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A NECESSARY CONDITION FOR LOWER FINITE-TIME RUIN PROBABILITY UNDER XL-REINSURANCE IN THE DISCRETE-TIME RISK PROCESS WITH EXPONENTIAL CLAIMS

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Abstract

In this paper, we consider the discrete-time risk process in insurance in the case of excess-of-loss (XL-reinsurance) as the following:

$$U_n(u, b) = u + nc(b) - \sum_{k=1}^{n} \min\{b, Y_k\}, \ n = 1, 2, 3, \ldots,$$

where $\{Y_n : n \in \mathbb{N}\}$ is an independent and identically distributed claim process such that Y_n is a random variable indicating claim severity at time n, $c(b)$ is a net income rate for one unit time with the

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retention level b of XL-reinsurance, and $u \geq 0$ is an initial capital. We obtain a necessary condition for reducing the finite-time ruin probability in the case of Y_n has an exponential distribution and $c(b)$ is calculated by the expected value principle. To illustrate these results some numerical examples are included.

1. Introduction

In the present moment, risk processes have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserves of insurance companies. The main interest from the view of an insurance company is claim arrivals and claim severities, which affect the capital of the company.

However, the insurance company can transfer the risk which is the effect of claim severity to the counterparty (reinsurance company). Therefore, reinsurance is a normally activity of insurance company because it can reduce the risk (ruin probability) arising from claim severities. For examples, insurance company holds the XL-reinsurance contract with retention level b, i.e., the quantity $h(b, y) = \min\{y, b\}$ is the part of the claim size y paid by the insurer and the remaining part $y - \min\{y, b\}$, called *reinsurance recovery*, paid by the reinsurer. Thus, there are many papers which studied their effect in the insurance business. For example, Dickson and Waters studied the effect of reinsurance on ruin probability, Asmussen et al. studied a model of a financial corporation which is related to an insurance company with the risk control method known in the industry as excess-of-loss reinsurance, Walhin and Paris calculated the probability of ruin in a discrete-time risk model using recursive methods, Taksar and Markussen considered the optimal reinsurance policy which minimizes the ruin probability of the cedent, Schäl and Angew studies an insurance model where the risk process can be controlled by reinsurance and by investment in a financial market, Li and Gu studied risk process and the optimization problem of maximizing the exponential utility of terminal wealth under the controls of excess-of-loss reinsurance and investment.

In 2006, Chan and Zhang [2] considered the discrete-time risk process as follows:

$$U_n = u + nc_0 - \sum_{k=1}^{n} Y_k, \quad n = 1, 2, 3, \ldots, \tag{1.1}$$

where $u \geq 0$ is an initial capital, c_0 is an net premium rate for one unit time and $\{Y_n : n \in \mathbb{N}\}$ is a sequence of independent and identically distributed (i.i.d.) claim random variables. This model is considered under the assumption that the insolvency (ruin) can occur at time instants $n = 1, 2, 3, \cdots$. Therefore, the ruin probability at one of the time $k = 1, 2, 3, \ldots, n$ is defined by

$$\varphi_n(u) = \Pr(U_k(u) < 0 \text{ for some } k = 1, 2, 3, \ldots, n). \tag{1.2}$$

In the case of exponential claim, they found the closed form for the ruin probability as follows:

$$\varphi_n(u) = \sum_{k=1}^{n} \frac{[\lambda(u + kc_0)]^{k-1}}{(k-1)! \cdot \exp(\lambda(u + kc_0))} \frac{u + c_0}{u + kc_0}, \; n = 1, 2, 3, \ldots. \tag{1.3}$$

In this paper, we shall consider the excess-of-loss reinsurance (XL-reinsurance) as control action, i.e., $h(b, y) = \min\{b, y\}$. Therefore, we consider the risk process given by

$$U_n(u, b) = u + nc(b) - \sum_{k=1}^{n} \min\{b, Y_k\}, \; n = 1, 2, 3, \ldots, \tag{1.4}$$

where $u \geq 0$ is an initial capital, b is a retention level of XL-reinsurer, $\{Y_n : n \in \mathbb{N}\}$ is i.i.d., and $c(b)$ is a net income rate for one unit time.

