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FINITE VOLUME ELEMENT METHODS AND ANALYSIS ALONG CHARACTERISTICS FOR THE ONE DIMENSIONAL CUBIC SEMICONDUCTOR DEVICE SIMULATION

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Abstract

Using Hermite cubic polynomial trial function and piecewise linear test function and combining the method of characteristics with finite volume element method, we give a full discrete finite volume element method along characteristics. Further, we get an optimal H^1 and L^2 error estimates. Finally, an experiment showing the advantage is provided.

1. Introduction

The mathematical model of the semiconductor device of heat conduction is described by the initial boundary value problem made up of a system of four quasi-linear partial differential equations. One equation of the elliptic type for the electric potential, two of convection-dominated diffusion type for the conservation of electron and hole concentration, and the last one for the heat conduction. The four equations relevant initial

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conduction and boundary condition make up a closed system. We consider the following one dimensional semiconductor problem [10] in $\Omega = [a, b]$:

$$-\frac{\partial^2 \psi}{\partial x^2} = \alpha(p - e + N(x)), \ (x, t) \in \Omega \times (0, \overline{T}], \ \overline{J} = (0, \overline{T}], \tag{1.1}$$

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial x}\left[D_e(x)\frac{\partial e}{\partial x} - \mu_e e \frac{\partial \psi}{\partial x}\right] - R(e, p, T), \ (x, t) \in \Omega \times \overline{J}, \tag{1.2}$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[D_p(x)\frac{\partial p}{\partial x} + \mu_p p \frac{\partial \psi}{\partial x}\right] - R(e, p, T), \ (x, t) \in \Omega \times \overline{J}, \tag{1.3}$$

$$\rho(x)\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = \left[\left(D_p(x)\frac{\partial p}{\partial x} + \mu_p(x) p \frac{\partial \psi}{\partial x}\right) - \left(D_e(x)\frac{\partial e}{\partial x} - \mu_e(x) e \frac{\partial \psi}{\partial x}\right)\right] \cdot \frac{\partial \psi}{\partial x},$$
$$(x, t) \in \Omega \times \overline{J}, \tag{1.4}$$

where (1.1) is the electrostatic potential equation, (1.2) and (1.3) are the electron and hole concentration equations and (1.4) is a temperature equation. The unknown functions here are the electrostatic potential ψ, the electron concentration e, the hole concentration p and the temperature T. All coefficients are positive. $D_s(x)(s = e, p)$ is the diffusion coefficient, $\mu_s(x)(s = e, p)$ are mobilities with the relation $D_s(x) = U_T \mu_s(x)$, where U_T is the thermal voltage. $N(x) = N_D(x) - N_A(x)$ is a given function, $N_D(x)$ and $N_A(x)$ being the donor and acceptor impurity concentrations. $R(e, p, T)$ is recombination term considering the temperature's effect. For e, p and T, the initial condition is

$$e(x, 0) = e_0(x), \ p(x, 0) = p_0(x), \ T(x, 0) = T_0(x), \ x \in \Omega. \tag{1.5}$$

The initial value of ψ can be calculated by (1.1) and (1.4). We consider two point boundary value problem, the boundary values are

$$\psi(a) = \psi'(b) = 0, \ e(a) = e'(b) = 0, \ p(a) = p'(b) = 0, \ T(a) = T'(b) = 0. \tag{1.6}$$

The work on the semiconductor device simulation is very important for the development of the semiconductor problem. Since Gummel [3] first proposed sequence iterative computation methods to treat this kind of problem, and thus opened a new field in 1964, there have been lots of numerical contributions towards about such problems. The main work now include: Douglas and Yuan [1], Yuan [9-11] and Zlámal [12] et al.

Finite volume element method, as a new method to solve differential equations, can preserve the desired conservation laws and preserve the main advantage both of the finite element method and also of the finite difference method. Compared with finite difference method, it can be parted freely especially in irregular domains. In this paper combining the modified method of characteristics, we get a full discrete characteristic finite volume element method for one dimensional cubic semiconductor device simulation which not only preserves the advantage of finite volume element method but also preserves the conceptual and computational advantages of the MMOC.

The outline of the paper is as follows. In Section 2 we introduce the algorithm of this paper and give the computational procedure. In Section 3 we study the estimate of rate of convergence on assumption of periodic boundary value condition. First some definitions and lemmas are given. Then we obtain an optimal prior error estimate in H^1 and in L^2. Finally, in Section 4, the numerical results are presented. In this paper, C stands for a general positive constant and ε is a general small positive constant having different meaning at different places.

2. Full Discrete Scheme of Characteristic Finite Volume Element Method

We use H^m to denote Sobolev space of m order. In $\Omega = [a, b]$, $(\cdot, \cdot)$ denotes $H^0 = L^2$ inter-integral, $\|\cdot\|_m$ denotes H^m norm and $\|\cdot\|_0 = \|\cdot\|$. Let $H_E^1 = \{u(x) \in H^1 \,|\, u(a) = 0\}$, $U = H^2 \cap H_E^1$. We divide $\Omega = [a, b]$ into J parts called *partition* T_h [4] with the nodes x_i, $i = 0, 1, \ldots, J$. Let $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$ and $h = \max_{1 \le i \le J}\{h_i\}$. Then we assume that the partition satisfies the regular condition $h_i \ge \mu h$, $\mu > 0$, $1 \le i \le J$. We introduce the dual partition T_h^* with nodes $x_{i-1/2}$, $i = 0, 1, \ldots, J$. The nodes satisfy

$$x_0 = x_{-1/2} \le x_{1/2} \le \cdots \le x_{i-1/2} \le x_{i+1/2} \le \cdots \le x_{J-1/2} = x_J.$$

Let $I_0^* = [x_0, x_{\frac{1}{2}}]$, $I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $I_J^* = [x_{J-\frac{1}{2}}, x_J]$, $1 \le i \le J-1$. All the $I_i^*(0 \le i \le J)$ made up of the cell dual partition T_h^*, I_i^* is also called *control volume*. We use Hermite cubic polynomial space U_h as trial function space due to T_h, the base functions about x_i are

$$\varphi_i^{(0)}(x) = \begin{cases} (1 - h_i^{-1}\,|\,x - x_i\,|)^2(2h_i|\,x - x_i\,| + 1), & x_{i-1} \le x \le x_i, \\ (1 - h_{i+1}^{-1}|\,x - x_i\,|)^2(2h_{i+1}|\,x - x_i\,| + 1), & x_i \le x \le x_{i+1}, \\ 0, & \text{others}, \end{cases}$$

$$\varphi_i^{(1)}(x) = \begin{cases} (x - x_i)(h_i^{-1}|\,x - x_i\,| - 1), & x_{i-1} \le x \le x_i, \\ (x - x_i)(h_{i+1}^{-1}|\,x - x_i\,| - 1), & x_i \le x \le x_{i+1}, \\ 0, & \text{others}. \end{cases}$$

We denote an $u_h \in U_h$, uniquely by $u_h = \sum_{i=0}^{n} [u_i \varphi_i^{(0)}(x) + u_i' \varphi_i^{(1)}(x)]$.

We use piecewise linear polynomial space V_h as a test function space due to T_h^*. The base functions about x_i are

$$\psi_i^{(0)}(x) = \begin{cases} 1, & x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}}, \\ 0, & \text{others}, \end{cases}$$

$$\psi_i^{(1)}(x) = \begin{cases} x - x_i, & x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}}, \\ 0, & \text{others}. \end{cases}$$

We denote a $v_h \in V_h$, uniquely by $v_h = \sum_{i=0}^{n} [v_i \psi_i^{(0)}(x) + v_i' \psi_i^{(1)}(x)]$.

Let the time step be $\Delta t = \overline{T}/N$, where N is a positive integer and $t^n = n\Delta t$, $n = 0, 1, \ldots, N$. Let $u(t^n) = u^n$. Define the interpolation operators $\prod_h : U \to U_h$ and $\prod_h^* : U \to V_h$.

