Advances and Applications in Fluid Mechanics

Published Online: December 2014
Available online at http://pphmj.com/journals/aafm.htm
Volume 17, Number 1, 2015, Pages 17-38 **ISSN: 0973-4686**

A NOTE ON BENJAMIN-FEIR INSTABILITY FOR WATER WAVES

Shahrdad G. Sajjadi

Department of Mathematics
Embry-Riddle Aeronautical University
FL, U. S. A.

and

Trinity College
University of Cambridge
U.K.

Abstract

The phenomenon of disintegration of wave trains, commonly known as the Benjamin-Feir instability, on deep as well as shallow water is reviewed covering both the original perturbation theory of Benjamin and Feir [4] as well as the alternative approach through the nonlinear Schrödinger equation, such as that given by Stuart and DiPrima [17]. Also presented is a short description of higher-order instability of wave trains whose astonishing result is due to the discovery made by Longuet-Higgins [9, 10]. The extension of the perturbation theory to instability of three-dimensional oblique waves [1] is also briefly explained, and a compelling suggestion is given for a possible application of this theory to fully three-dimensional waves on deep water.

Received: May 18, 2014; Accepted: July 19, 2014

2010 Mathematics Subject Classification: 76B07, 76B15, 76D33.

Keywords and phrases: Benjamin-Feir instability, water waves, three-dimensional waves.

1. Introduction

Finite amplitude water waves have been subject of scientific study by mathematicians and physicists for the last 150 years since the pioneering work of Stokes in 1847 [14]. A vast body of mathematical theory has been constructed, and in fact problems of water waves appear to have been responsible for many of the fundamental techniques developed in applied mathematics. The ingenious methods (such as inverse scattering, etc.) of solving exactly certain partial differential equations were first discovered for the shallow water wave approximation, namely Korteweg-de Vries equation, and much of the development arose from the cubic nonlinear Schrödinger equation which describes wave envelops for slow modulation of weakly nonlinear water waves. In recent years, water waves have provided a challenge to computing techniques and were the field of application of some of the first uses in continuum mechanics of ideas like Pade approximation. The mathematical richness of the field may seem surprising at first sight, since the governing differential equation of the classical problem is just Laplace's equation but of course the boundary conditions are nonlinear and this is the source of the wealth of problems. It is remarkable that in classical field as well studied as water waves, new physical phenomena are still being discovered, both theoretically and experimentally, and many open questions remain.

For the particular topic of two-dimensional periodic, irrotational surface waves of permanent form propagating under the effects of gravity on water of infinite and finite depth (Stokes waves), major advances were made over the last forty years by a powerful combination of analytical and numerical techniques. Quite apart from the mathematical questions of the rigorous existence of finite amplitude waves of permanent form (which has attracted and still attracts pure mathematicians), new phenomena of physical interests have been discovered and investigated. For example, instability of steady finite amplitude waves to long wave two-dimensional disturbances was predicted by Lighthill [8] by the use of Whitham's variational method and an approximate Lagrangian. Using Hamiltonian methods, Zakharov [20, 21] showed that weekly nonlinear gravity waves are unstable for modulations longer than a critical wavelength depending upon the wavelength. Both two- and three-dimensional disturbances were considered. Benjamin and Feir [4] examined the case of two-dimensional disturbances to weakly nonlinear waves employing standard perturbation methods and found instability to two-dimensional disturbances of sufficiently long wavelength. Using the numerical results obtained by Schwartz [13] for the shape of finite amplitude Stokes waves, Longuet-Higgins [9, 10] investigated the stability of finite amplitude water waves to superharmonic and subharmonic two-dimensional disturbances. This work extended the results of Zakharov and Benjamin and Feir to finite amplitude waves and disturbances of shorter wavelength. It confirmed Lighthill's prediction that the long wave instability disappears when the wave is steep, and gave values for growth rates that agree well with the observations of Benjamin and Feir (see Benjamin [3] and Lake and Yuen [5]). Longuet-Higgins also discovered that when the wave is sufficiently steep, two-dimensional subharmonic disturbances of twice the wavelength of the undisturbed wave become unstable and have growth rates substantially larger than the type studied by Lighthill, Zakharov and Benjamin and Feir.

Following these and other advances in two-dimensional water waves, interest has been directed to three-dimensional effects, where it might be anticipated that there is an even greater richness of phenomena. The objective of this paper is to establish analytically that three-dimensional progressive waves of finite amplitude on deep water (that is, three-dimensional Stokes waves) are unstable and determine an instability criterion for these types of waves. This proposition, like the Benjamin-Feir instability, implies that in practice, where perturbations from the ideal wave motion are inevitably present, a train of such waves will disintegrate if it travels far enough.