Recently, Intarasit et al. considered the risk process (1.4) to find the ruin probability at one of the time $k = 1, 2, 3, \ldots, n,$ denoted by

$$\Phi_n(u, b) = \Pr(U_k(u, b) < 0 \text{ for some } k = 1, 2, 3, \ldots, n). \tag{1.5}$$

They have considered this problem under the assumption $u \leq b - c(b)$ and found that

$$\Phi_n(u, b) = \sum_{k=1}^{n} \frac{[\lambda(u + kc(b))]^{k-1}}{(k-1)! \cdot \exp(\lambda(u + kc(b)))} \frac{u + c(b)}{u + kc(b)}, \; n = 1, 2, 3, \ldots. \quad (1.6)$$

This result is a very beautiful closed form, but it is quite strange because we observe that this formula is similar to the closed form of the case of without of reinsurance action as in Equation (1.3).

In this paper, we are interested in the effect of the assumption $u \leq b - c(b)$ to the finite-time ruin probability of the risk process (1.4). Finally, we give the numerical example in the last section.

2. Model Description

In this paper, we shall consider the risk process (1.4), and assume that all of random variables are defined on a probability space $(\Omega, \mathcal{F}, \Pr)$. Obviously, $\{U_n(u, b) : n \in \mathbb{N}\}$ is controlled by choosing the retention level b of XL-reinsurance for one period. For the retention level b, the insurer has to pay the premium rate to the reinsurer which is deduced from c_0, as a result of which insurer net income rate is represented by the function $c(b)$. The level $b = r_u$ stands for the control action without reinsurance, this means that $c_0 = c(r_u)$, and the level $b = r_l$ is the smallest retention level which can be chosen. Of course, we obtain the net income rate $c(b)$ such that

$$0 \leq c(r_l) \leq c(b) \leq c(r_u) = c_0 \quad (2.1)$$

for all $b \in [r_l, r_u]$. Throughout this paper, we shall consider the risk process (1.4) under the following assumptions:

Assumption 1. Independence Assumption

$\{Y_n : n \in \mathbb{N}\}$ is a sequence of i.i.d. non-negative random variables.

Assumption 2. Expected value premium principle

The premium rate for one unit time of insurer is calculated by the *expected value principle*, i.e.,

$$c_0 = (1 + \theta_0)\mathbf{E}[Y_1], \tag{2.2}$$

where $\theta_0 > 0$ is a safety loading of insurer. In addition, the premium rate of reinsurer is also computed by the expected value principle and deduced from c_0, i.e., the net income rate $c(b)$ is obtained by

$$c(b) = c_0 - (1 + \theta_1)\mathbf{E}[\max\{Y_1 - b,\ 0\}], \tag{2.3}$$

where $\theta_1 > 0$ is the safety loading of reinsurer and *b* is the retention level.

From Assumption 1, we found that $\{\min\{b,\ Y_n\} : n \in \mathbb{N}\}$ is also i.i.d., and using Assumption 2, we obtain that $c(b)$ is an increasing function on *b*, and found that

$$\lim_{b \to \infty} c_0 - (1 + \theta_1)\mathbf{E}[\max\{Y_1 - b,\ 0\}] = c_0.$$

In the case of XL-reinsurance, we therefore denote $r_u = \infty$.

3. Main Results

In this paper, we consider the finite-time ruin probability of the risk process in Equation (1.4) with the i.i.d. exponential claim process $\{Y_n : n \in \mathbb{N}\}$ such that Y_1 has an exponential distribution with intensity $\lambda > 0$, i.e., $\mathbf{E}[Y_1] = \dfrac{1}{\lambda}$.

Lemma 3.1. *Let u,* λ *be non-negative real numbers such that* $\lambda > 0$, *and n be a natural number. Then*

$$f_n(u,\ \delta) = \frac{[\lambda(u + n\delta)]^{n-1}(u + \delta)\exp(-\lambda(u + n\delta))}{(n-1)!(u + n\delta)} \tag{3.1}$$

is a decreasing function on $\delta > \dfrac{1}{\lambda}$.