For convenience, we rewrite (1.1)-(1.4) as

$$-\frac{\partial^2 \psi}{\partial x^2} = \alpha(p - e + N(x)), \;\; (x, t) \in \Omega \times \overline{J}, \; \overline{J} = (0, T], \tag{2.1}$$

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial x}\left[D_e \frac{\partial e}{\partial x}\right] + \mu_e u \frac{\partial e}{\partial x} + eu \frac{\partial \mu_e}{\partial x}$$

$$+ \alpha\mu_e e(p - e - N(x)) - R(e, p, T), \ (x, t) \in \Omega \times \bar{J}, \tag{2.2}$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[D_p \frac{\partial p}{\partial x}\right] - \mu_p u \frac{\partial p}{\partial x} - pu \frac{\partial \mu_p}{\partial x}$$

$$- \alpha\mu_p p(p - e - N(x)) - R(e, p, T), \ (x, t) \in \Omega \times \bar{J}, \tag{2.3}$$

$$\rho(x)\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = -D_p \frac{\partial p}{\partial x} \cdot u + \mu_p p|u|^2$$

$$+ D_e \frac{\partial e}{\partial x} \cdot u + \mu_e e|u|^2, \ (x, t) \in \Omega \times \bar{J}, \tag{2.4}$$

where

$$u = -\frac{\partial \psi}{\partial x}, \ (x, t) \in \Omega \times \bar{J}. \tag{2.5}$$

For problems (1.1)-(1.5), we assume the solution to be smooth due to its physical property and the coefficients satisfy

(1) $0 < C_* \le D_s(x), \mu_s(x) \le C^*$, $s = e, p$, C_*, C^* are positive constants.

(2) $\left|\frac{\partial \mu_s(x)}{\partial x}\right| \le C, \ s = e, p.$

(3) $R(e, p, T)$ satisfies the Lipschitz condition in three variables near ε_0 field. That is to say that there exists a $C \ge 0$, when $|\varepsilon_i| \le \varepsilon_0$, $(1 \le i \le 6)$, such that

$$|R(e(x, t) + \varepsilon_1, p(x, t) + \varepsilon_3, T(x, t) + \varepsilon_5)$$
$$- R(e(x, t) + \varepsilon_2, p(x, t) + \varepsilon_4, T(x, t) + \varepsilon_6)|$$
$$\le C\{|\varepsilon_1 - \varepsilon_2| + |\varepsilon_3 - \varepsilon_4| + |\varepsilon_5 - \varepsilon_6|\}, \ (x, t) \in \Omega \times \bar{J}. \tag{2.6}$$

Define

$$a(D; u, v) = -\int_a^b \frac{\partial}{\partial x}\left(D_u \frac{\partial}{\partial x} u\right) v dx,$$

$$a(1;\, u,\, v) = a(u,\, v) = -\int_a^b \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\, u\right) v dx.$$

Let $\tau_e(x)$ be the unit vector of $(-\mu_e u,\, 1)$ and $\tau_p(x)$ be the unit vector of $(\mu_p u,\, 1)$. Denote $\phi_s = [1 + \mu_s^2 |\, u\,|^2]^{1/2},\ s = e,\ p.$ Then the characteristic derivative is approximated by

$$\frac{\partial}{\partial \tau_e} = \frac{1}{\phi_e}\frac{\partial}{\partial t} - \frac{\mu_e u}{\phi_e}\frac{\partial}{\partial x},\quad \frac{\partial}{\partial \tau_p} = \frac{1}{\phi_p}\frac{\partial}{\partial t} + \frac{\mu_p u}{\phi_p}\frac{\partial}{\partial x}. \tag{2.7}$$

We use a backward difference quotient for $\left(\frac{\partial e^n}{\partial \tau_e}\right)(x) = \frac{\partial e}{\partial \tau_e}(x,\, t^n)$ along the characteristics. Specifically, we take

$$\phi_e(x)\frac{\partial e^n}{\partial \tau_e} = \phi_e(x)\frac{e^n(x) - e^{n-1}(x + \mu_e u^n \Delta t)}{[(x - \bar{x})^2 + \Delta t^2]^{1/2}} = \frac{e^n(x) - e^{n-1}(x + \mu_e u^n \Delta t)}{\Delta t}. \tag{2.8}$$

Let $\hat{x}_e = x + \mu_e u^n \Delta t$, $\hat{e}(x) = e(\hat{x})$. Then $\phi_e \frac{\partial e^n}{\partial \tau_e} \approx \frac{e^n - \hat{e}^{n-1}}{\Delta t}$; for hole concentration. Let $\hat{x}_p = x - \mu_p u^n \Delta t$, $\hat{p}(x) = p(\hat{x})$. Then $\phi_p \frac{\partial p^n}{\partial \tau_p} \approx \frac{p^n - \hat{p}^{n-1}}{\Delta t}$.

Denote $u_h = -\nabla\psi_h$, $\bar{x}_e^{n-1} = x + \mu_e u_h^n \Delta t$, $\bar{e}_h^{n-1} = e_h(\bar{x}_e^{n-1})$; $\bar{x}_p^{n-1} = x - \mu_p u_h^n \Delta t$, $\bar{p}_h^{n-1} = p_h(\bar{x}_p^{n-1})$. Then the full discrete finite volume method along characteristics can be given by

$$a(\psi_h^n,\, \chi_h) = \alpha(p_h^n - e_h^n + N^n,\, \chi_h),\quad \forall \chi_h \in V_h,\ 0 \le n \le N, \tag{2.9}$$

$$\begin{aligned}\left(\frac{e_h^n - \bar{e}_h^{n-1}}{\Delta t},\, \omega_h\right) &+ a(D_e;\, e_h^n,\, \omega_h) - \left(\bar{e}_h^{n-1} u_h^{n-1}\frac{\partial \mu_e}{\partial x},\, \omega_h\right)\\ &- (\alpha\mu_e e_h^n(\bar{p}_h^{n-1} - \bar{e}_h^{n-1} + N^n),\, \omega_h)\\ &= -(R(\bar{e}_h^{n-1},\, \bar{p}_h^{n-1},\, T_h^{n-1}),\, \omega_h),\quad \forall \omega_h \in V_h,\ 1 \le n \le N,\end{aligned} \tag{2.10}$$

$$\left(\frac{p_h^n - \bar{p}_h^{n-1}}{\Delta t},\, \omega_h\right) + a(D_p;\, p_h^n,\, \omega_h) + \left(\bar{p}_h^{n-1} u_h^{n-1}\frac{\partial \mu_p}{\partial x},\, \omega_h\right)$$

$$+ (\alpha\mu_p p_h^n(\overline{p}_h^{n-1} - \overline{e}_h^{n-1} + N^n), \omega_h)$$

$$= -(R(\overline{e}_h^{n-1}, \overline{p}_h^{n-1}, T_h^{n-1}), \omega_h), \quad \forall \omega_h \in V_h, 1 \le n \le N, \tag{2.11}$$

$$\left(\rho \frac{T_h^n - T_h^{n-1}}{\Delta t}, z_h\right) + a(T_h^n, z_h)$$

$$= -\left(D_p \frac{\partial p_h^n}{\partial x} \cdot u_h^n, z_h\right) + (\mu_p p_h^n | u_h^n |^2, z_h) + \left(D_e \frac{\partial e_h^n}{\partial x} \cdot u_h^n, z_h\right)$$

$$+ (\mu_e e_h^n | u_h^n |^2, z_h), \quad \forall z_h \in V_h. \tag{2.12}$$

$$e_h^0(x) = (\Pi_h e_0)(x), \; p_h^0(x) = (\Pi_h p_0)(x), \; T_h^0(x) = (\Pi_h T_0)(x), \; x \in \Omega. \tag{2.13}$$

The algorithm of the process is: the values of e_h^0, p_h^0, T_h^0 are defined by (2.13), by (2.9) we can obtain ψ_h^0; for $n = 1, 2, \ldots, N$, assuming that $\psi_h^{n-1}, e_h^{n-1}, p_h^{n-1}, T_h^{n-1}$ are obtained, by (2.10), (2.11) and (2.12) we can calculate the values of e_h^n, p_h^n and T_h^n, and then from (2.9) we get the value of ψ_h^n.