2. Instability of Two-dimensional Waves

Although the detailed analysis of the instability is necessarily complicated, however, the essential features can be explained with reference to Benjamin [3] and Benjamin and Feir [4] for water waves on water of finite depth h. The two-dimensional irrotational motion in an inviscid water with a free surface is governed by the following equations and boundary conditions:

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -h \le z \le 0, \quad -\infty < x < \infty, \tag{2.1}$$

$$\left.\begin{aligned} &\eta_t + \eta_x\phi_x - \phi_z = 0 \\ &g\eta + \phi_t + \frac{1}{2}|\nabla\phi|^2 = 0 \end{aligned}\right\} \text{ on } \quad z = \eta(x, t), \tag{2.2}$$

$$\phi_z = 0 \text{ on } z = -h, \tag{2.3}$$

where $\phi = \phi(x, z, t)$ is the velocity potential, $z = \eta(x, t)$ is the free surface elevation above its mean level at $z = 0$, g is the gravitational acceleration, the horizontal coordinates are (x, y) and the vertical coordinate z is positive upwards, and ∇^2 is the two-dimensional Laplacian.

The nonlinear boundary-value problem described by the preceding equations is known to have exact periodic solutions in the form $\phi = \Phi(x - ct, z)$ and $\eta = F(x - ct)$, where c is a constant phase velocity. The existence of such solutions was established by Struik [16] in shallow water by Levi-Civita [7] in deep water. They proved the convergence of the series expansion for ϕ and η whose leading terms agree with the Stokes [14] expansion. The results are

$$\phi = \Phi = \left(\frac{\omega a}{k}\right)\frac{\cosh k(z + h)}{\sinh kh}\sin\theta + \omega P a^2 \frac{\cosh 2k(z + h)}{\sinh 2kh}\sin 2\theta, \tag{2.4}$$

$$\eta = F = \overline{\eta} + a\cos\theta + ka^2 Q\cos 2\theta, \tag{2.5}$$

where

$$\theta = kx - \omega t, \quad P = \frac{3}{4}\coth kh \operatorname{cosech}^2 kh,$$

$$Q = \frac{1}{4}\coth kh(3\coth^2 kh - 1), \quad \overline{\eta} = -\frac{1}{2}ka^2 \operatorname{cosech} 2kh,$$

and

$$\omega^2 = gk \tanh kh\left[1 + k^2a^2\left\{1 + \operatorname{cosech}^2 kh + \frac{1}{8}(9 - 2\tanh^2 kh)\operatorname{cosech}^4 kh\right\}\right]. \tag{2.6}$$

This approximation is valid provided $ka \ll 1$, or alternatively if $ka \ll (kh)^3$ for small kh.

These results reduce to those of deep water as

$$\phi = \Phi = \left(\frac{\omega a}{k}\right)e^{kz} \sin\theta, \quad \eta = F = a\cos\theta + \frac{1}{2}ka^2\cos 2\theta, \tag{2.7}$$

$$\omega^2 = gk(1 + a^2k^2). \tag{2.8}$$

This describes the steady wave motion on deep water if $ka \ll 1$.

To investigate the stability of the steady wave motion described by (2.4) and (2.5), we write

$$\phi = \Phi + \varepsilon\phi' \quad \text{and} \quad \eta = F + \varepsilon\eta', \tag{2.9}$$

where ε is a small parameter so that the linear equations for ϕ' and η' can be obtained by neglecting the square of ε. The solutions ϕ', η' are assumed to consist of two sideband modes, together with the products of their interaction with the basic wave train, whose fundamental simple harmonic component has amplitude a and phase $\theta = kx - \omega t$. Since the system is nonlinear, harmonics with phases $2\theta, 3\theta, \ldots$ are traveling with the same phase velocity $c = \omega/k$ as the fundamental, and their amplitudes are assumed to decrease in relative order of magnitude like successive integral powers of $ak(\ll 1)$. A disturbance is introduced consisting of a pair of progressive wave modes with *sideband* frequencies and wavenumbers close to ω and k so that their phases are expressed in the form

$$\theta_1 = k(1 + \kappa)x - \omega(1 + \delta) - \gamma_1 \quad \text{(upper sideband)}, \tag{2.10}$$

$$\theta_2 = k(1 - \kappa)x - \omega(1 - \delta) - \gamma_2 \quad \text{(lower sideband)}, \tag{2.11}$$