Proof. Consider $\delta > \frac{1}{\lambda}$, i.e., $\lambda\delta - 1 > 0$, then

$$\frac{\partial}{\partial\delta} f_n(u, \delta)$$

$$= \frac{\lambda^{n-1}(u + n\delta)^{n-3}}{(n-1)!\, e^{\lambda(u+n\delta)}}[u + n\delta + n(n-2)(u+\delta) - (u + n\delta)\lambda n(u + \delta)]$$

$$= \frac{\lambda^{n-1}(u + n\delta)^{n-3}}{(n-1)!\, e^{\lambda(u+n\delta)}}[u + n\delta + n^2u + n^2\delta - 2nu - 2n\delta - \lambda nu^2$$

$$- \lambda nu\delta - \lambda n^2u\delta - \lambda n^2\delta^2]$$

$$= \frac{\lambda^{n-1}(u + n\delta)^{n-3}}{(n-1)!\, e^{\lambda(u+n\delta)}}[-u(2n-1) - n\delta - n^2u(\lambda\delta - 1 + \lambda) - n^2\delta(\lambda\delta - 1)]$$

$$< 0.$$

This means that $f_n(u, \delta)$ is a decreasing function on $\delta > \frac{1}{\lambda}$. This completes the proof. □

Theorem 3.2. *Under Assumptions* 1 *and* 2. *Let u be a non-negative real number and n be a natural number*. *If* Y_1 *has exponential distribution with intensity* $\lambda > 0$ *and* $u \le b - c(b)$, *then* $\Phi_n(u, b)$ *is a decreasing function on* $b \ge r_l$.

Proof. Let $u \le b - c(b)$. Using Equation (1.6), we therefore have

$$\Phi_n(u, b) = \sum_{k=1}^{n} \frac{[\lambda(u + kc(b))]^{k-1}}{(k-1)!\exp(\lambda[u + kc(b)])} \frac{u + c(b)}{u + kc(b)}$$

$$= \sum_{k=1}^{n} f_k(u, c(b)),$$

where

$$f_k(u, \delta) = \frac{[\lambda(u + k\delta)]^{k-1}}{(k-1)! \cdot \exp(\lambda[u + k\delta])} \frac{u + \delta}{u + k\delta}$$

for all $k = 1, 2, ..., n.$ From Assumption 2, we have $c_0 > \mathbf{E}[Y_1] = \frac{1}{\lambda}.$ Using Lemma 3.1, $f_k(u, \delta)$ is a decreasing function on δ and $c(b)$ is an increasing function on b, $f_k(u, c(b))$ is a decreasing function on b for all $k = 1, 2, ..., n.$ This implies that $\Phi_n(u, c(b), b)$ is also a decreasing function on b. The proof is completed. □

Using Equation (1.3) and setting $b = r_u,$ we have $\Phi_n(u, \infty) = \varphi_n(u).$ Under assumptions hypothesis of Theorem 3.2, we then have

$$\Phi_n(u, b) \leq \lim_{\delta \to \infty} \Phi_n(u, \delta) = \Phi_n(u, \infty) = \varphi_n(u)$$

for all $b > r_l$. This means that the net premium rate of insurer is reduced by choosing the retention level b of XL-reinsurance action, but the risk (ruin probability) is higher than the action without reinsurance. By Theorem 3.2, this leads to a necessary condition for reducing ruin probability under XL-reinsurance in the case of exponential claims as the following corollary.

Corollary 3.1. *Under Assumptions* 1 *and* 2. *Let u be non-negative real number and n be a natural number*. *If* Y_1 *has exponential distribution with intensity* $\lambda > 0$ *and* $\Phi_n(u, b)$ *is an increasing function on* $b > r_l$, *then* $u > b - c(b)$.

4. Applications

In this section, we provide numerical illustrations of the main results. We assume that the premium rate of insurer and reinsurer are calculated by the premium rate by using the expected value premium principle with safety loading of $\theta_0 = 0.20$ and $\theta_1 = 0.25,$ respectively, and the claim size Y_1 has an exponential distribution with intensity $\lambda = 1,$ i.e.,

$$c_0 = (1+\theta_0)\mathbf{E}[Y_1] = 1.2,$$

and

$$c(b) = c_0 - (1+\theta_1)\mathbf{E}[\max\{Y_1 - b,\ 0\}]$$

$$= 1.2 - 1.25\int_b^{\infty} y\exp(-y)dy$$

$$= 1.2 - 1.25\exp(-b). \tag{4.1}$$

Therefore, we obtain the graph $y = x - c(x)$ in Figure 1.

