Far East Journal of Mathematical Sciences (FJMS)
© 2014 Pushpa Publishing House, Allahabad, India
Published Online: January 2015
Available online at http://pphmj.com/journals/fjms.htm
Volume 95, Number 2, 2014, Pages 249-258

A COMPARISON OF PERFORMANCE OF CUSUM AND EWMA WITH PARAMETRIC AND NONPARAMETRIC FOR EXPONENTIAL DATA

Saowanit Sukparungsee

Department of Applied Statistics
Faculty of Applied Sciences
King Mongkut's University of Technology North Bangkok
Thailand 10800
e-mail: swns@kmutnb.ac.th

Abstract

The nonparametric CUSUM and EWMA control charts are proposed to study the performance in order to detect the deviation from the target value. These nonparametric control charts are effective alternatives to parametric control charts. The performance of the nonparametric control charts are compared with parametric CUSUM and EWMA control charts when observations are from exponential distribution. The numerical results found from Monte Carlo simulation with repeated 10^4 times. We find that the performance of nonparametric control charts are superior to parametric CUSUM and EWMA control charts. Furthermore, the nonparametric EWMA performs better than nonparametric CUSUM for all magnitudes of shift.

Received: October 13, 2014; Accepted: November 3, 2014

2010 Mathematics Subject Classification: 45B05, 60G40, 62J12.

Keywords and phrases: exponential distribution, performance, cumulative sum, exponentially weighted moving average, nonparametric control chart.

1. Introduction

Statistical Process Control (SPC) charts play a vital role in quality improvement and are widely used for monitoring, measuring, controlling and improving quality in many fields of application. An SPC is easily used to track the performance of a process in order to bring the process back to a target value as quickly as possible.

Quality control was introduced by Shewhart [8] in the twentieth century. Typically, SPC's are used in industry but they have also been applied in many other areas of application, for example, Health care (Frisën [5], Hawkins and Olwell [6]), Epidemiology (Sitter et al. [11]), Finance and Economics (Andersson [1]), Ergashev [4]), Environmental Sciences (Basseville and Nikiforov [2]).

A major aim of an SPC is to detect a change in a process as soon as possible, but at the same time it should only rarely give a false alarm. Generally, it is assumed that the parameter of an in-control process should be sustained at some specified target value. In practice, this parameter can change at an unknown point in time θ at which time the process will be out-of-control. However, the controller will usually have to observe the process up to a time τ before deciding that the process is out of control and stopping it.

The SPC is easily used to track the performance of a process in order to bring the process back to a target value as quickly as possible. To detect this change, one needs to apply statistical techniques and constrains. Generally, the most used ones are a mean of false alarm time or in-control Average Run Length (ARL_0) is the expectation of the time or observation before the control chart gives a false alarm that an in-control process has gone out-of-control. A second characteristic is an out-of-control Average Run Length (ARL_1) is the expectation of the time or observation between a process going out-of-control and the control chart giving the alarm that the process has gone out-of-control.

The Shewhart chart is the most commonly used for detecting a change, especially large shifts. However, in order to monitoring small and moderate shifts could be often occurred. Consequently, two effective alternatives to the Shewhart chart have been developed which overcome its shortcomings in the past few decades. These are Cumulative Sum (CUSUM) and Exponentially Weighted Moving Average (EWMA). The CUSUM chart was introduced by Page [9]. Next, the EWMA chart was initially presented by Roberts [10]. Both the CUSUM and EWMA control charts are known to be more sensitive to the detection of small to moderate changes because they pay attention to the historical observations.

Control charts are usually designed and evaluated under the assumption that the observations are from independent and identically distributed (i.i.d.) and from a normal distribution. In real applications, however, there are many situations in which the processes come from non-normal distributions, for example, Exponential, Laplace, Student-t or Gamma distributions (Borror et al. [3]; Mititelu et al. [7]) or unknown distributions. Therefore, the statistics as mean and variance could not be estimated and parametric control charts such traditional Shewhart control chat could not be used. Processes with data from unknown distributions need to be monitored by nonparametric control charts.

Recently, many types of nonparametric or distribution-free control charts are proposed as alternative effective to parametric control charts. On 2011, Yang and Cheng [12] proposed the nonparametric Cumulative Sum control chart and Yang et al. [13] also presented the nonparametric Exponentially Weighted Moving Average control chart which so-called EWMA Sign and Arcsine EWMA Sign for detecting the unknown distribution processes. They can be used with not only non-Normal observations but also when distribution of process is unknown. Furthermore, nonparametric control charts do not sensitive to unusual data such as an outlier.