where κ and δ are small fractions, and the respective amplitudes are ε_1 and ε_2, which are much smaller than a. Two particular products arise from the nonlinear interaction between the disturbance and the basic wave train, and these are the different components produced between the sidebands and the second harmonic. So the components generated have phases

$$2\theta - \theta_1 = \theta_2 + (\gamma_1 + \gamma_2), \quad 2\theta - \theta_2 = \theta_1 + (\gamma_1 + \gamma_2), \tag{2.12}$$

and amplitudes $\varepsilon_1 a^2 k^2$ and $\varepsilon_2 a^2 k^2$, respectively. Thus, if it happens that

$$\gamma = \gamma_1 + \gamma_2 \to \text{constant} \tag{2.13}$$

as the nonlinear processes develop in time, each mode will produce effects that become resonant with the other. Subsequently, if $\gamma \neq 0$ or π, each mode suffers as synchronous forcing effect proportional to the amplitude of the other so that two amplitudes can grow exponentially. Therefore, the basic wave train becomes unstable to this form of disturbance.

After a lengthy but straightforward calculation, Benjamin [3] obtained expansions for η_1' and ϕ_1' and hence found $\eta = \eta_1' + \eta_2'$ and $\phi = \phi_1' + \phi_2'$. Substituting these results into (2.2) leads to four equations with known parameters for the functions $\varepsilon_i(t)$ and $\gamma_i(t)$. They are

$$\frac{d\varepsilon_{1,2}}{dt} = \left[\frac{1}{2}\omega(ak)^2 X(K)\sin\gamma\right]\varepsilon_{2,1} \tag{2.14}$$

and

$$\frac{d\gamma}{dt} = \omega(ak)^2 X(K)\left[1 + \frac{(\varepsilon_1^2 + \varepsilon_2^2)}{2\varepsilon_1\varepsilon_2}\cos\gamma\right] - \omega\delta^2 Y(K), \tag{2.15}$$

where $\gamma = \gamma_1 + \gamma_2$, $K = kh$, and

$$X(K) = \frac{9 - 10\tanh^2 K + 9\tanh^4 K}{8\tanh^4 K} + \frac{4 + 2\,\mathrm{sech}^2 K + 3K\coth K\,\mathrm{sech}^2 K}{1 - 2K\tanh K\,\mathrm{sech}^2 K\cosh 2K + K^2\,\mathrm{sech}^4 K}, \tag{2.16}$$

$$Y(K) = 1 - \frac{8K(1 - K \tanh K)\operatorname{cosech} 2K}{(1 + 2K \operatorname{cosech} 2K)}. \tag{2.17}$$

It is worth noting that the last term in equation (2.15) represents the nonlinear effects possibly opposing the detuning effect of dispersion on the sideband modes.

It is noted that the function $Y(K)$ is proportional to the negative curvature of the dispersion relation according to the linearized theory, and positive for all nonzero values of *K*. Also, $Y(K) \to 0$ as $K \to 0$. For deep water waves, $Y(K) \to 1$ as $K \to \infty$, and hence equations (2.14) and (2.15) assume simple forms that admit explicit solutions.

The solutions of the pair of simultaneous differential equations (2.14) are

$$\varepsilon_{1,2}(t) = \varepsilon_{1,2}(0) \cosh\left[\frac{1}{2}\omega(ak)^2 X \int_0^t \sin \gamma dt\right] + \varepsilon_{2,1}(0) \sinh\left[\frac{1}{2}\omega(ak)^2 X \int_0^t \sin \gamma dt\right]. \tag{2.18}$$

Even though γ is as yet an unknown function of *t*, the amplitudes of the sideband modes undergo unbounded amplification if γ tends to constants other than 0 and π.

According to the argument given by Benjamin and Feir [4], the solutions $\varepsilon_i(t)$ are periodic and finitely bounded if

$$2(ak)^2 X(K) < \delta^2 Y(K). \tag{2.19}$$

This is the required condition of stability of the basic wave trains. In fact, the unbounded amplification of $\varepsilon_i(t)$ is impossible in this case because of equations (2.18) and (2.15), the former suggests that $\varepsilon_1 \to \varepsilon_2$ if $\varepsilon_2 \to \infty$, whereas the latter shows that $d\gamma/dt \to 0$, which is then impossible under the condition (2.19).