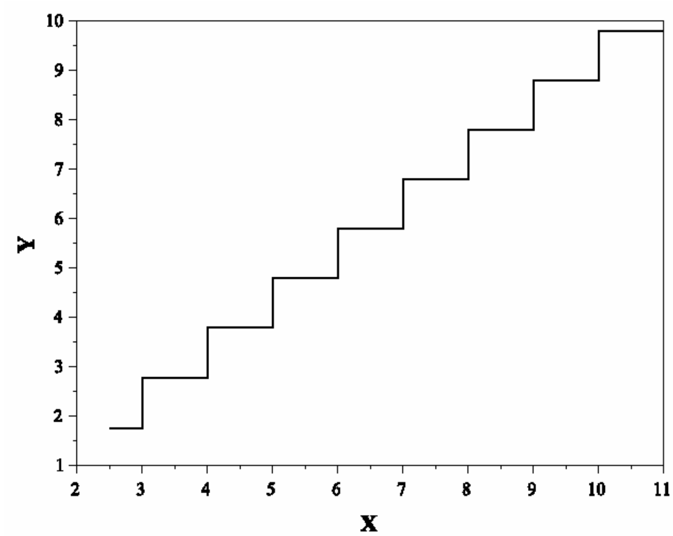

Figure 1. The graphs of the $y = x - c(x)$.

From Theorem 3.2, we then have $1 < b - c(b) < 10,$ for all $b \in [2.5, 10]$. Next, we shall show the results in the cases of initial capital $u = 1$ and 10 with time horizon $n = 100$.

Case $u = 1$. Since $1 \le b - c(b)$ for all $b \in [2.5, 10]$, by Theorem 3.2, we can calculate the ruin probability at one of the time $1, 2, 3, \ldots, n$ as the following formula:

$$\Phi_{100}(u, b) = \sum_{k=1}^{100} \frac{(1 + kc(b))^{k-1}}{(k-1)! \exp(1 + kc(b))} \frac{1 + c(b)}{1 + kc(b)}$$

$$= (2.2 + 1.25e^{-b}) \sum_{k=1}^{100} \frac{(1 + 1.2k + 1.25ke^{-b})^{k-2}}{\exp(1 + 1.2k + 1.25ke^{-b})(k-1)!}$$

which is shown in Figure 2 (left).

Case $u = 10$. Since the condition $10 > b - c(b)$ for all $b \in [2.5, 10]$ does not satisfy the assumption in Theorem 3.2, we cannot obtain the closed form as in Equation (1.3). However, we have approximated the finite-time ruin probability by using the simulation approach. The obtained result is carried out with 100,000 paths which are own in Figure 2 (right).

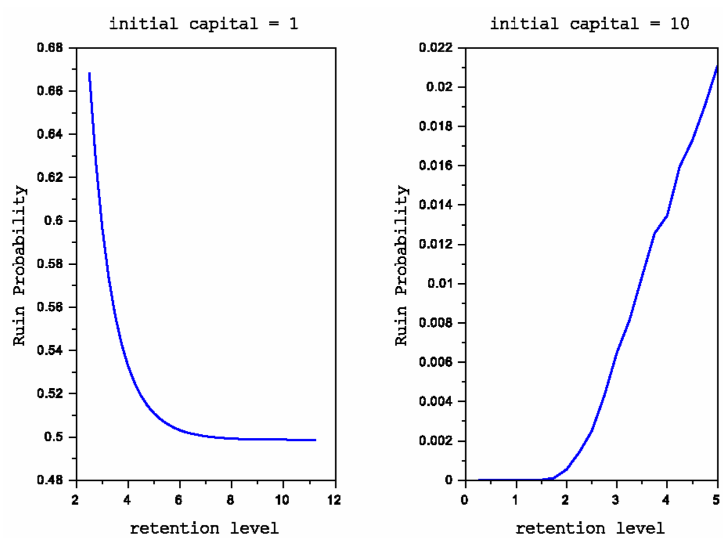

Figure 2. The graphs of the retention level and ruin probability of discrete-time surplus processes obtained from calculation.

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