3. Convergence Analysis

3.1. Definitions and lemmas

Definition 1 [4, 7]**.** We define the discrete norm on U_h : $\forall u_h \in U_h$,

$$\| u_h \|_{0,h}^2 = (u_h, u_h)_{0,h} = (\Pi_h^* u_h, \Pi_h^* u_h),$$

$$| u_h |_{1,h}^2 = \sum_{j=1}^{n} h_j \left[\left(\frac{u_j - u_{j-1}}{h_j} \right)^2 + u_{j-1}'^2 + u_j'^2 \right],$$

$$\| u_h \|_{1,h}^2 = | u_h |_{1,h}^2 + \| u_h \|_{0,h}^2.$$

Lemma 1. *For all* $u_h \in U_h$, *there exist two positive constants* C_1, C_2

independent of U_h, *such that*

$$C_1\| u_h \| \le \| u_h \|_{0,h} \le C_2\| u_h \|,$$

$$C_1\| u_h \|_1 \le \| u_h \|_{1,h} \le C_2\| u_h \|_1.$$

Lemma 2. *For all* $u_h \in U_h$, *and for h sufficiently small, there is*

$$a(D;\, u_h,\, \Pi_h^* u_h) \ge C\| u_h \|_1^2. \tag{3.1}$$

$a(D;\, u_h,\, \Pi_h^* v_h)$ *is denoted by* $a(D;\, u_h,\, \Pi_h^* v_h) = a_1(D;\, u_h,\, \Pi_h^* v_h) +$ $a_1(D;\, u_h,\, \Pi_h^* v_h)$, *where*

$$a_1(D;\, u_h,\, \Pi_h^* v_h) = a_1(D;\, v_h,\, \Pi_h^* u_h),\ \ \forall u_h,\, v_h \in U_h \tag{3.2}$$

and

$$C_1\| u_h \|_1^2 \le a_1(D;\, u_h,\, \Pi_h^* u_h) \le C_2\| u_h \|_1,$$

the second term satisfies

$$|\, a_2(D;\, u_h,\, \Pi_h^* v_h)\,| \le Ch\| u_h \|_1\| v_h \|_1,\ \ \forall u_h,\, v_h \in U_h. \tag{3.3}$$

Let $\|| \cdot \||_1 = [a_1(D;\, u_h,\, \Pi_h^* u_h)]^{1/2}$, $u_h \in U_h$. *Then* $\|| \cdot \||_1$ *is equivalent to* H^1 *norm* $\| \cdot \|_1$.

Proof. In this paper, $a_1(D;\, u_h,\, \Pi_h^* v_h)$ and $a_2(D;\, u_h,\, \Pi_h^* v_h)$ are the same as $B(D;\, u_h,\, \Pi_h^* v_h)$ and $E(D;\, u_h,\, \Pi_h^* v_h)$ in [7]. We only prove (3.3), the proof of the others can be seen in [7]. Now

$$\begin{aligned} a_2(D;\, u_h,\, \Pi_h^* v_h) &= a(D;\, u_h,\, \Pi_h^* v_h) - a_1(D;\, u_h,\, \Pi_h^* v_h) \\ &= \sum_{j=0}^{J} v_j' \int_{x_{j-1/2}}^{x_{j+1/2}} D'(x)(u_j - u_h)dx \\ &= \sum_{j=0}^{J} v_j' \int_{x_{j-1/2}}^{x_{j+1/2}} D'(x)u_h'(\vartheta_j)(x - x_j)dx. \end{aligned}$$

Following the analytic tradition of Li et al. [4], we know that in $I_j =$

$[x_{j-1}, x_j]$, there is

$$|u_h'(x)| \le C\left(\left|\frac{u_j - u_{j-1}}{h_j}\right| + |u_{j-1}'| + |u_j'|\right).$$

Using Hölder's inequality and the regularity of the partition, we have

$$|a_2(u_h, \Pi_h^* v_h)| \le C\left\{\sum_{j=0}^{J}(v_j')^2\right\}^{1/2}\left\{\sum_{j=0}^{J}[u_h'(\vartheta_j)]^2\right\}^{1/2} h$$

$$\le Ch|v_j|_{1,h}|u_j|_{1,h} \le Ch\|u_h\|_1\|v_h\|_1.$$

In the last step we use that $|\cdot|_1$ and $\|\cdot\|_1$ are equivalent in $H_E^1(\Omega)$.

Lemma 3. *Let* $(u_h, \Pi_h^* u_h) \equiv |\!|\!| u_h |\!|\!|_{0,h}^2$, $\forall u_h \in U_h$. *Then*

$$C_1\|u_h\| \le |\!|\!| u_h |\!|\!|_{0,h} \le C_2\|u_h\|. \tag{3.4}$$

Proof. Firstly from the interpolation theorem in Sobolev space, we have

$$\|\Pi_h^* u - u\| \le Ch^{4-m}\|u\|_m, \ 0 \le m \le 4, \ \forall u \in U.$$

By the triangle inequality

$$\|\Pi_h^* u\| \le \|u_h\| + \|\Pi_h^* u - u\| \le C\|u_h\|,$$

so that

$$(u_h, \Pi_h^* u) \le C\|u_h\|\|\Pi_h^* u\| \le C\|u_h\|^2.$$

Secondly Li et al. [4] have proved that $(u_h, \Pi_h^* u) \ge \beta\|u_h\|^2$, and thus

$$C_1\|u_h\| \le (u_h, \Pi_h^* u) \le C_2\|u_h\|.$$

The proof is complete.

Definition 2. For all $u_h \in U_h$, we introduce the negative norm as

$$\|u_h\|_{-1,h} = \sup_{\omega_h \in U_h \text{ and } \|\omega_h\|_1 \ne 0}\left|\frac{(u_h, \Pi_h^* \omega_h)}{\|\omega_h\|_1}\right|. \tag{3.5}$$

By Lemma 3 and the theory in [6] we can obtain that $\| u_h \|_{-1,h}$ and $\| u_h \|_{-1}$ are equivalent norms.

Note. In the following convergence analysis we do not declare when we use continuous norm instead of discrete norm.

3.2. H^1 norm error estimate

For convenience we suppose that the solution of (1.1)-(1.5) is Ω-periodic, and hence the corresponding discrete scheme (2.9)-(2.13) is also Ω-periodic. For ordinary boundary value problems we can get similar results dealing with the technique extension or mirror reflection (see [10]).

Now, we introduce the elliptic projection of the solution: let $\tilde{\psi} = \tilde{\psi}_h$, satisfy

$$a(\psi - \tilde{\psi}, \chi_h) = 0, \quad \forall \chi_h \in V_h, \tag{3.6}$$

and let $\tilde{e}, \tilde{p}, \tilde{T} \in V_h$, satisfy

$$a(D_e;\, e - \tilde{e}, \omega_h) + \lambda_e(e - \tilde{e}, \omega_h) = 0, \quad \forall \omega_h \in V_h, \tag{3.7}$$

$$a(D_p;\, p - \tilde{p}, \omega_h) + \lambda_p(p - \tilde{p}, \omega_h) = 0, \quad \forall \omega_h \in V_h, \tag{3.8}$$

$$a(T - \tilde{T}, \omega_h) + \lambda_T(T - \tilde{T}, \omega_h) = 0, \quad \forall \omega_h \in V_h, \tag{3.9}$$

where $\lambda_s (s = e, p, T)$ is a positive constant. It is well-known [2, 8] that for $\psi, e, p, T \in H^4(I) \cap H_E^1(I)$, we have

$$\| \psi - \tilde{\psi} \| + h \| \psi - \tilde{\psi} \|_1 \le Ch^4 \| \psi \|_4 \tag{3.10}$$

$$\| s - \tilde{s} \| + h \| s - \tilde{s} \|_1 + \left\| \frac{\partial (s - \tilde{s})}{\partial t} \right\| + h \left\| \frac{\partial (s - \tilde{s})}{\partial t} \right\|_1$$

$$\le C \left[\| s \|_4 + \left\| \frac{\partial s}{\partial t} \right\|_4 \right] h^4, \quad s = e, p, T. \tag{3.11}$$