On the other hand, if

$$2(ak)^2 X(K) > \delta^2 Y(K), \tag{2.20}$$

then for any initial values, with one exception, solutions of (2.14) and (2.15) have asymptotic properties as $t \to \infty$,

$$\gamma \sim \cos^{-1}\left[\left(\frac{\delta}{ak}\right)^2\left(\frac{Y}{X}\right) - 1\right] \text{ in } (0, \pi), \tag{2.21}$$

$$\varepsilon_1 \sim \varepsilon_2 \sim \exp\left[\frac{1}{2}(ak)^2 X\omega t \sin\gamma\right] = \exp\left[\frac{\delta}{2}\sqrt{Y}(2a^2k^2X - \delta^2Y)^{1/2}\omega t\right]. \tag{2.22}$$

The exceptional case arises when the initial values of ε_1 and ε_2 are equal and

$$\gamma(0) = \cos^{-1}\left[\left(\frac{\delta}{ka}\right)^2 \frac{Y}{X} - 1\right] \text{ in } (\pi, 2\pi);$$

the disturbance decays exponentially. However, this case has no physical significance.

If $2a^2k^2X(K) = \delta^2T(K)$, then ε_1 and ε_2 are found to have unbounded asymptotic growth, but this is linear rather than exponential in time. This is known as *marginal instability*. Thus, including this case, the instability condition becomes

$$2(ak)^2 X(K) \geq \delta^2 Y(K). \tag{2.23}$$

The stability or instability of the basic wave train depends on the sign of $X(K)$. If $X(K) > 0$, then there exists a range of values of δ so that the condition of instability (2.23) is satisfied. If, on the other hand, $X(K) > 0$, then the stability condition (2.20) remains valid for all values of the other parameters. Direct evaluation of $X(K)$ from (2.16) shows that $X(K)$ is positive or negative according to whether $kh > 1.363$ or < 1.363. In the case of infinitely deep water, the condition of instability becomes

$$2(\sqrt{2})ak \geq 2\delta > 0. \tag{2.24}$$

Thus, the Stokes wave trains on infinitely deep water are stable or unstable according to whether the wavelength $\lambda >$ or $< (2\pi/1.363)h = 4.61h.$ These results are in remarkable agreement with Whitham's [19] instability theory. Benjamin and Feir have also given very definite experimental evidence (see Figure 1) in favor of the instability of Stokes waves on deep water. Thus, the Benjamin-Feir instability is well established to affect waves on water of depths with $kh > 1.363$. Also, the theory of McLean [11, 12] and experiments of Su et al. [18] show that even for $kh < 1.363$, instabilities can grow. Nevertheless, by suitable choice of wave steepness and depth of water, it is possible to avoid this situation. It is important to point out that if $kh > 1.363$ there is a cutoff value δ_c or δ given by

$$\delta_c = (ak)\left(\frac{2X}{Y}\right)^{1/2}, \tag{2.25}$$

above which there is no unbounded growth of the sideband amplitudes. As kh is reduced towards 1.363, the unstable range $0 < \delta < \delta_c$ narrows down to a vanishing point. The asymptotic growth rate represented by (2.22) attains a maximum value for $\delta = \delta_c/\sqrt{2}$. The maximum value of the exponent is

$$\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dt}\right)_{\max} = \frac{1}{2}(ak)^2\omega X(K). \tag{2.26}$$

This becomes largest for deep water waves, since $X(K) = 1$, and tends to zero as kh is reduced towards 1.363.

It is also evident from the preceding discussion that Benjamin and Feir's results are only first approximations to the properties of unstable disturbances for small *ak*, and hence for small δ. But they are remarkably significant as the *exact* representations of these properties in the limit $ak \to 0$. As stated earlier, (2.23) is the precise instability condition for the Stokes wave trains of sufficiently small amplitude is an inviscid fluid, if a time $t \gg (\omega a^2 k^2)^{-1} = t_c$ or a distance $x \gg Ct_c = (\lambda/a^2 k^3)$ is allowed for instability to develop. In reality, the effect of viscosity is likely to suppress the instability if *ak* is extremely small, because the viscous dissipation rate is approximately independent of amplitude. However, it is also possible to assume that the mechanisms of viscous dissipation and of energy transfer to the sidebands are essentially independent provided both are weak. Benjamin [3] suggested that the practical condition of instability is

$$\frac{1}{2}(ak)^2\omega X(kh) > \beta, \tag{2.27}$$

where the left-hand side of this condition is the greatest possible rate of amplification for the inviscid fluid model, as given by (2.26), and β is the temporal dissipation rate (equal to the product of the group velocity and the spatial dissipation rate) that is experienced by the waves of sufficiently small amplitude at wavenumber k.

