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SHARPER ANALYSIS OF UPPER BOUND FOR THE ITERATION COMPLEXITY OF AN INTERIOR-POINT METHOD USING PRIMAL-DUAL FULL-NEWTON STEP ALGORITHM

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Abstract

The use of interior-point methods to solve linear optimization problems has become a great attention to the researchers. The most important thing is that the interior-point methods have the best complexity compared to other methods and also efficient in practice. The worst upper bound for the iteration complexity of this method is polynomial. Roos, Terlaky and Vial presented an interior-point method using primal-dual full-Newton step algorithm that requires the best known upper bound for the iteration complexity of an interior-point method. In this paper, we present their method with a slightly better iteration bound.

1. Introduction

Recently, the use of interior-point methods (IPMs) for solving linear

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optimization (LO) problems has become a major concern of the optimization researchers. This happens mainly due to the fact that interior-point methods have polynomial complexity, which is the best compared to other methods and these methods are also efficient in practice.

Interior-point method first appeared in 1984, when Karmarkar [4] proposed a polynomial-time method for LO problems. In the worst-case, for a problem with n inequalities and L bits of input data, Karmarkar's algorithm requires $O(n^{3.5}L)$ arithmetic operations on numbers with $O(L)$ bits.

In [8], Renegar improved the number of iterations to $O(\sqrt{n}L)$ iterations. Other variants of IPMs, called *potential reduction methods*, require also only $O(\sqrt{n}L)$ iterations. This was shown by Ye [13], Freund [1], Todd and Ye [12] and Kojima et al. [5].

Sonnevend [11] and Meggido [6] introduced a class of IPMs which uses the so-called central path as a guide line to the set of optimal solutions. These methods are called path-following methods. A variant of path-following methods was presented by Gonzaga [3], Monteiro and Adler [7] and Roos and Vial [10]. Their methods require $O(\sqrt{n}L)$ iterations, which is the best known upper bound for the iteration complexity of an IPM. Roos et al. in their book [9] obtained the same upper bound by using an algorithm which is a so-called primal-dual full-Newton step algorithm. Their upper bound for the number of iterations is

$$\left\lceil \sqrt{2n} \ln \frac{n\mu^0}{\varepsilon} \right\rceil, \tag{1.1}$$

where ε is the absolute accuracy of the objective function and $\mu^0 > 0$ denotes the initial value of the so-called barrier parameter.

In this paper, by careful analysis, we reduce the upper bound by a factor $\sqrt{2}$. The iteration upper bound that we obtained is

$$\left\lceil \sqrt{n} \log \frac{n\mu^0}{\varepsilon} \right\rceil.$$

2. Primal-dual IPM with Full-Newton Steps

The standard form of an LO problem is as follows:

$$\min\{c^T x : Ax = b,\ x \geq 0\}, \qquad (P)$$

where $c, x \in \mathcal{R}^n$, $b \in \mathcal{R}^m$ and $A \in \mathcal{R}^{m\times n}$. Any LO problem can be transformed into standard form, by introducing additional variables [2]. The problem (P) is often called the *primal* problem. Associated with any LO problem is another LO problem called the *dual* problem, which consists of the same data (A, b, c) arranged in a different way. The dual of (P) is

$$\max\{b^T y : A^T y + s = c,\ s \geq 0\}, \qquad (D)$$

where $s \in \mathcal{R}^n$ and $y \in \mathcal{R}^m$; (D) is called the *dual problem*.

The feasible regions of (P) and (D) are denoted by $\mathcal{P}$ and $\mathcal{D}$, respectively. The (relative) interiors of $\mathcal{P}$ and $\mathcal{D}$ are denoted by $\mathcal{P}^o$ and $\mathcal{D}^o$.

Finding an optimal solution of (P) and (D) is equivalent to solving the following system [9]:

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= 0, \end{aligned} \qquad (2.1)$$

where xs is the component-wise (or Hadamard) product of the vectors x and s and 0 denotes the zero vector. The first line in (2.1) is simply the feasibility constraint for the primal problem (P) and the second line represents the feasibility constraint for the dual problem (D). The last equation is the so-called complementarity condition.