Denote $\theta = \psi_h - \tilde{\psi}$, $\eta = \psi - \tilde{\psi}$; $\xi_e = e_h - \tilde{e}$, $\zeta_e = e - \tilde{e}$; $\xi_p = p_h - \tilde{p}$, $\zeta_p = p - \tilde{p}$; $\xi_T = T_h - \tilde{T}$, $\zeta_T = T - \tilde{T}$. First from (2.9) and (3.6) we can get the error estimate equation of electric potential

$$a(\theta^n, \chi_h) = \alpha(p_h^n - p^n + e^n - e_h^n, \chi_h). \tag{3.12}$$

We choose $\chi_h = \prod_h^* \theta^n$ as test function in (3.12). By Lemma 2, we have

$$\| \theta^n \|_1 \leq C\{\| \xi_e^n \| + \| \xi_p^n \| + h^4\}. \tag{3.13}$$

Next we examine electron concentration, for hole concentration we have the similar results. By (2.2), (2.10) and (3.7), we obtain

$$\begin{aligned}
&\left(\frac{\xi_e^n - \bar{\xi}_e^{n-1}}{\Delta t}, \omega_h\right) + a(D_e; \xi_e^n, \omega_h) \\
&= \left(\left[\frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t}\right], \omega_h\right) \\
&\quad + \alpha(\mu_e[e_h^n(\bar{p}_h^{n-1} - \bar{e}_h^{n-1} + N) - e^n(p^n - e^n + N)], \omega_h) \\
&\quad + \left([\bar{e}_h^{n-1} u_h^n - e^n u^n] \cdot \frac{\partial \mu_e}{\partial x}, \omega_h\right) + (R(e^n, p^n, T^n) - R(\bar{e}_h^{n-1}, \bar{p}_h^{n-1}, T_h^{n-1}), \omega_h) \\
&\quad + \lambda_e(e^n - \tilde{e}^n, \omega_h) + \left(\frac{\zeta_e^n - \bar{\zeta}_e^{n-1}}{\Delta t}, \omega_h\right).
\end{aligned} \tag{3.14}$$

For convenience, we rewrite (3.14) as

$$\begin{aligned}
&\left(\frac{\xi_e^n - \xi_e^{n-1}}{\Delta t}, \omega_h\right) + a(D_e; \xi_e^n, \omega_h) \\
&= \left(\left[\frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u_h^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t}\right], \omega_h\right) + \left(\frac{\bar{\xi}_e^{n-1} - \xi_e^{n-1}}{\Delta t}, \omega_h\right) \\
&\quad + \left(\frac{\zeta_e^n - \zeta_e^{n-1}}{\Delta t}, \omega_h\right) + \left(\frac{\zeta_e^{n-1} - \bar{\zeta}_e^{n-1}}{\Delta t}, \omega_h\right) \\
&\quad + \left([\bar{e}_h^{n-1} u_h^n - e^n u^n] \cdot \frac{\partial \mu_e}{\partial x}, \omega_h\right) + \alpha(\mu_e[e_h^n(\bar{p}_h^{n-1} - \bar{e}_h^{n-1} + N) \\
&\quad - e^n(p^n - e^n + N)], \omega_h) \\
&\quad + (R(e^n, p^n, T^n) - R(\bar{e}_h^{n-1}, \bar{p}_h^{n-1}, T_h^{n-1}), \omega_h) + \lambda_e(e^n - \tilde{e}^n, \omega_h).
\end{aligned} \tag{3.15}$$

We choose $\omega_h = \prod_h^* \frac{\xi_e^n - \xi_e^{n-1}}{\Delta t} = \prod_h^* d_t \xi_e^n.$ as test function in (3.15). Using Lemma 2 to the left hand of (3.15), we get

$$a\left(D_e;\, \xi_e^n,\, \prod_h^* \frac{\xi_e^n - \xi_e^{n-1}}{\Delta t}\right)$$

$$= \frac{1}{2\Delta t}\left(a(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^*(\xi_e^n - \xi_e^{n-1})) + a(D_e;\, \xi_e^n - \xi_e^{n-1},\, \prod_h^*(\xi_e^n - \xi_e^{n-1}))\right)$$

$$\geq \frac{1}{2\Delta t}\left(a(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^*(\xi_e^n - \xi_e^{n-1}))\right)$$

$$= \frac{1}{2\Delta t}\left(a_1(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^*(\xi_e^n - \xi_e^{n-1})) + a_2(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^*(\xi_e^n - \xi_e^{n-1}))\right)$$

$$\geq \frac{1}{2\Delta t}\left(|||\xi_e^n|||_1^2 - |||\xi_e^{n-1}|||_1^2\right) + a_2(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^* d_t\xi_e^n). \tag{3.16}$$

By Lemma 3, we obtain

$$|||\, d_t\xi_e^n\, |||_{0,h}^2 + \frac{1}{2\Delta t}\left(|||\, \xi_e^n\, |||_1^2 - |||\, \xi_e^{n-1}\, |||_1^2\right)$$

$$= \left(\left[\frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x}\cdot u_h^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t}\right],\, \prod_h^* d_t\xi_e^n\right) + \left(\frac{\bar{\xi}_e^{n-1} - \xi_e^{n-1}}{\Delta t},\, \prod_h^* d_t\xi_e^n\right)$$

$$+ \left(\frac{\zeta_e^n - \zeta_e^{n-1}}{\Delta t},\, \prod_h^* d_t\xi_e^n\right) + \left(\frac{\zeta_e^{n-1} - \bar{\zeta}_e^{n-1}}{\Delta t},\, \prod_h^* d_t\xi_e^n\right)$$

$$+ \left([\bar{e}_h^{n-1}u_h^n - e^n u^n]\cdot \frac{\partial \mu_e}{\partial x},\, \prod_h^* d_t\xi_e^n\right)$$

$$+ \alpha(\mu_e[e_h^n(\bar{p}_h^{n-1} - \bar{e}_h^{n-1} + N) - e^n(p^n - e^n + N)],\, \prod_h^* d_t\xi_e^n) + (R(e^n,\, p^n,\, T^n)$$

$$- R(\bar{e}_h^{n-1},\, \bar{p}_h^{n-1},\, T_h^{n-1}),\, \prod_h^* d_t\xi_e^n) + \lambda_e(e^n - \tilde{e}^n,\, \prod_h^* d_t\xi_e^n)$$

$$- a_2(D_e;\, \xi_e^n + \xi_e^{n-1},\, \prod_h^* d_t\xi_e^n). \tag{3.17}$$

Multiplying (3.17) by $2\Delta t,$ summing for $1 \leq n \leq N$ and using continuous norm instead of discrete norm, we have

$$2\sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t + \| \xi_e^N \|_1^2 - \| \xi_e^0 \|_1^2 \leq \sum_{i=1}^{9} T_i. \tag{3.18}$$

We now estimate the terms of the right hand of (3.18), for T_1,

$$| T_1 | \leq C_1 \sum_{n=1}^{N} \left\| \frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u_h^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t} \right\|^2 \Delta t + \varepsilon \sum_{n=1}^{N} \| \Pi_h^* d_t \xi_e^n \|^2 \Delta t. \tag{3.19}$$

Let $\hat{\phi}_e = [1 + \mu_e^2 | u_h^n |^2]^{1/2}$, so we have $\frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u_h^n = \hat{\phi}_e \frac{\partial e^n}{\partial \tau_e}$, and thus the following [8],

$$\left\| \frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u_h^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t} \right\|^2 \leq C \Delta t \int_{t^{n-1}}^{t^n} \int_{\Omega} \left| \frac{\partial^2 e}{\partial \tau_e^2} \right|^2 dx dt. \tag{3.20}$$

So that

$$| T_1 | \leq C \left\| \frac{\partial^2 e}{\partial \tau_e^2} \right\|_{L^2(J; L^2(\Omega))}^2 (\Delta t)^2 + \varepsilon \sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t. \tag{3.21}$$

