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A CHARACTERIZATION OF SMALL 3-COLORABLE GRAPHS

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Abstract

A k-coloring of a graph is a mapping $f : V(G) \to \{1, 2, ..., k\}$. A k-coloring is proper if adjacent vertices receive distinct colors. A graph is called k-colorable if it has a proper k-coloring. It is well-known that a graph is 2-colorable if and only if it does not contain any odd cycles as a subgraph. However, there is no result about forbidden subgraphs for 3-colorable graph.

In this paper, we show nine non-3-colorable graphs and prove that a graph with at most eight vertices is 3-colorable if and only if it does not contain the nine graphs as a subgraph.

1. Introduction

A *k-coloring* of a graph is a mapping $f : V(G) \to \{1, 2, ..., k\}$. A k-coloring is *proper* if adjacent vertices receive distinct colors. A graph is called *k-colorable* if it has a proper k-coloring. The chromatic number of

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a graph G, denoted by $\chi(G)$, is the smallest number k such that G is k-colorable.

It is well-known that a graph is 2-colorable if and only if it is bipartite. In other words, a graph is 2-colorable if and only if it does not contain any odd cycle as a subgraph. However, there is no characterization of 3-colorable graphs. Someone studies a sufficient condition for planar graph to be 3-colorable (see [1-3, 5-7]).

In this paper, we focus on forbidden subgraphs for 3-colorable graph. We find nine small non-3-colorable graphs as shown in Figure 1 and show that a graph with at most eight vertices is 3-colorable if and only if it does not contain the nine subgraphs.

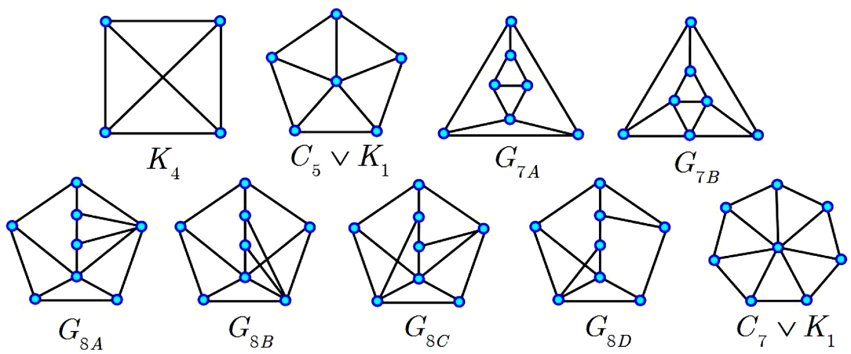


Figure 1. Small non-3-colorable graphs.

Throughout the paper, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G. Moreover, G_{7A}, G_{7B}, G_{8A}, G_{8B}, G_{8C} and G_{8D} are always the graphs in Figure 1.

An *independent set* in a graph is a set of pairwise nonadjacent vertices. For $X \subseteq V(G)$, $G - X$ is the graph obtained from deleting all vertices of X from G. In case $X = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by* X, denoted by $G[X]$ is the graph obtained from deleting all vertices of $V(G)$ outside X. A *cycle* C_n is a graph with n vertices which can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. We write the cycle $C_n = v_1 v_2 \cdots v_n$ if

$v_1, v_2, ..., v_n \in V(C_n)$ and $v_1v_2, v_2v_3, ..., v_nv_1 \in E(C)$. A *complete graph* K_n is a graph with n vertices such that vertices are pairwise adjacent. A *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices that are adjacent to v. The *join* of graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. For example, $C_5 \vee K_1$ and $C_7 \vee K_1$ in Figure 1. An *isomorphism* from a graph G_1 to a graph G_2 is a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. If there is an isomorphism from G_1 to G_2, we say G_1 is *isomorphic* to G_2, denoted by $G_1 \cong G_2$. For example, G_{7A} and G_{7B} have isomorphisms as shown in Figure 2.

