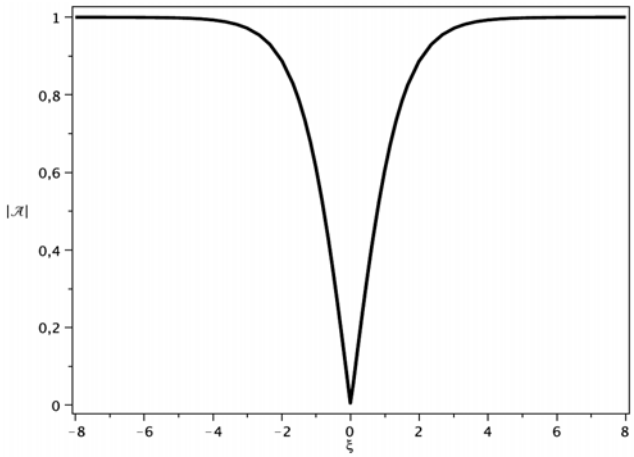


Figure 2. Dark soliton for $\gamma = -1$ and $\beta = 1$.

As an illustration we choose $\gamma = -1$, $\beta = 1$, the wave amplitude is the modulus of *A*. The wave amplitude with respect to ξ can be seen in Figure 2. The solution is not suitable for surface water wave equation. Therefore for γ negative, the AB equation was not describing solitary wave. The solution (25) is called *dark soliton*.

While for the case of V_+, it is required that the water is in rest at infinity, then *f* and f' are zero for ξ tends to infinity, therefore it gives $E = 0$, and we get

$$\beta \frac{1}{2}(f')^2 = \pm\frac{1}{2} f^2 - \gamma \frac{1}{4} f^4 + E = \left(\frac{1}{2} f^2 - \gamma \frac{1}{4} f^4\right). \tag{26}$$

Choosing $g = \sqrt{\frac{\gamma}{2}} f$ and doing some modification, we can rewrite (26) as

$$(g')^2 = \frac{1}{\beta}(1 - g^2)g^2. \tag{27}$$

From the expression (27), we get

$$g(\xi) = \operatorname{sech}\left(\frac{\xi}{\sqrt{\beta}}\right). \tag{28}$$

So, the solution of (16) for γ positive is given by

$$\mathcal{A}(\xi, \tau) = \sqrt{\frac{2}{\gamma}}\ \operatorname{sech}\left(\frac{\xi}{\sqrt{\beta}}\right)e^{i\tau}. \tag{29}$$

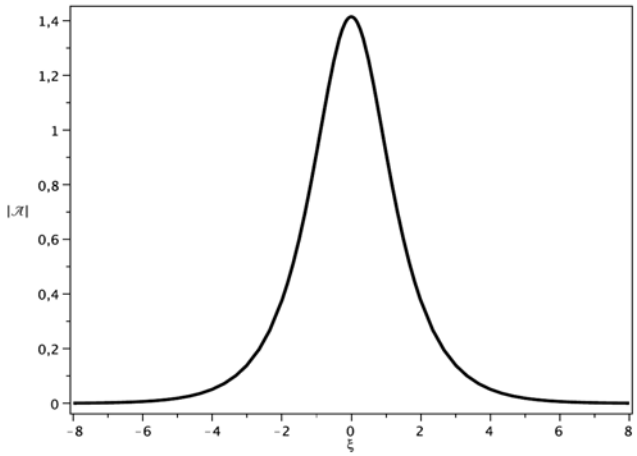


Figure 3. Bright soliton for $\gamma = 1$ and $\beta = 1$.

By the same way as for the solution (25) but here $\gamma = 1$, the solution (29) is called "*bright soliton*" or soliton (see Figure 3).

4. Comparison of the NLS-type Equation

For the modified KdV (mKdV), the equation with exact dispersion relation (see [1, 5]) is

$$\partial_t \eta + i\Omega(-i\partial_x)\eta + \frac{3}{4}\partial_x \eta^2 = 0. \tag{30}$$

The coefficients β and γ of its NLS-type equation are given by

$$\beta = \frac{1}{2}\Omega''(k_0) \text{ and } \gamma = \frac{9}{4}k_0(\sigma_0 + \sigma_2)$$

with

$$\sigma_0 = \frac{1}{\Omega'(k_0) - \Omega'(0)} \text{ and } \sigma_2 = \frac{k_0}{2\Omega(k_0) - \Omega(2k_0)}.$$

Equation (30) is in non-dimension form. The relation between the normalize variables and the physical variables is given by $\eta_{lab} = h\eta$, $x_{lab} = hx$, $t_{lab} = t\sqrt{\left(\frac{h}{g}\right)}$, $k_{lab} = \frac{k}{h}$, $\omega_{lab} = \omega\sqrt{\left(\frac{g}{h}\right)}$.

By using the above relation for the AB equation, comparison between coefficient of NLS-type of AB and KdV can be seen in Figure 4. The coefficient β of NLS_{AB} is the same as of NLS_{KdV} (see Figure 4) since their dispersion relations are the same. The coefficient β is always positive.

Next let us consider the value of γ. For long wave case (small k), the coefficients γ of NLS_{AB} and NLS_{KdV} are really closed. However, for the short wave case, the coefficient γ of NLS_{AB} is larger than γ of NLS_{KdV}. The order of coefficient γ for NLS_{AB} and NLS_{KdV} are $\mathcal{O}(k^{5/2})$ and $\mathcal{O}(k^{3/2})$, respectively, such that the order $\mathcal{O}\left(\frac{\gamma\text{NLS}_{AB}}{\gamma\text{NLS}_{KdV}}\right) \approx k$.