To estimate the rest of terms of T_1, suppose the partition parameter satisfies $\Delta t = o(h)$. We introduce the hypothesis [10]:

$$\sup_{0 \leq n \leq N} \{ \| \xi_e^{n-1} \|_\infty + \| \xi_p^{n-1} \|_\infty + \| \xi_T^{n-1} \|_\infty \} \to 0, \ (h, \Delta t) \to 0 \tag{3.22}$$

$$\sup_{0 \leq n \leq N} \| \nabla \theta^n \|_\infty \to 0, (h, \Delta t) \to 0. \tag{3.23}$$

So that when $(h, \Delta t)$ is sufficiently small, we have

$$| e_h^n | + | p_h^n | + | T_h^n | \leq C, \ | u_h^n | \leq C. \tag{3.24}$$

To handle T_2, using the hypothesis and the result of [8]: $\left\| \frac{\bar{\xi}_e^{n-1} - \xi_e^{n-1}}{\Delta t} \right\|^2 \leq C \| \nabla \xi_e^{n-1} \|^2$, we have

$$
\begin{aligned}
|T_2| &\le C\sum_{n=1}^{N}\|\nabla\xi_e^{n-1}\|^2\Delta t + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t \\
&\le C\sum_{n=1}^{N}\|\xi_e^{n-1}\|_1^2\Delta t + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t. \qquad (3.25)
\end{aligned}
$$

In treating T_3 using (3.11) and T_4 similar to T_2, we obtain

$$
|T_3| \le C\sum_{n=1}^{N}\left\|\frac{\partial\zeta_e^n}{\partial t}\right\|\|\Pi_h^* d_t\xi_e^n\| \le Ch^8 + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t. \qquad (3.26)
$$

$$
|T_4| \le C\sum_{n=1}^{N}\|\nabla\zeta_e^{n-1}\|^2\Delta t + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t \le Ch^6 + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t. \qquad (3.27)
$$

By (3.5), (3.6) and (3.24) we examine T_5, T_6,

$$
\begin{aligned}
|T_5| &= 2\sum_{n=1}^{N}\left|\left(\frac{\partial\mu_e}{\partial x}[u_h^{n-1}(\bar{e}_h^{n-1} - e^n) + e^n(u_h^{n-1} - u^n)], \Pi_h^* d_t\xi_e^n\right)\right|\Delta t \\
&\le C_1\sum_{n=1}^{N}(\|\xi_e^{n-1}\|^2 + \|\xi_p^{n-1}\|^2)\Delta t + C_2(h^8 + h^6 + (\Delta t)^2) \\
&\quad + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t. \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
|T_6| &= 2\sum_{n=1}^{N}|\alpha\mu_e(e_h^n(\bar{p}_h^{n-1} - p^n) + e_h^n(e^n - \bar{e}_h^{n-1}) \\
&\quad + (e_h^n - e^n)(p^n - e^n + N), \Pi_h^* d_t\xi_e^n)|\Delta t \\
&\le C_1\sum_{n=1}^{N}(\|\xi_e^n\|^2 + \|\xi_p^{n-1}\|^2)\Delta t + C_2(h^8 + (\Delta t)^2) + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t. \qquad (3.29)
\end{aligned}
$$

By (2.6), we have

$$
\begin{aligned}
|T_7| &\le C\sum_{n=1}^{N}(\|e^n - \bar{e}_h^{n-1}\|^2 + \|p^n - \bar{p}_h^{n-1}\|^2 + \|T^n - T_h^{n-1}\|^2)\Delta t \\
&\quad + \varepsilon\sum_{n=1}^{N}\|d_t\xi_e^n\|^2\Delta t
\end{aligned}
$$

$$\leq C_1 \sum_{n=1}^{N} (\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2) \Delta t$$

$$+ C_2 (h^8 + (\Delta t)^2) + \varepsilon \sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t. \tag{3.30}$$

$$| T_8 | \leq C \sum_{n=1}^{N} \| \zeta_e^n \| \| \Pi_h^* d_t \xi_e^n \| \Delta t \leq C h^8 + \varepsilon \sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t. \tag{3.31}$$

To estimate T_9, we use (3.3) and the inverse argument of finite element method [2]:

$$| T_9 | \leq C \sum_{n=1}^{N} h \| \xi_e^n + \xi_e^{n-1} \|_1 \| d_t \xi_e^n \|_1 \Delta t \leq C \sum_{n=1}^{N} \| \xi_e^n + \xi_e^{n-1} \|_1 \| d_t \xi_e^n \| \Delta t$$

$$\leq C \sum_{n=1}^{N} \| \xi_e^n \|_1^2 \Delta t + \varepsilon \sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t. \tag{3.32}$$

By the estimate of all T_i $(1 \leq i \leq 9)$ and (3.18), make ε properly small and note that $\xi_e^0 = \xi_p^0 = 0$, so that

$$\| \xi_e^N \|_1^2 + \sum_{n=1}^{N} \| d_t \xi_e^n \|^2 \Delta t \leq C_1 \sum_{n=1}^{N} (\| \xi_e^{n-1} \|_1^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2) \Delta t$$

$$+ C_2 (h^6 + (\Delta t)^2). \tag{3.33}$$

For hole concentration, choosing $\omega_h = \Pi_h^* \dfrac{\xi_p^n - \xi_p^{n-1}}{\Delta t} = \Pi_h^* d_t \xi_p^n$ as test function, we have

$$\| \xi_p^N \|_1^2 + \sum_{n=1}^{N} \| d_t \xi_p^n \|^2 \Delta t \leq C_1 \sum_{n=1}^{N} (\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|_1^2 + \| \xi_T^{n-1} \|^2) \Delta t$$

$$+ C_2 (h^6 + (\Delta t)^2). \tag{3.34}$$

At last we discuss the error estimate of temperature. By (2.4), (2.12) and (3.9), we obtain

$$\left(\rho \frac{\xi_T^n - \xi_T^{n-1}}{\Delta t}, z_h \right) + a(\xi_T^n, z_h) = \left(\rho \left[\frac{\partial T}{\partial t} - \frac{T^n - T^{n-1}}{\Delta t} \right], z_h \right)$$

$$+ \left(\rho \frac{\zeta_T^n - \zeta_T^{n-1}}{\Delta t}, z_h \right) + \lambda_T (T - \widetilde{T}, z_h)$$

$$
\begin{aligned}
&+\left(D_p\left(\frac{\partial p_h^n}{\partial x}\cdot u_h^n-\frac{\partial p^n}{\partial x}\cdot u^n\right), z_h\right)\\
&+(\mu_p(p_h^n|u_h^n|^2-p^n|u^n|^2), z_h)\\
&-\left(D_e\left(\frac{\partial e_h^n}{\partial x}\cdot u_h^n-\frac{\partial e^n}{\partial x}\cdot u^n\right), z_h\right)\\
&+(\mu_e(e_h^n|u_h^n|^2-e^n|u^n|^2), z_h). \qquad (3.35)
\end{aligned}
$$

We choose $z_h = \prod_h^* \frac{\xi_T^n-\xi_T^{n-1}}{\Delta t} = \prod_h^* d_t\xi_T^n$ as test function in (3.35). By Lemmas 2 and 3 and similar to (3.17), we have

$$
\begin{aligned}
&|\| d_t\xi_T^n \||_{0,h}^2+\frac{1}{2\Delta t}(|\| \xi_T^n \||_1^2-|\| \xi_T^{n-1} \||_1^2)\\
\le &\left(\rho\left[\frac{\partial T}{\partial t}-\frac{T^n-T^{n-1}}{\Delta t}\right], \prod_h^* d_t\xi_T^n\right)+\left(\rho\,\frac{\zeta_T^n-\zeta_T^{n-1}}{\Delta t}, \prod_h^* d_t\xi_T^n\right)\\
&+\lambda_T(T-\widetilde{T}, \prod_h^* d_t\xi_T^n)+\left(D_p\left(\frac{\partial p_h^n}{\partial x}\cdot u_h^n-\frac{\partial p^n}{\partial x}\cdot u^n\right), \prod_h^* d_t\xi_T^n\right)\\
&+(\mu_p(p_h^n|u_h^n|^2-p^n|u^n|^2), \prod_h^* d_t\xi_T^n)-\left(D_e\left(\frac{\partial e_h^n}{\partial x}\cdot u_h^n-\frac{\partial e^n}{\partial x}\cdot u^n\right), \prod_h^* d_t\xi_T^n\right)\\
&+(\mu_e(e_h^n|u_h^n|^2-e^n|u^n|^2), \prod_h^* d_t\xi_T^n)-a_2(\xi_T^n+\xi_T^{n-1}, \prod_h^* d_t\xi_T^n). \qquad (3.36)
\end{aligned}
$$