Consequently, this paper aims to compare the performance of the CUSUM and EWMA for both parametric and nonparametric control charts when the observations are from exponential distribution which usually is represented as lifetime of products.

2. The Parametric and Nonparametric CUSUM and EWMA Charts and Theirs Properties

Let $X_1, X_2, \ldots, X_t, \ldots$ be observed independent random variables. The change-point model is as follows:

$$\begin{cases} \xi_t \sim \exp(\alpha_0); & t = 1, 2, \ldots, \theta - 1; \\ \xi_t \sim \exp(\alpha); & t = \theta, \theta + 1, \ldots, \quad \alpha \neq \alpha_0. \end{cases}$$

We use notation $\theta = \infty$ for the case when there is no change in the distribution of observed data. Note that if $\theta = 1,$ then the change occurs at the very beginning.

2.1. The CUSUM control chart

In 1954, the CUSUM chart was first proposed by Page [9]. The CUSUM chart is used to detect shifts in the process mean and each point is based on information from all samples up to and including the current sample. The standard CUSUM chart was denoted for the discrete time case defined by statistic X_t with the following recursion equation:

$$X_t = (X_{t-1} - \xi_t + a), \quad t = 1, 2, \ldots, \quad X_0 = x, \tag{1}$$

where ξ_t has the distribution $F(x, \alpha)$ and a is a constant. The first passage time of the CUSUM chart is shown an out-of-control as well as this equation

$$\tau_b = \inf\{n \geq 0;\ X_t \geq b\},\ b \text{ is the boundary.}$$

2.2. The EWMA control chart

The Exponentially Weighted Moving Average (EWMA) chart, initiated by Roberts [8] is an effective alternative to Shewhart chart for detecting small shifts. The EWMA for discrete time case is defined by the following recursion:

$$Z_t = (1 - \lambda) Z_{t-1} + \lambda g(\xi_t), \quad t = 1, 2, \ldots . \tag{2}$$

Typically, $\lambda \in (0, 1)$ is a weighting factor for previous observations. To start the recursion, an initial value is required for Z_0 is usually assumed that $Z_0 = \alpha_0$. If the anticipated shift in the mean value is positive, then we take the decision that the process is out-of-control when for the first time $Z_t > H$ as follows:

$$\tau_H = \inf\{t \in N : Z_t > H\}.$$

2.3. The nonparametric control chart

Assume that a quality characteristic, ξ, has a target value T. Let $Y = \xi - T$, be the different value between observation and target value. Then it could be positive or negative value as statistic sign test in nonparametric. The process proportion $p = P(Y > 0)$, where $p = 0.5$ for the in-control process. If the process is out-of-control (the observations deviate from the target value), then process proportion has changed to $p \neq 0.5$. In order to detect, the deviation from the process target at any unexpected times, a random sample of size n, $\xi_1, \xi_2, \ldots, \xi_n$, is taken from ξ. The statistics Y_j is defined as follows:

$$Y_j = \xi_j - T \text{ and } I_j = \begin{cases} 1; & Y_j > 0; \\ 0; & \text{otherwise.} \end{cases} \tag{3}$$

Let M be the sum number of $Y_j > 0$. Then $M = \sum_{j=0}^{n} I_j$ will follow a binomial distribution with parameter $(n, 0.5)$ when the process is in-control.

2.3.1. The nonparametric CUSUM control chart

Substitute the statistic M into Equation (1) then the nonparametric CUSUM can be rewritten as

$$X_t = (X_{t-1} - M_t + a), \quad t = 1, 2, \ldots, \quad X_0 = x,$$

where M has binomial distribution with parameter $(n, 0.5)$ for in-control process.

2.3.2. The nonparametric EWMA control chart

The statistic of EWMA sign control chart is

$$EWMA_{M_i} = \lambda M_i + (1 - \lambda) EWMA_{M_{i-1}},$$

where M is the ith sequentially recorded number of $Y_j > 0,$ from the process. Let the initial value of the statistic of EWMA sign control chart be $EWMA_{M_0} = n/2.$ The mean and variance of $EWMA_M$ are $E(EWMA_{M_i})$ $= n/2$ and $Var(EWMA_{M_i}) = n/4\left(\frac{\lambda}{2-\lambda}\right)$ if $t \to \infty.$ Therefore, the control limits for the EWMA sign control chart are as follows:

$$UCL_{EWMA_M} = \frac{n}{2} + k\sqrt{\frac{\lambda}{2-\lambda}\left(\frac{n}{4}\right)},$$

$$LCL_{EWMA_M} = \frac{n}{2} - k\sqrt{\frac{\lambda}{2-\lambda}\left(\frac{n}{4}\right)},$$

where k is the width of control limit.