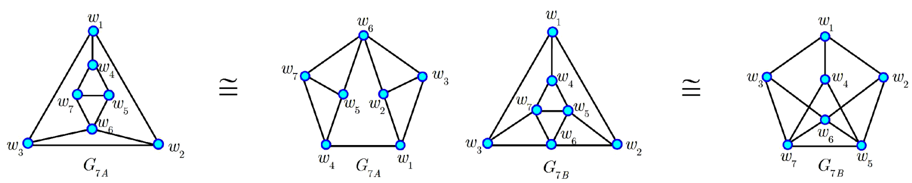


Figure 2. Isomorphisms of G_{7A} and G_{7B}.

We start showing that the chromatic number of the nine graphs is four.

Lemma 1.1. *If G is a graph in Figure* 1, *then* $\chi(G) = 4$.

Proof. Figure 3 shows that $\chi(G) \leq 4$.

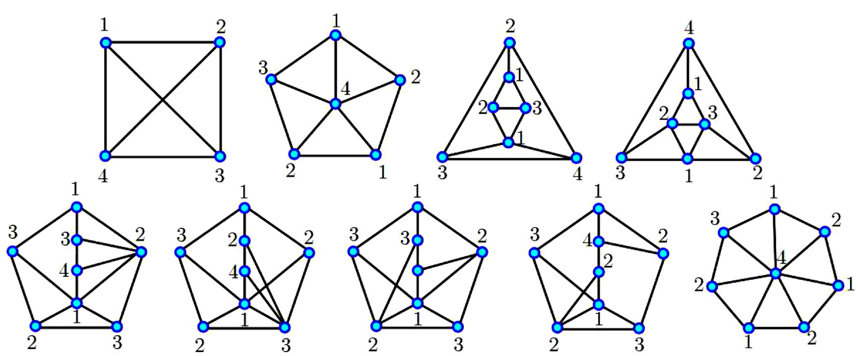


Figure 3. 4-colorings of the nine graphs.

It remains to show the nine graphs are not 3-colorable.

(a) All four vertices of K_4 need distinct colors; hence, K_4 is not 3-colorable.

(b) Since C_5 and C_7 require at least 3 colors and the centers require one more color, both $C_5 \vee K_1$ and $C_7 \vee K_1$ are not 3-colorable.

(c) Suppose that G_{7A} and G_{7B} are 3-colorable. Without loss of generality, w_4, w_6, w_5 and w_7 are labeled by colors 1, 1, 2 and 3, respectively (see Figure 2). For G_{7A}, the vertices w_1, w_2, w_3 cannot be labeled by color 1 and need three different colors, a contradiction. For G_{7B}, the vertices w_2, w_3 must be labeled by colors 3, 2, respectively. Then w_1 cannot be labeled by colors 1, 2 or 3, a contradiction.

(d) To prove G_{8A}, G_{8B}, G_{8C} and G_{8D} are not 3-colorable, we assign names of all vertices as shown in Figure 4.

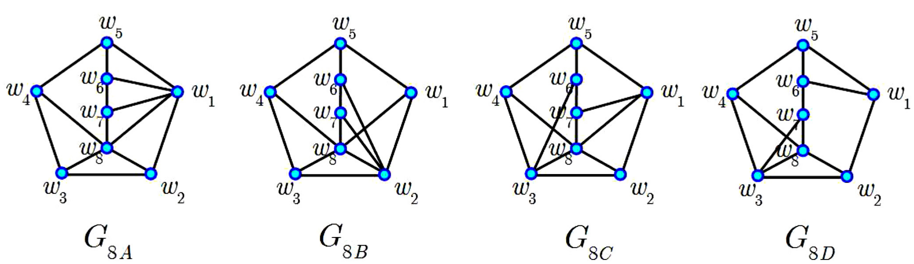


Figure 4. Assign names to all vertices of G_{8A}, G_{8B}, G_{8C}, G_{8D}.

Suppose that the four graphs are 3-colorable. Without loss of generality, suppose that w_8, w_3, w_2 and w_4 are labeled by colors 1, 2, 3 and 3, respectively. For G_{8A}, G_{8B} and G_{8C}, w_1 and w_5 must be labeled by color 2 and color 1, respectively. Hence, the remaining two vertices cannot be labeled, a contradiction. For G_{8D}, the vertex w_7 must be labeled by color 3. Then w_1, w_5 and w_6 cannot be labeled by color 3. However, the three vertices need three different colors, a contradiction. $\square$

To prove our main results, we need the following tools: Lemma 1.3, Theorem 1.4 and Theorem 1.5.