Multiplying (3.36) by $2\Delta T$, summing for $1\le n\le N$ and using continuous norm instead of discrete norm and noting that $\xi_T^0=0$, we have

$$\|\xi_T^N\|_1^2+2\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t\le\sum_{i=1}^{8}Q_i. \qquad (3.37)$$

To deal with the right hand terms of (3.37), we obtain

$$|Q_1+Q_2+Q_3|\le C((\Delta t)^2+h^8)+\varepsilon\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t. \qquad (3.38)$$

By the hypothesis (3.24), the bounds for Q_4, Q_5 are

$$|Q_4| = 2\sum_{n=1}^{N}\left|\left(\left(D_p\left(\frac{\partial p_h^n}{\partial x} - \frac{\partial p^n}{\partial x}\right)\cdot u_h^n + \frac{\partial p^n}{\partial x}(u_h^n - u^n)\right), \Pi_h^* \xi_T^n\right)\right|\Delta t$$

$$\le C_1\sum_{n=1}^{N}(\|\xi_e^n\|_1^2 + \|\xi_p^n\|_1^2)\Delta t + C_2(h^6 + (\Delta t)^2) + \varepsilon\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t. \quad (3.39)$$

$$|Q_5| = 2\sum_{n=1}^{N}|(\mu_p(p_h^n - p^n)|u_h^n|^2 + \mu_p p^n(|u_h^n|^2 - |u^n|^2), \Pi_h^* d_t\xi_T^n)|\Delta t$$

$$\le C_1\sum_{n=1}^{N}(\|\xi_e^n\|^2 + \|\xi_p^n\|^2)\Delta t + C_2(h^6 + (\Delta t)^2) + \varepsilon\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t. \quad (3.40)$$

For Q_6, Q_7, using the same estimates as that of Q_5, Q_6, we obtain

$$|Q_6 + Q_7| \le C_1\sum_{n=1}^{N}(\|\xi_e^n\|_1^2 + \|\xi_p^n\|_1^2)\Delta t$$

$$+ C_2(h^6 + (\Delta t)^2) + \varepsilon\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t. \quad (3.41)$$

By (3.3) and inverse arguments of finite element method, we have

$$|Q_8| \le C_1\sum_{n=1}^{N}\|\xi_T^n\|_1^2\Delta t + \varepsilon\sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t. \quad (3.42)$$

Put all the Q_i $(1 \le i \le 8)$ into (3.37), hide ε terms, we have

$$\|\xi_T^N\|_1^2 + \sum_{n=1}^{N}\|d_t\xi_T^n\|^2\Delta t \le C_1\sum_{n=1}^{N}(\|\xi_e^n\|_1^2 + \|\xi_p^n\|_1^2 + \|\xi_T^{n-1}\|_1^2)\Delta t$$

$$+ C_2(h^6 + (\Delta t)^2). \quad (3.43)$$

Combining (3.33) and (3.34), we obtain

$$\|\xi_e^N\|_1^2 + \|\xi_p^N\|_1^2 + \|\xi_T^N\|_1^2 + \sum_{n=1}^{N}(\|d_t\xi_e^n\|_1^2 + \|d_t\xi_p^n\|_1^2 + \|d_t\xi_T^n\|^2)\Delta t$$

$$\le C_1\sum_{n=1}^{N}(\|\xi_e^{n-1}\|_1^2 + \|\xi_p^{n-1}\|_1^2 + \|\xi_T^{n-1}\|_1^2)\Delta t + C_2\{h^6 + (\Delta t)^2\}. \quad (3.44)$$

Applying the discrete Gronwall lemma, we conclude that

$$\|\xi_e^N\|_1^2 + \|\xi_p^N\|_1^2 + \|\xi_T^N\|_1^2 + \sum_{n=1}^{N}(\|d_t\xi_e^n\|_1^2 + \|d_t\xi_p^n\|_1^2 + \|d_t\xi_T^n\|^2)\Delta t$$

$$\leq C\{h^6 + (\Delta t)^2\}. \tag{3.45}$$

Now, we prove the hypotheses (3.22), (3.23).

(i) It is easy to see that $\theta^0 = 0$, $\xi_e^0 = \xi_p^0 = \xi_T^0 = 0$ when $n = 0$. So (3.22) and (3.23) are true.

(ii) Assume (3.22) and (3.23) are true when $n = L - 1$. By (3.13) and (3.45), we have

$$\|\xi_e^L\|_\infty + \|\xi_p^L\|_\infty + \|\xi_T^L\|_\infty \leq Ch^{-1/2}(h^3 + \Delta t) \to 0, \ \ (h, \Delta t) \to 0$$

$$\|\nabla\theta^L\|_\infty \leq Ch^{-1/2}\|\theta^L\|_1 \leq Ch^{-1/2}(h^3 + \Delta t) \to 0, \ \ (h, \Delta t) \to 0.$$

So (3.22) and (3.23) are true for all the $n\,(1 \leq n \leq N)$.

Thus, we conclude the theorem.

Theorem 1. *Let* $\{e, p, \psi\}$ *be the solution of* (1.1)-(1.5) *and have some smoothness,* $\{e_h^n, p_h^n, \psi_h^n\}$ *be the solution of the full discrete characteristic finite volume element scheme* (2.9)-(2.13). *When* h *sufficiently small and satisfies* $\Delta t = o(h)$, *then we can obtain* H^1 *norm error estimate as*

$$\sup_{0\leq n\leq N}\{\|e^n - e_h^n\|_1 + \|p^n - p_h^n\|_1 + \|T^n - T_h^n\|_1\} \leq C(h^3 + \Delta t). \tag{3.46}$$

3.3. L^2 norm error estimate

We choose $\omega_h = \prod_h^* \xi_e^n$ as the test function in (3.15).

$$\left(\frac{\xi_e^n - \xi_e^{n-1}}{\Delta t}, \prod_h^* \xi_e^n\right) + a(D_e; \xi_e^n, \prod_h^* \xi_e^n)$$

$$= \left(\left[\frac{\partial e^n}{\partial t} - \mu_e \frac{\partial e^n}{\partial x} \cdot u_h^n - \frac{e^n - \bar{e}^{n-1}}{\Delta t}\right], \prod_h^* \xi_e^n\right) + \left(\frac{\bar{\xi}_e^{n-1} - \xi_e^{n-1}}{\Delta t}, \prod_h^* \xi_e^n\right)$$

$$+\left(\frac{\zeta_e^n - \zeta_e^{n-1}}{\Delta t}, \Pi_h^* \xi_e^n\right) + \left(\frac{\zeta_e^{n-1} - \overline{\zeta}_e^{n-1}}{\Delta t}, \Pi_h^* \xi_e^n\right)$$

$$+ ([\overline{e}_h^{n-1} u_h^n - e^n u^n] \cdot \frac{\partial \mu_e}{\partial x}, \Pi_h^* \xi_e^n) + \alpha(\mu_e [e_h^n (\overline{p}_h^{n-1} - \overline{e}_h^{n-1} + N)$$

$$- e^n (p^n - e^n + N)], \Pi_h^* \xi_e^n)$$

$$+ (R(e^n, p^n, T^n) - R(\overline{e}_h^{n-1}, \overline{p}_h^{n-1}, T_h^{n-1}), \Pi_h^* \xi_e^n)$$

$$+ \lambda_e (e^n - \widetilde{e}^n, \Pi_h^* \xi_e^n). \tag{3.47}$$

To handle the left hand of (3.47), we apply Lemma 3 and (3.1) in Lemma 2, and obtain

$$\left(\frac{\xi_e^n - \xi_e^{n-1}}{\Delta t}, \Pi_h^* \xi_e^n\right) + a(D_e; \xi_e^n, \Pi_h^* \xi_e^n)$$

$$\geq \frac{1}{2\Delta t} (\|| \xi_e^n \||_{0,h}^2 - \|| \xi_e^{n-1} \||_{0,h}^2) + C_0 \| \xi_e^n \|_1^2. \tag{3.48}$$