2.4. The average run length

Let $E_\theta(\cdot)$ denote the expectation under distribution $F(x, p_0)$ that the change-point occurs at point $\theta.$ In the literature on quality control the quantity $E_\infty(\tau_b) = A$ is the so-called "Average Run Length," which is a method frequently used in SPC charts for evaluation of the performance of various control charts. There are various characteristics that correspond with the performance of SPC chart, however, the average run length is still the most popular and commonly used characteristic for evaluating the performance of the SPC chart. There are two cases of the average run length as follows:

The first one shows that the performance of the SPC chart is in-control. It is called “in-control Average Run Length” or “ ARL_0, ” denoted as

$$ARL_0 \equiv E_\theta(\tau) = T, \quad \theta = \infty,$$

where θ is a change-point time, τ is the first exit time, $E_\theta(\tau)$ is the expectation under distribution $F(x, p_0)$ that the change-point occurs at point θ, and T is constant.

The next one shows that the performance of the SPC chart is out-of-control. It is called “out-of-control Average Run Length” or “ ARL_1, ” which depends on parameter θ. The ARL_1, was denoted as

$$ARL_1 \equiv E_\theta(\tau - \theta + 1 | \tau \geq \theta), \quad \theta = 1,$$

where $E_\theta(\tau - \theta + 1 | \tau \geq \theta)$ is the expectation under distribution $F(x, p \neq p_0)$ that the change-point occurs at point θ, respectively.

3. The Numerical Results and Performance Comparison

The performance of parametric and nonparametric of both CUSUM and EWMA control charts are compared when observation are from exponential distribution. The numerical results of approximations are based on Monte Carlo simulation which repeated 10,000 times. We given $\mathrm{ARL}_0 = 370$ and 500 and sample sizes $n = 20$ and 100 the magnitudes of change are $\delta = (0.00, 1)0.01$ for parametric control charts and $p = (0.50, 0.60)0.01$ for nonparametric control charts. The reference value of CUSUM statistic $k = 0.5$ and the weighted parameter of EWMA statistic $\lambda = 0.05$. The best performance of any control chart will give a minimum ARL_1 as shown on Tables 1 and 2 for fixed $\mathrm{ARL}_0 = 370$ and 500, respectively.

Table 1. Comparison of numerical results of ARL between exponential parametric and nonparametric CUSUM and EWMA when fixed $\text{ARL}_0 = 370$, $n = 20$ and 100

δ	CUSUM		EWMA		p_0	Non-CUSUM		Non-EWMA	
	$n = 20$	$n = 100$	$n = 20$	$n = 100$		$n = 20$	$n = 100$	$n = 20$	$n = 100$
0.00	371.557	367.105	374.633	372.471	0.50	366.580	365.105	371.010	370.730
0.01	282.267	293.657	75.635	23.271	0.51	199.828	188.254	31.05*	11.02**
0.02	211.430	160.093	25.298	8.622	0.52	138.697	96.593	10.57*	3.87**
0.03	155.387	89.041	16.335	5.776	0.53	107.346	71.149	7.082*	2.77**
0.04	110.033	65.207	10.665	3.822	0.54	87.678	60.853	4.564*	1.59**
0.05	64.924	32.678	8.405	3.057	0.55	71.187	47.556	3.541*	1.182**
0.06	47.555	13.941	6.770	2.356	0.56	65.206	39.286	2.301*	1.058**
0.07	18.669	6.146	5.510	1.951	0.57	56.214	34.833	1.925*	1.026**
0.08	11.201	2.848	4.593	1.644	0.58	51.087	30.528	1.574*	0.925**
0.09	2.059	1.199	2.225	1.076	0.59	45.716	27.239	1.025*	0.514**
0.10	0.253	0.779	1.789	0.977	0.60	39.792	24.693	0.115*	0.258**

*Minimum ARL_1 for $n = 20$ **Minimum ARL_1 for $n = 100$

Table 2. Comparison of numerical results of ARL between exponential parametric and nonparametric CUSUM and EWMA when fixed $\text{ARL}_0 = 500$, $n = 20$ and 100

