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MODIFIED PROJECTION METHOD EXTENDED TO STRONGLY PSEUDOMONOTONE KY FAN INEQUALITIES

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Abstract

The purpose of this paper is to present modified projection methods for solving strongly pseudomonotone Ky Fan inequalities. Error estimates for the sequences generated by this method are established under few assumptions. We also analyze the strong convergence of the proposed algorithm in a real Hilbert space and report some computational results.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let C be a nonempty closed convex subset of $\mathcal{H}$. Recall that a mapping $T : C \to C$ called to be *nonexpansive* if $\| T(x) - T(y) \| \leq \| x - y \|$ for all $x, y \in C$. Denote by $Fix(T)$ the set of fixed points of T.

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We consider the typical form of equilibrium problems which are formulated by the Ky Fan inequalities [13] as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \tag{1.1}$$

where f is a bifunction from $C \times C$ to $\mathcal{R}$ such that $f(x, x) = 0$ for all $x \in C$. Let $Sol(f, C)$ denote the set of solutions of problem (1.1). When $f(x, y) = \langle F(x), y - x \rangle$ for all $x, y \in C$ with $F : C \to \mathcal{H}$, problem (1.1) can be formulated as variational inequalities:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C. \tag{1.2}$$

In recent years, problem (1.1) becomes an attractive field for many researchers both in theory and applications in the electricity market, the transportation, economics and networks (see [1, 2, 5, 6, 8, 11, 10, 16]). Methods for solving problem (1.1) have studied extensively in many different ways. There exist several methods for solving (1.2) with F being a monotone mapping. The simplest method is the projection one. At each iteration k of this method, a point $x^k \in C$ and then the vector $x^k - \lambda_k F(x^k)$ with stepsize $\lambda_k > 0$, is projected on the feasible domain C. Recently, motivated by the basic projection method for solving problem (1.2) with a fixed stepsize, Khanh and Vuong [14] showed that the iteration sequences proposed by the method converge linearly to a unique solution of problem (1.2), where the function F only requires Lipschitz continuous and the strongly pseudomonotone assumptions. In the special case $f(x, y) = \langle F(x), y - x \rangle$ for all $x, y \in C$, and the subproblem needed to solve in the projection method is of the form

$$\begin{aligned} y^k &= Pr_C(x^k - \lambda_k F(x^k)) \\ &= \operatorname{argmin}\left\{\frac{1}{2}\| y - x^k \|^2 + \lambda_k \langle F(x^k), y - x^k \rangle : y \in C\right\} \\ &= \operatorname{argmin}\left\{\frac{1}{2}\| y - x^k \|^2 + \lambda_k f(x^k, y) : y \in C\right\}, \end{aligned}$$

where Pr_C is the metric projection on C. This strategy has also been recently incorporated in this method by Mastroeni [16] via the following the auxiliary principle:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \text{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \|^2 : y \in C\right\}. \end{cases} \tag{1.3}$$

The author showed that if f is strongly monotone and Lipschitz-type continuous on C, then $\{x^k\}$ converges to the unique solution to problem (1.1) in $\mathcal{R}^n$.

Otherwise, among other generalized monotone Ky Fan inequalities, the class of strongly pseudomonotone Ky Fan inequalities has been investigated by many researchers (see [3, 4, 17-20]). It is well-known to see that the class of strongly pseudomonotone Ky Fan inequalities virtually contains the class of strongly monotone Ky Fan inequalities. As the above, the projection method is an efficient and very simple method for solving strongly monotone Ky Fan inequalities, a natural question arises: *Could the iteration sequence of the projection method* (1.3) *be strongly convergent in a real Hilbert space only under Lipschitz-type continuous and strongly pseudomonotone assumptions of the bifunction* f? In this paper, we will give a complete solution for the above question by showing that the projection method not only works well for strongly monotone Ky Fan inequalities but also succeeds in solving strongly pseudomonotone Ky Fan inequalities. We also analyze the strong convergence of the algorithm in a real Hilbert space.

The rest of the paper is organized as follows: In Section 2, we give formal definitions of our target problem (1.1) and the strong pseudomonotonicity of f, describe the projection method for strongly pseudomonotone Ky Fan inequalities and the proof of its convergence. In the last section, we apply the extragradient method for the Nash market equilibrium model, and the numerical results are reported.