Multiplying (3.48) by $2\Delta t$, summing for $1 \leq n \leq N$ and using continuous norm instead of discrete norm and noting that $\xi_e^0 = \xi_p^0 = 0$, we have

$$\| \xi_e^N \|^2 + C_0 \sum_{n=1}^{N} \| \xi_e^n \|_1^2 \Delta t \leq \sum_{i=1}^{8} T_i. \tag{3.49}$$

To treat the first three terms in the same way as done in H^1 norm error estimate, we obtain

$$| T_1 | \leq C_1 (\Delta t)^2 + C_2 \sum_{n=1}^{N} \| \xi_e^n \|^2 \Delta t. \tag{3.50}$$

$$| T_2 | \leq \varepsilon \sum_{n=1}^{N} \| \xi_e^{n-1} \|_1^2 \Delta t + C \sum_{n=1}^{N} \| \xi_e^n \|^2 \Delta t. \tag{3.51}$$

$$| T_3 | \leq C_1 h^8 + C_2 \sum_{n=1}^{N} \| \xi_e^n \|^2 \Delta t. \tag{3.52}$$

To handle T_4, we cannot proceed in a way similar to that in (3.27) as the term $\| \nabla \zeta_e^{n-1} \|$ leading to h^6 order will appear which we do not want. To avoid this thing, we use Definition 2 of H^{-1} norm estimate. Consider the transformation: $f(x) = x + \mu_e u_h^n \Delta t$, it follows from [8] that f is a bijection of Ω onto itself. We see that

$$|\det(Df)(x) - 1| \le C\Delta t, \quad \forall x \in \Omega. \tag{3.53}$$

$$\begin{aligned}
&\left\| \frac{\zeta_e^n - \zeta_e^{n-1}}{\Delta t} \right\|_{-1} \\
&= \frac{1}{\Delta t} \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \left[\int_\Omega \zeta_e^{n-1}(x) g(x) dx - \int_\Omega \zeta_e^{n-1}(f(x)) g(x) dx \right] \right) \\
&= \frac{1}{\Delta t} \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \left[\int_\Omega \zeta_e^{n-1}(x) g(x) dx - \int_\Omega \zeta_e^{n-1}(x) g(f - 1(x)) \det(Df)(x)^{-1} dx \right] \right) \\
&\le \frac{1}{\Delta t} \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \left[\int_\Omega \zeta_e^{n-1}(x) \{ g(x) - g(f^{-1}(x)) \} dx \right] \right) \\
&\quad + \frac{1}{\Delta t} \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \left[\int_\Omega \zeta_e^{n-1}(x) g(f^{-1}(x)) [1 - \det(Df)(x)^{-1}] dx \right] \right) \\
&\equiv W_1 + W_2.
\end{aligned} \tag{3.54}$$

Following the results of [8],

$$\| g - g \circ f^{-1} \| \le C\Delta t \| (\nabla g) \circ f^{-1} \| \le C \| g \|_1 \Delta t. \tag{3.55}$$

Combining with (3.53), we obtain

$$W_1 \le C \frac{1}{\Delta t} \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \| \zeta_e^{n-1} \| \| g - g \circ f^{-1} \| \right) \le C \| \zeta_e^{n-1} \|. \tag{3.56}$$

$$W_2 \le C \sup_{g \in U_h} \left(\frac{1}{\| g \|_1} \| \zeta_e^{n-1} \| \| g \circ f^{-1} \| \right) \le C \| \zeta_e^{n-1} \|. \tag{3.57}$$

By (3.54), (3.56) and (3.57), we have shown that

$$\begin{aligned}
| T_4 | &\le C \sum_{n=1}^{N} \left\| \frac{\zeta_e^n - \zeta_e^{n-1}}{\Delta t} \right\|_{-1} \| \xi_e^n \|_1 \Delta t \le C \sum_{n=1}^{N} \| \zeta_e^{n-1} \| \Delta t \\
&\quad + \varepsilon \sum_{n=1}^{N} \| \xi_e^n \|_1^2 \Delta t \le Ch^8 + \varepsilon \sum_{n=1}^{N} \| \xi_e^n \|_1^2 \Delta t.
\end{aligned} \tag{3.58}$$

To estimate T_5, we use partition integral and transfer the spacial derivative of $u_h^{n-1} - u^n = \dfrac{\partial(\psi_h^{n-1} - \psi^n)}{\partial x}$ to ξ_e^n, and get

$$| T_5 | = 2\sum_{n=1}^{N}\left|\left(\frac{\partial \mu_e}{\partial x}[u_h^{n-1}(\bar{e}_h^{n-1} - e^n) + e^n(u_h^{n-1} - u^n)], \Pi_h^* \xi_e^n\right)\right|\Delta t$$

$$\leq C_1\sum_{n=1}^{N}(\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2)\Delta t + C_2(h^8 + (\Delta t)^2) + \varepsilon\sum_{n=1}^{N}\| \xi_e^n \|_1^2 \Delta t. \quad (3.59)$$

The remaining terms are bounded as in H^1 norm error estimate, and we obtain

$$| T_6 + T_7 + T_8 | \leq C_1\sum_{n=1}^{N}(\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2)\Delta t + C_2(h^8 + (\Delta t)^2). \quad (3.60)$$

By all the error estimates of T_i and (3.49), hiding ε term in the left hand of (3.49) yields

$$\| \xi_e^N \|^2 + \sum_{n=1}^{N}\| \xi_e^n \|_1^2 \Delta t \leq C_1\sum_{n=1}^{N}(\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2)\Delta t + C_2(h^8 + (\Delta t)^2). \quad (3.61)$$

For hole concentration we have the similar result

$$\| \xi_p^N \|^2 + \sum_{n=1}^{N}\| \xi_p^n \|_1^2 \Delta t \leq C_1\sum_{n=1}^{N}(\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2)\Delta t + C_2(h^8 + (\Delta t)^2). \quad (3.62)$$

Now, to treat temperature equation (3.35), in (3.35) choosing $Z_h = \Pi_h^* \xi_T^n$ as test function, we have

$$\left(\rho\frac{\xi_T^n - \xi_T^{n-1}}{\Delta t}, \Pi_h^* \xi_T^n\right) + a(\xi_T^n, \Pi_h^* \xi_T^n)$$

$$= \left(\rho\left[\frac{\partial T}{\partial t} - \frac{T^n - T^{n-1}}{\Delta t}\right], \Pi_h^* \xi_T^n\right) + \left(\rho\frac{\zeta_T^n - \zeta_T^{n-1}}{\Delta t}, \Pi_h^* \xi_T^n\right)$$

$$+\lambda_T(T-\widetilde{T},\ \Pi_h^*\ \xi_T^n)+\left(D_p\left(\frac{\partial p_h^n}{\partial x}\cdot u_h^n-\frac{\partial p^n}{\partial x}\cdot u^n\right),\ \Pi_h^*\ \xi_T^n\right)$$

$$+(\mu_p(p_h^n|u_h^n|^2-p^n|u^n|^2),\ \Pi_h^*\ \xi_T^n)-\left(D_e\left(\frac{\partial e_h^n}{\partial x}\cdot u_h^n-\frac{\partial e^n}{\partial x}\cdot u^n\right),\ \Pi_h^*\ \xi_T^n\right)$$

$$+(\mu_e(e_h^n|u_h^n|^2-e^n|u^n|^2),\ \Pi_h^*\ \xi_T^n). \tag{3.63}$$

Multiplying (3.63) by $2\Delta t$, summing for $1\leq n\leq N$ and using the norm equivalent and noting that $\xi_T^0=0$, we have

$$\|\xi_T^N\|^2+C_0\sum_{n=1}^N\|\xi_T^n\|_1^2\Delta t\leq\sum_{i=1}^7 Q_i. \tag{3.64}$$