Lemma 1.2. *Let $S \subset V(G)$ be an independent set. If G is not 3-colorable, then $G - S$ contains an odd cycle.*

Proof. Let $S \subset V(G)$ be an independent set. Assume that $G - S$ does not contain an odd cycle. Then $G - S$ is bipartite; hence, $G - S$ is 2-colorable. All vertices from S can be labeled by the third color. Therefore, G is 3-colorable. □

Lemma 1.3. *Let v be a vertex of a graph G. If G is not 3-colorable, then $N(v)$ contains an odd cycle or $V(G) - N(v) - \{v\}$ is not an independent set.*

Proof. Let v be a vertex of a graph G. Assume that $N(v)$ does not contain any odd cycle and $S = V(G) - N(v) - \{v\}$ is an independent set of G. We first label all vertices in $N(v)$ by using two colors and label v and all vertices in S by using the third color. Hence, G is 3-colorable. □

Theorem 1.4 [4]**.** *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 1.5. *Let G be a graph with at most seven vertices. Then G is 3-colorable if and only if K_4, $C_5 \vee K_1$, G_{7A} and G_{7B} are not subgraphs of G.*

Proof. Let G be a graph with at most seven vertices.

Necessity. Suppose that K_4, $C_5 \vee K_1$, G_{7A} or G_{7B} is a subgraph of G. Since the four graphs are not 3-colorable, nor is G.

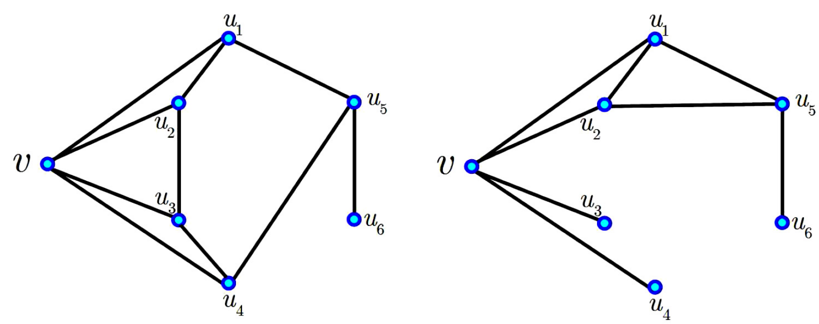
Sufficiency. Let v be a vertex of G with $d(v) = \Delta(G)$ and $S = V(G) - N(v) - \{v\}$. Suppose that G is not 3-colorable. If G is a complete graph, then K_4 is a subgraph of G. If $N(v)$ contains C_3 or C_5, then K_4 or $C_5 \vee K_1$ is a subgraph of G. Assume that G is not a complete graph and $N(v)$ does not contain any odd cycle. By Theorem 1.4, $d(v) = \Delta(G) \geq \chi(G) \geq 4$. By

Lemma 1.3, S is not an independent set. Then $|S| \geq 2$. Hence, $|S| = 2$ and $d(v) = 4$ because G has at most seven vertices.

Let $N(v) = \{u_1, u_2, u_3, u_4\}$ and $S = \{u_5, u_6\}$ such that u_5, u_6 are adjacent. By Lemma 1.2, both $G - v - u_5$ and $G - v - u_6$ contain odd cycles.

Case 1. $G - v - u_5$ or $G - v - u_6$ contains C_5. Without loss of generality, suppose that $C_5 = u_1u_2u_3u_4u_5$ is a subgraph of $G - v - u_6$ as shown in Figure 5(a).

Since G is not 3-colorable, $d(u_6) \geq 3$. Then u_6 is adjacent to at least two vertices from u_1, u_2, u_3, u_4. If u_6 is adjacent to u_2, u_3, then G_{7B} is a subgraph of G; otherwise, G_{7A} is a subgraph of G.