2. Modified Projection Method

Let C be a nonempty closed convex subset of $\mathcal{H}$. For each $x \in \mathcal{H}$, there exists a unique point in C, denoted by $Pr_C(x)$ such that

$$Pr_C(x) = \text{argmin}\{\| x - y \| : y \in C\}.$$

Let $g : C \to \mathcal{R}$ be convex and subdifferentiable on C. Then a point $x^* \in C$ is a solution to the following convex problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the (outward) normal cone of C at $x^* \in C$.

We recall some well-known definitions on monotonicity and Lipschitz-type continuity of a bifunction.

Definition 2.1. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. The bifunction $f : C \times C \to \mathcal{R}$ is said to be

(i) γ-*strongly monotone* on $C \times C$ if for each $x, y \in C$,

$$f(x, y) + f(y, x) \leq -\gamma \| x - y \|^2;$$

(ii) *monotone on* $C \times C$ if for each $x, y \in C$,

$$f(x, y) + f(y, x) \leq 0;$$

(iii) β-*strongly pseudomonotone* on $C \times C$ if for each $x, y \in C$,

$$f(x, y) \leq 0 \Rightarrow f(y, x) \leq -\beta \| x - y \|^2;$$

(iv) *Lipschitz-type continuous* with constants $c_1 > 0$ and $c_2 > 0$ on $C \times C$ if for each $x, y, z \in C$,

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \| x - y \|^2 - c_2 \| y - z \|^2.$$

Choosing the real Hilbert space $\mathcal{H}$ and the subset C satisfies the following:

$$\mathcal{H} := \left\{ x = (x_1, x_2, \ldots) : \sum_{k=0}^{\infty} x_k^2 < +\infty \right\}$$

and

$$C := \{x \in \mathcal{H} : \| x \| \le a\}.$$

By using $f(x, y) = (b - \| x \|)\langle x, y - x\rangle$, for $a \in \left(\frac{b}{2}, b\right)$, Khanh and Vuong [14] showed that f is strongly pseudomonotone on $C \times C$ with constant $\beta = b - a$. But it is neither strongly monotone nor monotone on $C \times C$. Moreover, we will show that f is Lipschitz-type with constants $c_1 = \frac{(b + 2a)^2}{2(b - a)}$ and $c_2 = \frac{b - a}{2}$. Indeed, for each $x, y, z \in C$, we have

$$\begin{aligned}
& f(x, y) - f(x, z) \\
&= (b - \| x \|)\langle x, y - x\rangle - (b - \| x \|)\langle x, z - x\rangle \\
&= (b - \| x \|)\langle x, y - z\rangle \\
&= (b - \| y \|)\langle y, z - y\rangle + \langle (b - \| x \|)x - (b - \| y \|)y, z - y\rangle \\
&= -f(y, z) + \left\langle \frac{1}{\sqrt{b - a}}(b - \| x \|)x - (b - \| y \|)y, \sqrt{b - a}(y - z)\right\rangle \\
&\ge -f(y, z) - \frac{1}{2}(b - a)\| y - z \|^2 \\
&\quad - \frac{1}{2(b - a)}\| (b - \| x \|)x - (b - \| y \|)y \|^2.
\end{aligned} \tag{2.1}$$

From [14], it implies that

$$\| (b - \| x \|)x - (b - \| y \|)y \| \le (b + 2a)\| x - y \|, \forall x, y \in C. \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$f(x, y) - f(x, z) \geq -f(y, z) - \frac{1}{2}(b-a)\| y - z \|^2 - \frac{(b+2a)^2}{2(b-a)}\| x - y \|^2$$

and hence

$$f(x, y) + f(y, z) \geq f(x, z) - c_2\| y - z \|^2 - c_1\| x - y \|^2, \forall x, y \in C,$$

where $c_1 = \frac{(b+2a)^2}{2(b-a)}$ and $c_2 = \frac{b-a}{2}$.

Let $\varepsilon > 0$. As usual, we say that a point $\bar{x} \in C$ is an ε-*solution* to problem (1.1) if there exists $x^* \in Sol(f, C)$ such that $\| \bar{x} - x^* \| \leq \varepsilon$. The following algorithm is designed for solving the Ky Fan inequalities, where the bifunction *f* is γ-strongly pseudomonotone and Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$ on $C \times C$.

Algorithm 2.2. Choosing positive sequence $\{\lambda_k\}$ satisfies the following conditions:

$$0 < a \leq \lambda_k \leq b < \frac{1}{2c_1}, c_2 < \gamma, \varepsilon > 0, \delta = \frac{1}{\sqrt{1 + 2a(\gamma - c_2)}}. \quad (2.3)$$

Step 0. $k = 0$, find an initial point $x^0 \in C$.