The similar estimates to Q_1, Q_2, Q_3 in H^1 norm error estimates, we have

$$|Q_1+Q_2+Q_3|\leq C_1(h^8+(\Delta t)^2)+C_2\sum_{n=1}^N\|\xi_T^n\|^2\Delta t. \tag{3.65}$$

To deal with Q_4, with the technique of partition integral transferring the spacial derivative of some terms to ξ_T^n and using hypothesis (3.22), (3.23) yields

$$|Q_4|=C\sum_{n=1}^N\left|\left(\left(D_p\left(\frac{\partial p_h^n}{\partial x}-\frac{\partial p^n}{\partial x}\right)\cdot u_h^n+\frac{\partial p^n}{\partial x}(u_h^n-u^n)\right),\ \Pi_h^*\ \xi_T^n\right)\right|\Delta t$$

$$\leq C_1(h^8+(\Delta t)^2)+C_2\sum_{n=1}^N(\|\xi_e^n\|^2+\|\xi_p^n\|^2+\|\xi_T^n\|^2)\Delta t$$

$$+\varepsilon\sum_{n=1}^N(\|\xi_p^n\|_1^2+\|\xi_T^n\|_1^2)\Delta t. \tag{3.66}$$

$$|Q_5|=2\sum_{n=1}^N|(\mu_p(p_h^n-p^n)|u_h^n|^2+\mu_p p^n(|u_h^n|^2-|u^n|^2),\ \Pi_h^*\ \xi_T^n)|\Delta t$$

$$\leq C_1(h^8+(\Delta t)^2)+C_2\sum_{n=1}^N(\|\xi_e^n\|^2+\|\xi_p^n\|^2+\|\xi_T^n\|^2)\Delta t$$

$$+\varepsilon\sum_{n=1}^N\|\xi_T^n\|_1^2\Delta t. \tag{3.67}$$

For $Q_6, Q_7,$ we handle in the same way as $Q_4, Q_5.$ We deduce

$$| Q_6 + Q_7 | \le C_1(h^8 + (\Delta t)^2) + C_2 \sum_{n=1}^{N} (\| \xi_e^n \|^2 + \| \xi_p^n \|^2 + \| \xi_T^n \|^2) \Delta t$$

$$+ \varepsilon \sum_{n=1}^{N} (\| \xi_p^n \|_1^2 + \| \xi_T^n \|_1^2) \Delta t. \tag{3.68}$$

Putting (3.65)-(3.68) into (3.64), we have

$$\| \xi_T^N \|^2 + C_0 \sum_{n=1}^{N} \| \xi_T^n \|_1^2 \Delta t$$

$$\le C_1(h^8 + (\Delta t)^2) + C_2 \sum_{n=1}^{N} (\| \xi_e^n \|^2 + \| \xi_p^n \|^2 + \| \xi_T^n \|^2) \Delta t$$

$$+ \varepsilon \sum_{n=1}^{N} (\| \xi_p^n \|_1^2 + \| \xi_T^n \|_1^2) \Delta t. \tag{3.69}$$

Combining (3.61) and (3.62) and hiding ε term, we have

$$\| \xi_e^N \|^2 + \| \xi_p^N \|^2 + \| \xi_T^N \|^2 + \sum_{n=1}^{N} (\| \xi_e^n \|_1^2 + \| \xi_p^n \|_1^2 + \| \xi_T^n \|_1^2) \Delta t$$

$$\le C_1 \sum_{n=1}^{N} (\| \xi_e^{n-1} \|^2 + \| \xi_p^{n-1} \|^2 + \| \xi_T^{n-1} \|^2) \Delta t + C_2(h^8 + (\Delta t)^2). \tag{3.70}$$

By the discrete Gronwall lemma, we conclude that

$$\| \xi_e^N \|^2 + \| \xi_p^N \|^2 + \| \xi_T^N \|^2 + \sum_{n=1}^{N} (\| \xi_e^n \|_1^2 + \| \xi_p^n \|_1^2 + \| \xi_T^n \|_1^2) \Delta t$$

$$\le C(h^8 + (\Delta t)^2). \tag{3.71}$$

So that we can obtain the result for L^2 error estimate.

Theorem 2. *Let* $\{e, p, \psi\}$ *be the solution of* (1.1)-(1.5) *and have some smoothness,* $\{e_h^n, p_h^n, \psi_h^n\}$ *be the solution of the full discrete characteristic finite volume element scheme* (2.9)-(2.13). *When h is sufficiently small and satisfies* $\Delta t = o(h)$, *then we can obtain* L^2 *norm error estimate as*

$$\sup_{0 \le n \le N} \{\| e^n - e_h^n \| + \| p^n - p_h^n \| + \| T^n - T_h^n \|\} \le C(h^4 + \Delta t). \tag{3.72}$$

4. Numerical Experiment

Consider the following simple example

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - \sin(x)\dfrac{\partial u}{\partial x} + \cos(x)u = f(x, t), & 0 < t \le T,\ 0 \le x \le \pi, \\ u(x, 0) = \sin(x), & 0 \le x \le \pi, \\ u(0, t) = u(\pi, t) = 0, & 0 < t \le T. \end{cases} \tag{4.1}$$

The exact solution is $u = e^{(-t)}\sin(x)$. Using Hermite cubic polynomial full discrete characteristic finite volume element method we obtain the approximate solution U_h. We get ideal results related to different spacial steps and time steps in the examination from which we verify the feasibility and effectiveness of this method. We perform two kinds of computations: Table 1, the comparison of different nodes between the discrete approximate solution (FVEM) and the exact solution (TS) when $h = \pi/16$, $dt = h^3$; Table 2, the comparison of different nodes between the discrete approximation solution (FVEM) and the exact solution (TS) when $h = \pi/16$, $dt = h^4$.

Table 1. The comparison of the nodes between the discrete solution and the true solution when $h = \pi/16$, $dt = h^3 = 0.007570$

Node	$x = \pi/8$	$x = \pi/4$	$x = \pi/2$	$x = 5\pi/8$	$x = 3\pi/4$	$x = 7\pi/8$
$\text{FVEM}\vert_{t=0.1(0.098409)}$	0.346943	0.641072	0.906615	0.837603	0.641073	0.346943
$\text{TS}\vert_{t=0.1(0.098409)}$	0.346818	0.640836	0.906279	0.837292	0.640836	0.346818
$\text{FVEM}\vert_{t=0.2(0.196817)}$	0.314535	0.581200	0.821950	0.759385	0.581207	0.314539
$\text{TS}\vert_{t=0.2(0.196817)}$	0.314314	0.580776	0.821341	0.758820	0.580776	0.314314
$\text{FVEM}\vert_{t=0.4(0.393634)}$	0.258507	0.477693	0.675599	0.624184	0.477732	0.258533
$\text{TS}\vert_{t=0.4(0.393634)}$	0.258158	0.477015	0.674601	0.623250	0.477015	0.258158

Table 2. The comparison of the nodes between the discrete solution and the true solution when $h = \pi/16$, $dt = h^4 = 0.001486$

Node	$x = \pi/8$	$x = \pi/4$	$x = \pi/2$	$x = 5\pi/8$	$x = 3\pi/4$	$x = 7\pi/8$
$\mathrm{FVEM}\|_{t=0.1(0.099585)}$	0.346435	0.640129	0.905280	0.836370	0.640130	0.346435
$\mathrm{TS}\|_{t=0.1(0.099585)}$	0.346410	0.640082	0.905213	0.836308	0.640082	0.346410
$\mathrm{FVEM}\|_{t=0.2(0.199170)}$	0.313619	0.579495	0.819532	0.757149	0.579497	0.313620
$\mathrm{TS}\|_{t=0.2(0.199170)}$	0.313575	0.579411	0.819410	0.757037	0.579411	0.313575
$\mathrm{FVEM}\|_{t=0.4(0.399827)}$	0.256634	0.474205	0.670635	0.619589	0.474213	0.256639
$\mathrm{TS}\|_{t=0.4(0.399827)}$	0.256565	0.474070	0.670436	0.619402	0.474070	0.256565

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