(a) (b)

Figure 5. $G - v - u_6$ contains C_5.

Case 2. $G - v - u_5$ and $G - v - u_6$ contain C_3.

Suppose that $C_3 = u_1u_2u_5$ is a subgraph of $G - v - u_6$ as shown in Figure 5(b). Since $d(u_1), d(u_2) \leq 4$ but C_3 is a subgraph of $G - v - u_5$, the C_3 must be $u_3u_4u_6$. Hence, G_{7A} is a subgraph of G. □

2. Main Results

It is obvious that a graph containing one of the nine non-3-colorable subgraphs is not 3-colorable but it needs several pages to prove that a non-3-

colorable graph with at most eight vertices must contain one of the nine graphs. Then the proof is divided into three lemmas in order to shorten and easily be followed.

Lemma 2.1. *Let G be a non-3-colorable graph with eight vertices and $\Delta(G) = 5$. Then K_4, $C_5 \vee K_1$, G_{7A}, G_{7B}, G_{8A}, G_{8B}, G_{8C}, G_{8D} or $C_7 \vee K_1$ is a subgraph of G.*

Proof. Let G be a non-3-colorable graph with eight vertices and $\Delta(G) = 5$. Let v be a vertex of G with $d(v) = 5$ and $S = V(G) - N(v) - \{v\}$. Let $N(v) = \{u_1, u_2, u_3, u_4, u_5\}$ and $S = \{u_6, u_7\}$.

If $N(v)$ contains C_3, C_5 or C_7, then G contains K_4, $C_5 \vee K_1$ or $C_7 \vee K_1$. Assume that $N(v)$ does not contain any odd cycle. By Lemma 1.3, we obtain that u_6, u_7 are adjacent and $G - v - u_6$, $G - v - u_7$ contain odd cycles.

Case 1. C_3 is a subgraph of $G - v - u_6$ and $G - v - u_7$. There are three possible ways that C_3 is a subgraph of $G - v - u_6$ and $G - v - u_7$ as shown in Figure 6.

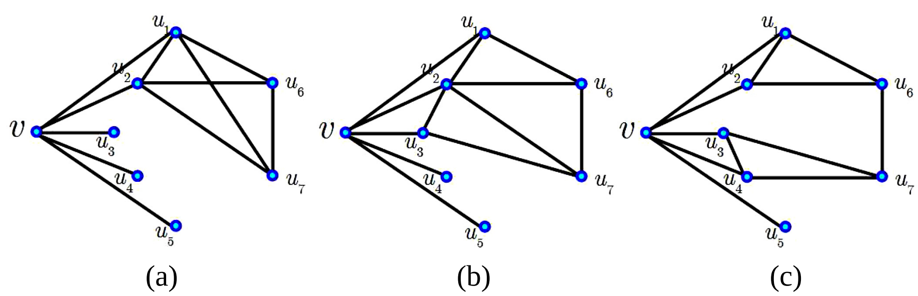


Figure 6. C_3 is a subgraph of $G - v - u_6$ and $G - v - u_7$.

In Figure 6(a), $G[\{u_1, u_2, u_6, u_7\}] \cong K_4$.

In Figure 6(b), $G - u_4 - u_5 \cong C_5 \vee K_1$.

In Figure 6(c), $G - u_5 \cong G_{7A}$.