Step 1. Solve the strongly convex program:

$$x^{k+1} = \operatorname{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \|^2 : y \in C\right\}.$$

Step 2. If $\| x^k - x^{k+1} \| < \frac{\varepsilon(1-\delta)}{\delta}$, then terminate: x^{k+1} is an ε-solution to (1.1). Otherwise, set $k := k + 1$ and go back to Step 1.

Let us discuss the global convergence of Algorithm 2.2.

Theorem 2.3. *Let* $f : C \times C \to \mathcal{R}$ *be* γ*-strongly pseudomonotone on* C, *Lipschitz-type continuous bifunction with constants* $c_1 > 0$ *and* $c_2 > 0$. *For each* $x \in C$, *let* $f(x, \cdot)$ *be convex and subdifferentiable on* C. *Then* $Sol(f, C)$ *has a unique point* x^* *and*

$$[1 + 2\lambda_k(\gamma - c_2)]\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2, \forall k \geq 0. \tag{2.4}$$

Moreover, *the sequence* $\{x^k\}$ *generated by Algorithm* 2.2 *converges strongly to the point* x^* *with computational error as follows*:

$$\| x^{k+1} - x^* \| \leq \delta^{k+1}\| x^0 - x^* \| \leq \frac{\delta^{k+1}}{1 - \delta}\| x^0 - x^* \|$$

and

$$\| x^{k+1} - x^* \| \leq \delta\| x^k - x^* \| \leq \frac{\delta}{1 - \delta}\| x^k - x^{k+1} \|.$$

Proof. Using the well-known necessary and sufficient condition for optimality of convex programing, we see that

$$x^{k+1} = \text{argmin}\left\{\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \|^2 : y \in C\right\}$$

if and only if

$$0 \in \partial_2\left(\lambda_k f(x^k, y) + \frac{1}{2}\| y - x^k \|^2\right)(x^{k+1}) + N_C(x^{k+1}).$$

By the well-known Moreau-Rockafellar theorem, we have

$$0 \in x^{k+1} - x^k + \lambda_k \partial_2 f(x^k, x^{k+1}) + N_C(x^{k+1}).$$

This implies that

$$0 = x^{k+1} - x^k + \lambda_k w^1 + w^2, \tag{2.5}$$

where $w^1 \in \partial_2 f(x^k, x^{k+1})$ and $w^2 \in N_C(x^{k+1})$. By the definition of the normal cone N_C, we have

$$\langle w_2,\ y - x^{k+1}\rangle \le 0,\ \forall y \in C.$$

Combining this and (2.5), we get

$$\langle x^k - x^{k+1} - \lambda_k w^1,\ y - x^{k+1}\rangle \le 0,\ \forall y \in C.$$

Replacing y by $x^* \in C,$ we get

$$\langle x^k - x^{k+1},\ x^* - x^{k+1}\rangle \le \lambda_k \langle w^1,\ x^* - x^{k+1}\rangle. \tag{2.6}$$

Using the definition of $w^1 \in \partial_2 f(x^k,\ x^{k+1}),$ we have

$$f(x^k,\ x) - f(x^k,\ x^{k+1}) \ge \langle w^1,\ x - x^{k+1}\rangle,\ \forall x \in C.$$

Applying this for $x = x^*,$ we obtain

$$f(x^k,\ x^*) - f(x^k,\ x^{k+1}) \ge \langle w^1,\ x - x^{k+1}\rangle. \tag{2.7}$$

From (2.6) and (2.7), it implies that

$$\langle x^k - x^{k+1},\ x^* - x^{k+1}\rangle \le \lambda_k [f(x^k,\ x^*) - f(x^k,\ x^{k+1})]. \tag{2.8}$$

Since the bifunction f is Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$ on C, and $x^* \in C,\ x^k \in C$ for all $k \ge 0,$ we have

$$\begin{aligned} &f(x^k,\ x^*) - f(x^k,\ x^{k+1}) \\ &\le f(x^{k+1},\ x^*) + c_1\| x^k - x^{k+1} \|^2 + c_2\| x^{k+1} - x^* \|^2. \end{aligned} \tag{2.9}$$