δ	CUSUM		EWMA		p_0	Non-CUSUM		Non-EWMA	
	$n = 20$	$n = 100$	$n = 20$	$n = 100$		$n = 20$	$n = 100$	$n = 20$	$n = 100$
.00	518.654	498.672	505.611	505.575	0.50	509.689	493.766	500.71	503.01
.01	445.204	374.75	56.41	32.299	0.51	388.913	317.162	33.152*	10.536**
.02	317.788	218.854	23.686	10.601	0.52	289.445	206.172	11.091*	4.887**
.03	256.329	155.896	14.95	7.184	0.53	231.026	154.223	7.435*	2.952**

.04	187.375	115.423	11.07	4.899	0.54	192.941	132.657	4.424*	1.514**
.05	131.084	64.457	8.949	3.803	0.55	168.122	104.572	3.695*	1.022**
.06	89.387	38.974	7.258	3.072	0.56	147.998	87.035	2.814*	0.952**
.07	82.672	11.992	5.963	2.463	0.57	133.224	74.843	2.247*	0.428**
.08	21.953	7.495	4.921	2.021	0.58	114.956	67.997	2.015*	0.159**
.09	5.556	1.499	3.145	1.128	0.59	106.549	59.377	1.051*	0.078**
.10	1.691	0.901	1.825	0.521	0.60	91.999	55.275	0.852*	0.005**

*Minimum ARL_1 for $n = 20$ **Minimum ARL_1 for $n = 100$

4. Conclusions

In this work, the nonparametric CUSUM and EWMA control charts are studied the performance in order to detect the deviation from the target value of process. These nonparametric control charts are based on sign test in nonparametric which binomial distribution will be concerned. Furthermore, the performance of those nonparametric control charts are compared with parametric CUSUM and EWMA control charts when observations are exponential distributed. The Monte Carlo simulation is constructed to find out the numerical results with 10^4 times repeated. We found that the nonparametric is superior to parametric control chart. In addition, the nonparametric EWMA performs better than nonparametric CUSUM for small shifts $(p \leq 0.58)$, otherwise they are in the same manner of detecting the large shifts $(p > 0.58)$ when fixed $\text{ARL}_0 = 370$. The nonparametric EWMA is superior to other control charts for all magnitudes of shifts for given $\text{ARL}_0 = 500$.

Acknowledgement

The author would like to express his gratitude to Faculty of Applied Science, King Mongkut’s University of Technology, North Bangkok, Thailand for supporting research grant with Contract no. 5745103.

References

[1] E. Andersson, Monitoring cyclical processes: a non-parametric approach, Journal of Applied Statistics 29 (2002), 973-990.

[2] M. Basseville and I. Nikiforov, Detection of Abrupt Changes: Theory and Applications, Prentice-Hall, Englewood Cliffs, 1993.

[3] C. M. Borror, D. C. Montgomery and G. C. Runger, Robustness of the EWMA control chart to non-normality, Journal of Quality Technology 31 (1999), 309-316.

[4] B. A. Ergashev, On a CAPM Monitoring Based on the EWMA Procedure, In Working Paper, Presented at 9th International Conference of the Society for Computational Economics and Finance, 2003.

[5] M. Frisën, Evaluations of Methods for Statistical Surveillance, Statistics in Medicine 11 (1992), 1489-1502.

[6] D. M. Hawkins and D. H. Olwell, Cumulative Sum Charts and Charting for Quality Improvement, Springer, New York, 1998.

[7] G. Mititelu, Y. Areepong, S. Sukparungsee and A. A. Novikov, Explicit Analytical Solutions for the Average Run Length of CUSUM and EWMA Charts, East-West Journal of Mathematics 1 (2010), 253-265.

[8] D. C. Montgomery, Introduction to Statistical Quality Control, 5th ed., Wiley, New York, 2005.

[9] E. S. Page, Continuous inspection schemes, Biometrika 41 (1954), 100-114.

[10] S. W. Roberts, Control chart tests based on geometric moving average, Technometrics 1 (1959), 239-250.

[11] R. R. Sitter, L. Hanrahan, D. DeMets and H. Anderson, A monitoring system to detect increased rates of cancer incidence, American Journal of Epidemiology 132 (1990), 123-130.

[12] S. F. Yang and S. W. Cheng, A new non-parametric CUSUM mean chart, Quality and Reliability Engineering International 27(7) (2011), 867-875.

[13] S. F. Yang, J. S. Lin and S. W. Cheng, A new nonparametric EWMA sign control chart, Expert Systems with Applications 38 (2011), 6239-6243.