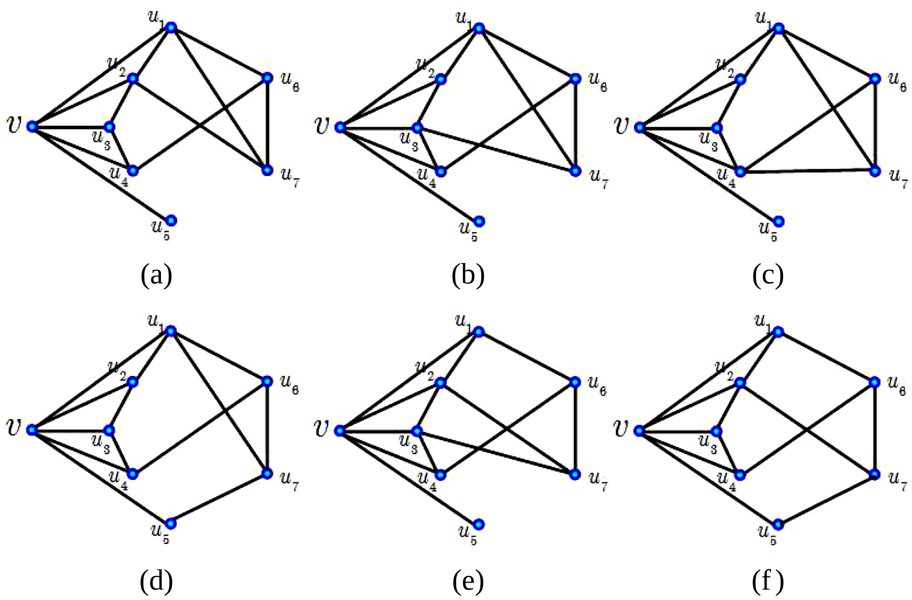


Figure 7. C_5 is a subgraph of $G - v - u_6$.

Case 2. C_5 is a subgraph of $G - v - u_6$ or $G - v - u_7$. Without loss of generality, suppose that $C_5 = u_1u_2u_3u_4u_6$ is a subgraph of $G - v - u_7$. Moreover, C_3 or C_5 is a subgraph of $G - v - u_6$. That is, the vertex u_7 must be in the odd cycle and it must be adjacent to exactly two vertices from $N(v)$ in the cycle. There are six possible ways to classify the two vertices as shown in Figure 7.

In Figures 7(a) and 7(c), G_{7A} is a subgraph of $G - u_5$.

In Figure 7(b), $N(v)$ does not contain any odd cycle. Then $G - v - u_6$ does not contain a cycle, a contradiction.

In Figure 7(e), $G - u_5 \cong G_{7B}$.

In Figure 7(d), if C_3 is a subgraph of $G - v - u_6$, then the cycle must be $u_1u_7u_5$; hence, $G \cong G_{8A}$. If C_5 is a subgraph of $G - v - u_6$, then the cycle must be $u_1u_7u_5u_3u_2$ or $u_1u_7u_5u_3u_4$. Then there exists the edge whose endpoints u_3 and u_5. Hence, $G - vu_1 \cong G_{8D}$.

In Figure 7(f), if C_3 is a subgraph of $G - v - u_6$, then the cycle must be $u_2u_7u_5$; hence, $G \cong G_{8B}$. If C_5 is a subgraph of $G - v - u_6$, then the cycle must be $u_2u_7u_5u_4u_3$ or $u_2u_7u_5u_4u_1$. Then there exists the edge whose endpoints u_4 and u_5. Hence, $G \cong G_{8C}$. $\square$

Lemma 2.2. *Let G be a non-3-colorable graph with eight vertices and* $\Delta(G) = 4$. *Let v be a vertex with degree* 4. *If* $V(G) - N(v) - \{v\}$ *is not a clique, then* K_4, $C_5 \vee K_1$, G_{7A}, G_{7B} *or* $G_7 \vee K_1$ *is a subgraph of G.*

Proof. Let G be a non-3-colorable graph with eight vertices and $\Delta(G) = 4$. Let v be a vertex with degree 4. Let $N(v) = \{u_1, u_2, u_3, u_4\}$ and $S = V(G) - N(v) - \{v\} = \{u_5, u_6, u_7\}$. If $N(v)$ contains C_3, then K_4 is a subgraph of G. Assume that $N(v)$ does not contain any odd cycle and $G[S]$ is not a clique.

Case 1. $G[S]$ has no edge. Then $S \cup \{v\}$ is an independent set. By Lemma 1.2, G is 3-colorable; a contradiction.

Case 2. $G[S]$ has exactly one edge, say u_5u_6. Claim that $G - u_7$ is not 3-colorable. Suppose that $G - u_7$ is 3-colorable. Since $N(u_7) \subset N(v)$ does not contain any odd cycle, it is 2-colorable. We can label u_7 by the third color. Then G is also 3-colorable, a contradiction. By Theorem 1.5, K_4, $C_5 \vee K_1$, G_{7A} or G_{7B} is a subgraph of $G - u_7$. Therefore, it is a subgraph of G.