Since x^* is a unique solution to problem (1.1) and $x^{k+1} \in C,$ we have $f(x^*,\ x^{k+1}) \ge 0.$ By the strong pseudomonotonicity of f, we get

$$f(x^{k+1},\ x^*) \le -\gamma\| x^{k+1} - x^* \|^2. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\begin{aligned} &f(x^k,\ x^*) - f(x^k,\ x^{k+1}) \\ &\le -\gamma\| x^{k+1} - x^* \|^2 + c_1\| x^k - x^{k+1} \|^2 + c_2\| x^{k+1} - x^* \|^2 \\ &= -(\gamma - c_2)\| x^{k+1} - x^* \|^2 + c_1\| x^k - x^{k+1} \|^2. \end{aligned}$$

Combining this and (2.8), we have

$$\langle x^k - x^{k+1}, x^* - x^{k+1}\rangle \le \lambda_k[-(\gamma - c_2)\| x^{k+1} - x^* \|^2 + c_1\| x^k - x^{k+1} \|^2].$$

Using the assumption $\lambda_k \in \left(0, \frac{1}{2c_1}\right)$, we get

$$[1 + 2\lambda_k(\gamma - c_2)]\| x^{k+1} - x^* \|^2$$

$$\le \| x^k - x^* \|^2 - (1 - 2\lambda_k c_1)\| x^{k+1} - x^k \|^2 \le \| x^k - x^* \|^2.$$

This implies (2.4).

From (2.3) and (2.4), it follows that

$$[1 + 2a(\gamma - c_2)]\| x^{k+1} - x^* \|^2 \le \| x^k - x^* \|^2, \forall k \ge 0.$$

Then

$$\| x^{k+1} - x^* \|^2 \le \delta\| x^k - x^* \|^2, \forall k \ge 0,$$

where $\delta = \frac{1}{\sqrt{1 + 2a(\gamma - c_2)}} \in (0, 1)$. This implies that

$$\| x^{k+1} - x^* \| \le \delta\| x^k - x^* \| \le \delta^2\| x^{k-1} - x^* \|$$

$$\le \cdots \le \delta^{k+1}\| x^0 - x^* \|.$$

Otherwise,

$$\| x^k - x^* \| \le \| x^k - x^{k+1} \| + \| x^{k+1} - x^* \| \le \| x^k - x^{k+1} \| + \delta\| x^k - x^* \|,$$

and so $\| x^k - x^* \| \le \frac{1}{1 - \delta}\| x^k - x^{k+1} \|$. Therefore, we have

$$\| x^{k+1} - x^* \| \le \delta^{k+1}\| x^0 - x^* \| \le \frac{\delta^{k+1}}{1 - \delta}\| x^0 - x^* \|$$

and

$$\| x^{k+1} - x^* \| \le \delta\| x^k - x^* \| \le \frac{\delta}{1 - \delta}\| x^k - x^{k+1} \|.$$

The proof is complete. □

For each $x, y \in C$, by using $f(x, y) := \langle F(x), y - x \rangle$, we will give the iteration projection for strongly pseudomonotone $VI(F, C)$ and its convergence is linearly convergent, provided that the given problem has a solution.

Corollary 2.4. *Let C be a nonempty closed convex set of a real Hilbert space $\mathcal{H}$ and let $F : C \to \mathcal{H}$ be strongly pseudomonotone with a modulus γ and Lipschitz continuous with a constant $L > 0$. Let $0 < a \le \lambda_k \le b < \frac{2\gamma}{L^2}$, and the sequence $\{x^k\}$ be defined by*

$$x^0 \in C, \quad x^{k+1} = Pr_C(x^k - \lambda_k F(x^k)), \ \forall k \ge 0.$$

Then the sequence $\{x^k\}$ converges strongly to the unique solution x^* of problem $VI(F, C)$ with computational error as follows:

$$\| x^{k+1} - x^* \| \le \delta^{k+1} \| x^0 - x^* \| \le \frac{\delta^{k+1}}{1-\delta} \| x^0 - x^* \|$$

and

$$\| x^{k+1} - x^* \| \le \delta \| x^k - x^* \| \le \frac{\delta}{1-\delta} \| x^k - x^{k+1} \|.$$

3. Illustrative Examples and Numerical Results

We consider a Nash equilibrium model (see [15]) as an example for problem (1.1). In the model, it is assumed that there are *n*-firms producing a common homogeneous commodity and that the price p_i of the firm i depends on the total quantity $\sigma_x = \sum_{i=1}^{n} x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i. Suppose that the profit of the firm i is given by

$$f_i(x_1, ..., x_n) := x_i p_i(\sigma_x) - h_i(x_i), \ i = 1, ..., n, \tag{3.1}$$

where h_i is the cost function of the firm i that is assumed to be dependent only on its production level.