By using path-following IPM, the complementarity condition in (2.1) is replaced by $xs = \mu e$, where μ is any positive number and e is the all-one vector. This new constraint is referred to as the centering condition with

respect to μ. The resulting system is

$$Ax = b, \quad x \geq 0,$$

$$A^T y + s = c, \quad s \geq 0,$$

$$xs = \mu e. \tag{2.2}$$

If system (2.2) has a solution for some $\mu > 0$, then a solution exists for every $\mu > 0$ [9]. This happens if and only if interior point condition (IPC) is satisfied. The solutions are denoted as $x(\mu)$, $y(\mu)$ and $s(\mu)$. We call $x(\mu)$ the μ-*center of* (P) and $(y(\mu), s(\mu))$ the μ-*center of* (D).

When μ runs through $(0, \infty)$, then $x(\mu)$ runs through a curve in $\mathcal{P}^o$ which is called the *central path of* (P). Similarly, the set $\{(y(\mu), s(\mu)) : \mu \in (0, \infty)\}$ is called the *central path of* (D). If $\mu \to 0$, then $x(\mu)$, $y(\mu)$ and $s(\mu)$ converge to a solution of (2.1), which means that the central path converges to the set of optimal solutions of (P) and (D). On the other hand, if $\mu \to \infty$, then $x(\mu)$ and $(y(\mu), s(\mu))$ converge to the so-called *analytic center* of (P) and (D), respectively.

Next, it will describe how Newton's method can be used to obtain an approximate solution of system (2.2), for fixed μ. Given a primal-dual feasible pair $(x, (y, s))$, we want to define search direction Δx, Δy and Δs such that $(x + \Delta x, y + \Delta y, s + \Delta s)$ satisfy (2.2).

Since $Ax = b$ and $A^T y + s = c$, system (2.2) is equivalent to the following system:

$$A\Delta x = 0,$$

$$A^T \Delta y + \Delta s = 0,$$

$$s\Delta x + x\Delta s + \Delta x \Delta s = \mu e - xs.$$

The third equation is nonlinear, due to the quadratic term $\Delta x \Delta s$. By neglecting this quadratic term, according to Newton's method for solving nonlinear equations, we obtain the linear system

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{2.3}$$

The resulting directions of (2.3) are known as the primal-dual Newton directions. By taking a step along these directions, one finds new iterates $(x^+, (y^+, s^+))$ such that x^+ and s^+ are positive. The new iterates are given by

$$x^+ = x + \Delta x, \quad y^+ = y + \Delta y, \quad s^+ = s + \Delta s.$$

In the process of following the central path to the optimal solution, by using Newton steps, we generate a sequence of points within the neighborhood of central path. We need a quantity to measure the proximity of $(x, (y, s))$ to the μ-center.

Before defining this proximity measure, we reformulate the linear system (2.3), by scaling Δx, Δy and Δs to d_x, d_y and d_s as follows:

$$d_x = \frac{v\Delta x}{x}, \quad d_y = \frac{\Delta y}{\sqrt{\mu}}, \quad d_s = \frac{v\Delta s}{s},$$

where

$$v = \sqrt{\frac{xs}{\mu}}.$$

If we define $D = \text{diag}(\sqrt{x/s})$, then system (2.3) is equivalent to

$$\begin{aligned} ADd_x &= 0, \\ (AD)^T d_y + d_s &= 0, \\ d_x + d_s &= v^{-1} - v. \end{aligned} \tag{2.4}$$

The first two equations of system (2.4) show that the vectors d_x and d_s belong to the null space and the row space of the matrix AD, respectively. These two spaces are orthogonal, therefore, d_x and d_s are orthogonal. The orthogonality of d_x and d_s implies

$$\| d_x \|^2 + \| d_s \|^2 = \| d_x + d_s \|^2$$
$$= \| v^{-1} - v \|^2.$$

Note that d_x, d_s (and also d_y) are zero if and only if $v^{-1} - v = 0,$ which happens only if $v = e,$ and then x, y and s coincide with the respective μ-centers. Therefore, to measure the 'distance' of $(x, (y, s))$ to the μ-center, we use the quantity $\delta(x, s; \mu)$ defined by

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\| v^{-1} - v \|^2. \tag{2.5}$$

For any $\tau \geq 0,$ the τ-neighborhood of the μ-center is given by the set

$$\{(x, y, s) : x \in \mathcal{P}, (y, s) \in \mathcal{D}, \delta(x, s; \mu) \leq \tau\}.$$

After a full-Newton step, the duality gap at the new iterates always assumes the same value as at the μ-centers, i.e., $(x^+)^T s^+ = n\mu$ (cf. [9, Theorem II.47]). An IPM with full-Newton steps can be described as in Figure 1.