Case 3. $G[S]$ has exactly two edges, say u_5u_6 and u_6u_7.

By Lemma 1.2, $G - u_5 - u_7$ and $G - u_6$ must contain odd cycles, say C_1 and C_2, respectively. Under the condition, $N(v)$ does not contain any odd cycle, $u_6 \in V(C_1)$ as shown in Figure 8.

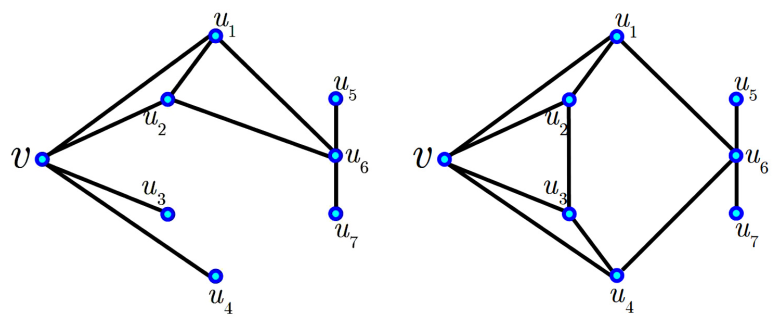


Figure 8. $G[S]$ has exactly two edges.

Either u_5 or u_7 is in $V(C_2)$ because $\Delta(G) = 4$, $N(v)$ does not contain any odd cycle and $\{u_5, u_6, u_7\}$ is not a clique. Without loss of generality, suppose that $u_5 \in V(C_2)$. Clearly, $N(v)$ needs two colors. Then u_5 and u_6 require two new colors. Then $G - u_7$ is not 3-colorable; hence, the proof is done by Theorem 1.5. □

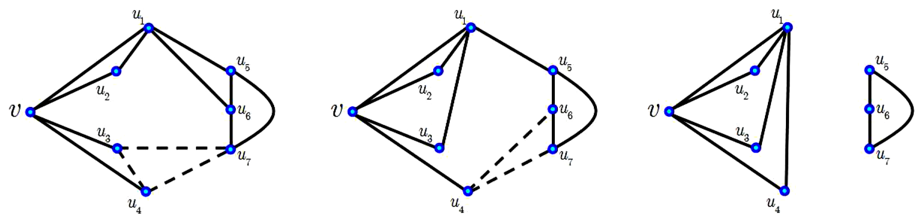
Lemma 2.3. *Let G be a non-*3*-colorable graph with eight vertices and* $\Delta(G) = 4$. *Let v be a vertex with degree* 4. *If* $V(G) - N(v) - \{v\}$ *is a clique, then* K_4, $C_5 \vee K_1$, G_{7A}, G_{7B}, $C_5 \vee K_1$, G_{8A}, G_{8B}, G_{8C}, G_{8D} *or* $C_7 \vee K_1$ *is a subgraph of G.*

Proof. Let G be a non-3-colorable graph with eight vertices and $\Delta(G) = 4$. Let v be a vertex with degree 4. Let $N(v) = \{u_1, u_2, u_3, u_4\}$ and $S = V(G) - N(v) - \{v\} = \{u_5, u_6, u_7\}$. Assume that S is a clique. If there is a vertex w with degree at most two, then $G - w$ is not 3-colorable; hence, we apply Theorem 1.5. Otherwise, suppose that every vertex has degree at least three.

Case 1. $d(u_1)$, $d(u_2)$, $d(u_3)$ or $d(u_4)$ is 4. Without loss of generality, assume that $d(u_1) = 4$.

Case 1.1. u_1 is adjacent to u_5, u_6, u_7. Then the four vertices become a K_4.

Case 1.2. u_1 is adjacent to u_2, u_5, u_6. If $\{u_3, u_4, u_7\}$ is not a clique, then the proof is done by Lemma 2.2. Suppose that $\{u_3, u_4, u_7\}$ is a clique as shown in Figure 9(a). Then $G - u_2$ is G_{7A}.