Since the solution is occurred for $\beta\gamma > 0$, and we know that $\beta > 0$, so there is critical wave number k_{crit} such that for $k > k_{crit}$, $\beta\gamma > 0$. For NLS_{AB}, $k_{crit} \approx 1.47$ while for NLS_{KdV}, $k_{crit} = 1.15$ and $1.46 \le k_{crit} \le 4.24$ for BBM (see [1]), which has dispersion relation $\omega = k\left(1 + \frac{1}{6}k^2\right)^{-1}$.

Observe that although the dispersion relations of mKdV and AB equation are the same, but their k_{crit} are different. This is caused by the effect of nonlinearity term of each other, as we know that their nonlinearity are

different. k_{crit} of NLS_{AB} is closer to the value ≈ 1.36 which is found when the wave groups are considered for the full set of equations describing surface waves on a layer fluid [4] than of NLS_{KdV}.

According to sign β and γ where wave groups that are relevant for laboratory experiment only exist when their sign are positive, Table 1 shows summarize of sign of β and γ for each equation.

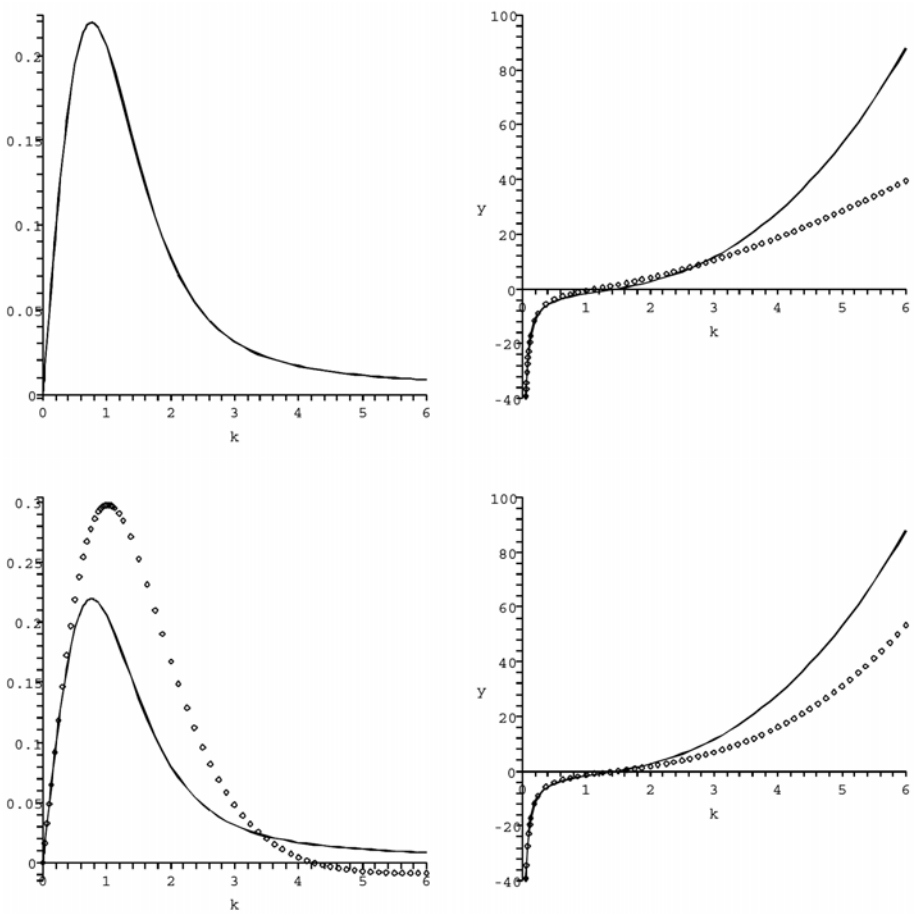


Figure 4. Parameters β (left) and γ (right) for NLS_{AB} (line) and NLS_{KdV} (dot) at the top, and for NLS_{AB} (line) and NLS_{BBM} (dot) in normalize parameter.

Table 1. The table of k_{crit} for various models with their dispersion relation

Model	Dispersion relation	Laboratory wave groups
KdV	$k\left(1-\frac{1}{6}k^2\right)$	impossible
BBM	$k\left(1-\frac{1}{6}k^2\right)^{-1}$	for $1.46 \le k \le 4.24$
mKdV	$k\sqrt{(\tanh(k)/k)}$	$k \ge k_{crit} \approx 1.15$
AB	$k\sqrt{(\tanh(k)/k)}$	$k \ge k_{crit} \approx 1.47$

5. Conclusion

In this paper, we studied the derivation of envelop equation which is called *NLS equation*. By using the AB equation as a uni-directional wave equation, it was shown that the amplitude equation for wave groups not only depends on the dispersive properties, but also depends on nonlinearity of the models. Although the dispersive property of AB equation is really the same as the modified KdV equation in [1], the critical k for each equation is different. The critical k for AB equation is closer to the value that found when wave groups are considered for the full set of equations describing surface waves on a layer fluid [4]. It is caused by the nonlinearity of the AB equation which is different from the KdV equation.

Acknowledgments

The authors wish to thank Professor E. van Groesen for many rewarding discussions. The research is the fundamental research scheme funded by DIKTI with the contract number of DIPA UNSOED No.: 63/UN23.10/PN/ 2013.

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