Let $C_i \subset \mathcal{R}_+ := \{x \in \mathcal{R} : x \geq 0\} (i = 1, ..., n)$ denote the strategy set of the firm i. Each firm seeks to maximize its own profit by choosing the corresponding production level under the presumption that the production of the other firms is parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus, under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, ..., x_n^*)^T \in C := C_1 \times \cdots \times C_n$ is said to be a *Nash equilibrium point* if

$$f_i(x_1^*, ..., x_{i-1}^*, y_i, x_{i+1}^*, ..., x_n^*) \leq f_i(x_1^*, ..., x_n^*), \forall y_i \in C_i, \forall i = 1, ..., n. \quad (3.2)$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in the n-person noncooperative game theory.

Set

$$\phi(x, y) := -\sum_{i=1}^{n} f_i(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \quad (3.3)$$

and

$$f(x, y) := \phi(x, y) - \phi(x, x). \quad (3.4)$$

Then it has been proved in [15] that the problem of finding an equilibrium point of this model can be formulated as problem (1.1). We consider classical Cournot-Nash models (see [15]), the price and the cost functions for each firm are assumed to be affine with the following forms:

$$p_i(\sigma_x) := p(\sigma_x) = \alpha_0 - \chi\sigma_x \text{ with } \alpha_0 \geq 0, \chi > 0,$$

$$h_i(x_i) = \mu_i x_i + \xi_i \text{ with } \mu_i \geq 0, \xi_i \geq 0 \ (i = 1, ..., n).$$

Combining this with (3.1), (3.2), (3.3) and (3.4), we get

$$f(x, y) = \langle Ax + \chi^n(y + x) + \mu - \alpha, y - x\rangle, \quad (3.5)$$

where

$$A = \begin{pmatrix} 0 & \chi & \chi & \cdots & \chi \\ \chi & 0 & \chi & \cdots & \chi \\ \chi & \chi & 0 & \cdots & \chi \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \chi & \chi & \cdot & \cdots & 0 \end{pmatrix}_{n\times n}, \quad \alpha = (\alpha_0, ..., \alpha_0)^T, \quad \mu = (\mu_1, ..., \mu_n)^T. \tag{3.6}$$

Example 3.1. Consider now a numerical example with $n = 5$ and

$$C = \begin{cases} x \in \mathcal{R}^5_+, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 10, \\ 2x_1 + x_2 - x_3 + x_4 + 3x_5 \leq 15, \\ x_1 + x_2 + x_3 + x_4 + 0.5x_5 \geq 4. \end{cases}$$

The bifunction f is extensively defined (3.5) as follows:

$$f(x, y) = \langle Ax + \chi^n(y + x) + \mu - \alpha, y - x\rangle - \| x - y \|^2.$$

Then, from (3.6), it follows that

$$\begin{aligned} & f(x, y) + f(y, x) \\ &= \langle Ax + \chi^n(y + x) + \mu - \alpha, y - x\rangle + \langle Ay + \chi^n(x + y) + \mu - \alpha, x - y\rangle \\ &\quad - 2\| x - y \|^2 \\ &= -\langle A(y - x), y - x\rangle - 2\tau\| x - y \|^2 \\ &\leq -2\| x - y \|^2. \end{aligned}$$

So, f is strongly pseudomonotone on $C \times C$ with constant $\gamma = 2$. It is easy to compute that f is Lipschitz-type with constants $c_1 \geq \frac{\gamma}{2} + \frac{1}{5}\| A \|$ and $c_2 \geq \frac{\gamma}{2} + \frac{1}{5}\| A \|$. The parameters are taken as follows: $\varepsilon = 10^{-4}$, $x^0 =$

$(1, 2, 1, 1, 1)^T \in C$, $\chi = 1$, $\alpha_0 = 2$, $\mu = (3, 4, 5, 7, 6)^T$. Choose $c_2 = \tau + \frac{1}{2}\|A\| = 1.8$ and $c_1 = 6.2$. Since $0 < a \le \lambda_k \le b < \frac{1}{2c_1} = 0.2778$, we choose $\lambda_k = a = b = 0.2$ for all $k \ge 0$. Then we have $\delta = 0.9623$.