(a) (b) (c)

Figure 9. Graphs with $\Delta(G) = 4$ and $d(v) = 4$.

Case 1.3. u_1 is adjacent to u_2, u_3, u_5. If $\{u_4, u_6, u_7\}$ is not a clique, then the proof is done by Lemma 2.2. Suppose that $\{u_4, u_6, u_7\}$ is a clique as shown in Figure 9(b). If u_2 is adjacent to u_3, then $\{v, u_1, u_2, u_3\}$ is a K_4. If u_2 is adjacent to u_4, then the proof is done by Case 1.2. If u_2 is adjacent to u_5, then $G - u_3$ is G_{7A}. According to $d(u_2) \geq \delta(G) = 3$, u_2 is adjacent to u_6 or u_7. Similarly, u_3 is adjacent to u_6 or u_7. Suppose that u_2 and u_3 are adjacent to u_6 and u_7, respectively.

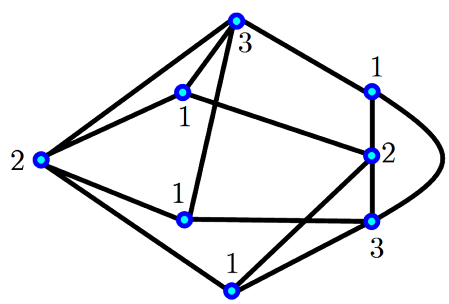


Figure 10. A 3-coloring.

Hence, G must be the graph shown in Figure 10 which is 3-colorable, a contradiction.

Case 1.4. u_1 is adjacent to u_2, u_3, u_4 (see Figure 9(c)). If $\{u_2, u_3, u_4\}$ is not an independent set, then K_4 is a subgraph of G; otherwise, assume that $\{u_2, u_3, u_4\}$ is an independent set. If u_2 is adjacent to two vertices from u_5, u_6, u_7, then the proof is done by Case 1.2. Suppose that u_2 is adjacent to at most one vertex from u_5, u_6, u_7. Similarly, each of u_3 and u_4 is adjacent to at most one vertex from u_5, u_6, u_7. Since $d(u_5)$, $d(u_6)$, $d(u_7)$ ≥ 3, we assume that there are edges u_2u_5, u_3u_6 and u_4u_7.

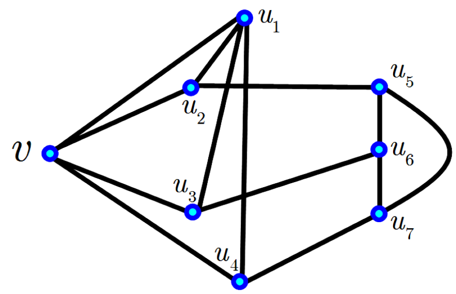


Figure 11. The graph G_{8D}.

Hence, G must be the graph shown in Figure 11 which is G_{8D}.

Case 2. $d(u_1)$, $d(u_2)$, $d(u_3)$ and $d(u_4)$ is 3. Since $\sum_{x \in V(G)} d(x)$ is an even number, so is $d(u_5) + d(u_6) + d(u_7)$.

Case 2.1. $d(u_5) = d(u_6) = d(u_7) = 4$. Then there are exactly six edges whose an endpoint from u_1, u_2, u_3, u_4 and the other endpoint from u_5, u_6, u_7. Recall that there are exactly four edges whose endpoint from u_1, u_2, u_3, u_4 and the other endpoint is v. Since $d(u_1) = d(u_2) = d(u_3) = d(u_4)$ $= 3$, we have $d(u_1) + d(u_2) + d(u_3) + d(u_4) = 12$. Then $G[\{u_1, u_2, u_3, u_4\}]$ has exactly one edge, say u_1u_2. Since $d(u_3) = d(u_4) = 3$, each of u_3 and u_4 is adjacent to exactly two vertices from u_5, u_6, u_7. Suppose that u_4 is not adjacent to u_7. Now, we focus on u_7 and its neighborhood. Since $d(u_7)$ $= 4$ and $V(G) - \{u_7\} - N(u_7)$ is not a clique, the proof is done by Lemma 2.2.