Primal-dual IPM with full-Newton steps

Input:

an accuracy parameter $\varepsilon > 0$;

a proximity parameter τ, $0 \leq \tau < 1$;

strictly feasible (x^0, y^0, s^0) with $(x^0)^T s^0 = n\mu^0$ and $\delta(x^0, s^0; \mu^0) \leq \tau$;

a barrier update parameter θ, $0 < \theta < 1$.

begin

$x := x^0$; $s := s^0$; $y = y^0$; $\mu := \mu^0$;

while $n\mu \geq \varepsilon$ **do**

$\mu := (1 - \theta)\mu$;

$x := x + \Delta x$;

$y := y + \Delta y$;

$s := s + \Delta s$;

endwhile

end

Figure 1. Primal-dual IPM with full-Newton steps.

3. Analysis of the Primal-dual IPM with Full-Newton Steps

We first deal with the effect of a full-Newton step on the proximity measure. The next lemma implies that when $\delta(x, s; \mu)$ is small enough then the primal-dual Newton step is quadratically convergent, as stated in Corollary 3.1.

Lemma 3.1 (cf. [9, Theorem II.50])**.** *If* $\delta := \delta(x, s; \mu) \leq 1$, *then the primal-dual Newton step is feasible*, *i.e.*, x^+ *and* s^+ *are nonnegative*. *Moreover*, *if* $\delta < 1$, *then* x^+ *and* s^+ *are positive and*

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

Corollary 3.1. *If* $\delta := \delta(x, s; \mu) \leq \frac{1}{\sqrt{2}}$, *then* $\delta(x^+, s^+; \mu) \leq \delta^2$.

Initially, the duality gap is $n\mu^0$. In each iteration, it is reduced by the factor $1 - \theta$. Using this, one easily proves the following lemma.

Lemma 3.2 (cf. [9, Lemma II.17])**.** *After at most*

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

iterations, *the algorithm stops and we have* $n\mu \leq \varepsilon$.

We have the following lemma that will be used in proving the next two theorems.

Lemma 3.3 (cf. [9, Lemma II.54])**.** *Let* (x, s) *be a positive primal-dual pair and* $\mu > 0$ *such that* $x^T s = n\mu$. *Moreover*, *let* $\delta := \delta(x, s; \mu)$ *and let* $\mu^+ = (1 - \theta)\mu$. *Then*

$$\delta(x, s; \mu^+)^2 = (1 - \theta)\delta^2 + \frac{\theta^2 n}{4(1 - \theta)}.$$

The next theorems present iteration bound of the primal-dual IPM with full-Newton steps.

Theorem 3.1. *If* $\tau = 1/\sqrt{2}$ *and* $\theta = 1/\sqrt{n+1}$, *then the algorithm requires at most*

$$\left\lceil \sqrt{n+1} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

iterations. *The output is a primal-dual pair* (x, s) *such that* $x^T s \leq \varepsilon$.

Proof. Let us take $\tau = 1/\sqrt{2}$. By using Corollary 3.1, since we have $\delta(x, s; \mu) \leq 1/\sqrt{2}$, after the primal-dual Newton step we have $\delta(x^+, s^+; \mu) \leq 1/2$. Then, after the update of the barrier parameter to $\mu^+ = (1 - \theta)\mu$ with $\theta = 1/\sqrt{n+1}$, by using Lemma 3.3, we get the following upper bound of $\delta(x^+, s^+; \mu^+)^2$:

$$\delta(x^+, s^+; \mu^+)^2 \leq \frac{1 - \theta}{4} + \frac{\theta^2 n}{4(1 - \theta)} = \frac{1}{2}.$$

The last equality follows by substituting $\theta = 1/\sqrt{n+1}$. Hence, we obtain $\delta(x^+, s^+; \mu^+) \leq 1/\sqrt{2} = \tau$. This means that the property

$$\delta(x, s; \mu) \leq \tau$$

is maintained after each iteration. Therefore, combining this with Lemma 3.2, we obtain the theorem. □

Note that Theorem 3.1 holds for every $n \geq 1$. In practice, n is much larger. For such cases it is worth mentioning that slightly better iteration bounds can be obtained, as the following theorem.

Theorem 3.2. *If* $\tau \in [0.6687, 0.6773]$ *and* $\theta = 1/\sqrt{n}$, *then for* $n \geq 47$, *the algorithm requires at most*

$$\left\lceil \sqrt{n} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

iterations.