The approximate solution obtained after 11 iterations is (see Table 1):

$$x^{11} = (0.9467, 0.9447, 0.9426, 0.9405, 0.4510)^T.$$

The computations are performed by Matlab R2008a running on a PC Desktop Intel(R) Core(TM)i5 650@3.2GHz 3.33GHz 4Gb RAM.

Table 1. The sequence $\{x^k\}$ given by Algorithm 2.2, $n = 5$

Iteration (k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k
0	1	2	1	1	1
1	0.5763	1.9897	0.6786	0.9821	0.9922
2	0.7892	1.3473	0.7345	0.9808	0.8327
3	0.8945	1.2437	0.8009	0.8992	0.7033
4	0.9008	1.1751	0.9008	0.9112	0.6670
5	0.9045	1.0437	0.9109	0.9210	0.5033
6	0.9338	0.9751	0.9298	0.9278	0.4670
7	0.9428	0.9540	0.9387	0.9366	0.4559
8	0.9455	0.9475	0.9414	0.9393	0.4524
9	0.9464	0.9456	0.9422	0.9402	0.4514
10	0.9466	0.9449	0.9425	0.9404	0.4511
11	0.9467	0.9448	0.9426	0.9405	0.4510

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References

[1] P. N. Anh, A logarithmic quadratic regularization method for solving pseudomonotone equilibrium problems, Acta Math. Vietnam. 34 (2009), 183-200.

[2] P. N. Anh, An LQP regularization method for equilibrium problems on polyhedral, Vietnam J. Math. 36 (2008), 209-228.

[3] P. N. Anh and H. A. Le Thi, An Armijo-type method for pseudomonotone equilibrium problems and its applications, J. Global Optim. 57(3) (2013), 803-820.

[4] P. N. Anh and J. K. Kim, Outer approximation algorithms for pseudomonotone equilibrium problems, Comput. Math. Appl. 61 (2011), 2588-2595.

[5] P. N. Anh and T. Kuno, A cutting hyperplane method for generalized monotone nonlipschitzian multivalued variational inequalities, Modeling, Simulation and Optimization of Complex Processes, H. G. Bock, H. X. Phu, R. Rannacher and J. P. Schloder, eds., Springer-Verlag, Berlin, Heidelberg, 2012.

[6] P. N. Anh, L. D. Muu, V. H. Nguyen and J. J. Strodiot, Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities, J. Optim. Theory Appl. 124 (2005), 285-306.

[7] A. S. Antipin, The convergence of proximal methods to fixed points of extremal mappings and estimates of their rates of convergence, Comput. Math. Math. Phys. 35 (1995), 539-551.

[8] E. Blum and W. Oettli, From optimization and variational inequality to equilibrium problems, Math. Student 63 (1994), 127-149.

[9] G. Cohen, Auxiliary problem principle extended to variational inequalities, J. Optim. Theory Appl. 59 (1988), 325-333.

[10] S. Dafermos, Exchange price equilibria and variational inequalities, Math. Program. 46 (1990), 391-402.

[11] P. Daniele, F. Giannessi and A. Maugeri, Equilibrium Problems and Variational Models, Kluwer, 2003.

[12] F. Facchinei and J. S. Pang, Finite-dimensional Variational Inequalities and Complementary Problems, Springer-Verlag, New York, 2003.

[13] K. Fan, A minimax inequality and applications, Inequality III, O. Shisha, ed., Academic Press, New York, 1972, pp. 103-113.

[14] P. D. Khanh and P. T. Vuong, Modified projection method for strongly pseudomonotone variational inequalities, J. Global Optim. 58 (2014), 341-350.

[15] I. V. Konnov, Combined Relaxation Methods for Variational Inequalities, Springer-Verlag, Berlin, 2000.

[16] G. Mastroeni, On auxiliary principle for equilibrium problems, Nonconvex Optimization and its Applications, P. Daniele, F. Giannessi and A. Maugeri, eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[17] A. Moudafi, Proximal point algorithm extended to equilibrium problem, J. Nat. Geom. 15 (1999), 91-100.

[18] M. A. Noor, Auxiliary principle technique for equilibrium problems, J. Optim. Theory Appl. 122 (2004), 371-386.

[19] M. V. Solodov and B. F. Svaiter, A new projection method for variational inequality problems, SIAM J. Control Optim. 37 (1999), 765-776.

[20] T. D. Quoc, P. N. Anh and L. D. Muu, Dual extragradient algorithms to equilibrium problems, J. Global Optim. 52 (2012), 139-159.