Case 2.2. $d(u_5) = 4$ and $d(u_6) = d(u_7) = 3.$ Without loss of generality, suppose that u_5 is adjacent to u_1 and u_2. If u_3 is not adjacent to u_4, then $\{v, u_3, u_4\}$ is not a clique; hence, we focus on u_5 and apply Lemma 2.2. Suppose that u_3 and u_4 are adjacent.

Notice that $d(u_1) + d(u_2) + d(u_3) + d(u_4) = 12$, $d(u_5) = 4$ and $d(u_6) = d(u_7) = 3.$ Then $G[\{u_1, u_2, u_3, u_4\}]$ has exactly two edges. The first edge is u_3u_4 and there are two possible ways for the remaining edge as shown in Figure 12. Moreover, u_6 and u_7 have only one choice left. Hence, G must be a graph in Figure 12 which is 3-colorable, a contradiction. □

It remains to combine the three lemmas and tools to obtain the main result.

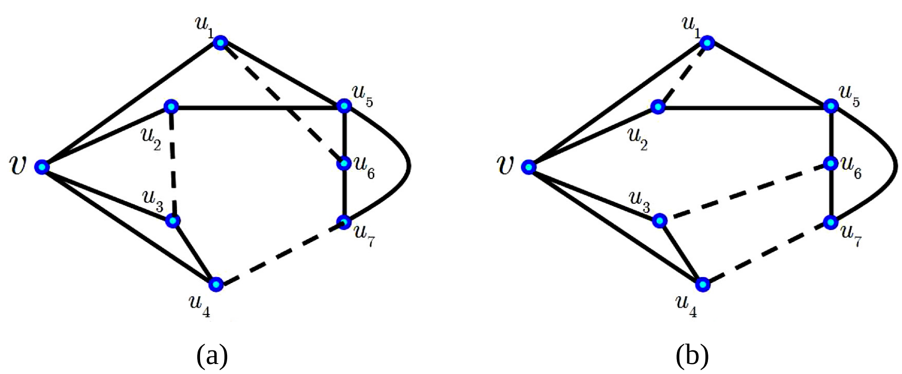


Figure 12. Graphs with $d(u_5) = 4$ and $d(u_6) = d(u_7) = 3.$

Theorem 2.4. *Let G be a graph with at most eight vertices. Then G is* 3-*colorable if and only if G contains neither* K_4, $C_5 \vee K_1$, G_{7A}, G_{7B}, G_{8A}, G_{8B}, G_{8C}, G_{8D} *nor* $C_7 \vee K_1$.

Proof. If G is a graph with at most seven vertices, then the statement is true by Theorem 1.5. Let G be a graph with eight vertices.

Necessity. Suppose that G contains K_4, $C_5 \vee K_1$, G_{7A}, G_{7B}, G_{8A}, G_{8B}, G_{8C}, G_{8D} or $C_7 \vee K_1$. Since the nine graphs are not 3-colorable, nor is G.

Sufficiency. Suppose that G is not 3-colorable. Let v be a vertex of G with $d(v) = \Delta(G)$ and $S = V(G) - \{v\} - N(v)$. If G is a complete graph, then K_4 is a subgraph of G. If $N(v)$ contains an odd cycle, then K_4, $C_5 \vee K_1$ or $C_7 \vee K_1$ is a subgraph of G. Assume that G is not a complete graph and $N(v)$ does not contain any odd cycle. By Theorem 1.4, $d(v) = \Delta(G) \geq 4$.

Case 1. $d(v) \geq 6$. Since $N(v)$ does not contain any odd cycle, G is 3-colorable, a contradiction.

Case 2. $d(v) = 5$. Then $|S| = 2$ because of $|V(G)| = 8$. Hence, the statement is true by Lemma 2.1.

Case 3. $d(v) = 4$. Then $|S| = 3$ because of $|V(G)| = 8$. Hence, the statement is true by Lemma 2.2 and Lemma 2.3. □

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