Proof. Let us take $\theta = 1/\sqrt{n}$. By using Lemma 3.1 and Lemma 3.3, we can verify that if

$$\frac{(1-1/\sqrt{n})\tau^4}{2(1-\tau^2)} + \frac{1}{4(1-1/\sqrt{n})} \leq \tau^2, \tag{3.1}$$

then $\delta(x, s; \mu) \leq \tau$ is maintained. The region in the (τ, n)-space defined by (3.1) is depicted in Figure 2, where the smallest value of n is around 46.6274 at $\tau = 0.6731$. Therefore, $n = 47$ is the smallest integer value of n which satisfies (3.1). We can find out that for $n \geq 47$, the inequality (3.1) holds for $\tau \in [0.6687, 0.6773]$. Then we have the theorem. □

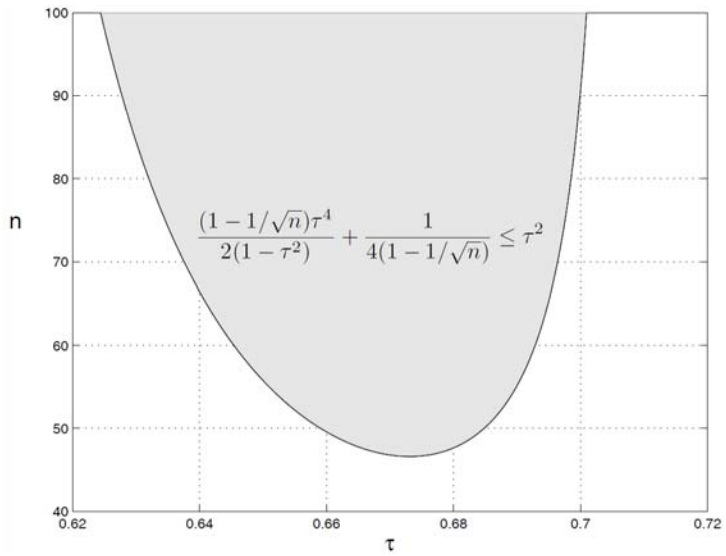

Figure 2. The region defined by (3.1).

The value of n in Theorem 3.2 can be improved to $n \geq 6$, as stated in the following theorem.

Theorem 3.3. *If* $\tau \in [0.7433, 0.8289]$ *and* $\theta = 1/\sqrt{n}$, *then for* $n \geq 6$, *the algorithm requires at most*

$$\left\lceil \sqrt{n} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

iterations.

Proof. We use [9, Theorem II.52], a sharper quadratic convergence result of a primal-dual Newton step. This theorem states that if $\delta := \delta(x, s; \mu) < 1$, then

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^4)}}.$$

Then, by using Lemma 3.3, we obtain that for $\theta = 1/\sqrt{n}$ the property $\delta(x, s; \mu) \leq \tau$ is maintained if

$$\frac{(1-1/\sqrt{n})\tau^4}{2(1-\tau^4)} + \frac{1}{4(1-1/\sqrt{n})} \le \tau^2. \tag{3.2}$$

Figure 3 depicts the region defined by (3.2). The smallest value of n is around 5.1971 at $\tau = 0.7968$, and we can verify that for the smallest integer value $n = 6$, the inequality (3.2) holds for $\tau \in [0.7433, 0.8289]$. Thus, we get the theorem. □

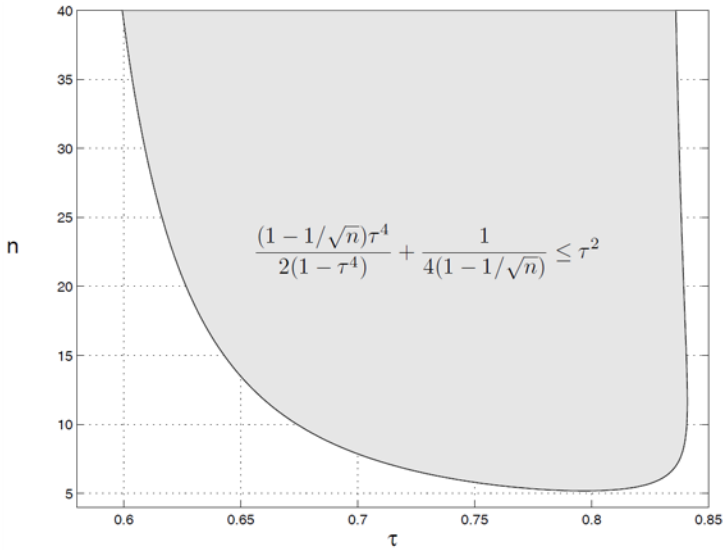

Figure 3. The region defined by (3.2).

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