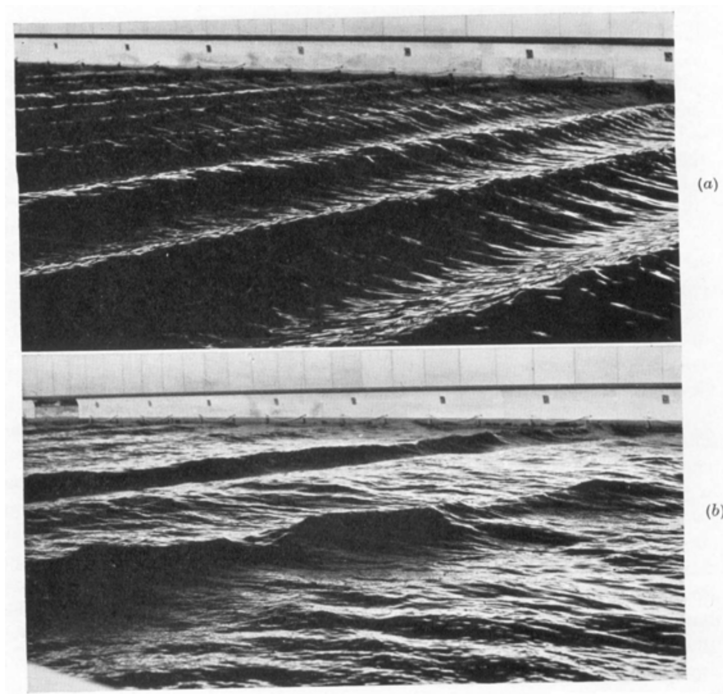

Figure 1. Photographs of a progressive wave train at two stations, showing disintegration due to instability: (a) view near to wavemaker, (b) view at 200 ft away from the wavemaker. Fundamental wavelength, 7.2ft (from Benjamin [3]).

3. Nonlinear Schrödinger Equation Approach

An alternative treatment of the Benjamin-Feir instability mechanism of the two-dimensional Stokes waves on deep water was given by Stuart and DiPrima [17]. They used the method of the nonlinear Schrödinger equation that allows the analysis of the sideband perturbations. The main idea of the Benjamin-Feir mechanism can be described by considering a small-amplitude wave represented by $a\exp[i(kx-\omega t)]$. All the harmonics of this mode will appear because of nonlinear interactions; in particular, the first harmonic is proportional to $a^2\exp[2i(kx-\omega t)]$. We assume two modal perturbations represented by $a_1\exp[i(k_1x-\omega t)]$ (the upper sideband), and $a_2\exp[i(k_2x-\omega_2 t)]$ (the lower sideband), where the amplitudes a_1 and a_2 are smaller than a. It is noted that the nonlinear interaction between the first harmonic and the upper sideband generates

$$a^2a_1\exp[i\{(2k-k_1)x-(2\omega-\omega_1)t\}], \tag{3.1}$$

together with another component not stated explicitly. Simultaneously, the nonlinear interaction between the first harmonic and the lower sideband produces

$$a^2a_2\exp[i\{(2k-k_2)x-(2\omega-\omega_1)t\}], \tag{3.2}$$

along with another component not stated explicitly. If we write

$$k_1=k_2=2k \quad \text{and} \quad \omega_1+\omega_2=2\omega, \tag{3.3}$$

the preceding results give

$$a^2a_1\exp[i(k_2x-\omega_2 t)] \tag{3.4}$$

and

$$a^2a_2\exp[i(k_1x-\omega_1 t)]. \tag{3.5}$$

Clearly, the former is proportional to the lower sideband, whereas the latter is proportional to the upper sideband. The simultaneous presence of the upper and lower sidebands combined with the first harmonic is likely to produce a resonant phenomenon. The exponential growth in time originating from this synchronous resonance leads to the instability of the Stokes waves. Thus, the resonance phenomenon must satisfy the two conditions (3.3), which must be consistent with the associated dispersion relation. For Stokes waves on deep water, the dispersion relation is $\omega^2 = gk$. If k_1, k_2 are close to k and ω_1, ω_2 are also close to ω, and if we write

$$k_1 = k(1+\kappa), \quad \omega_1 = \omega(1+\delta), \tag{3.6}$$

$$k_2 = k(1+\kappa), \quad \omega_2 = \omega(1+\delta), \tag{3.7}$$

then

$$\omega_1 = \sqrt{gk_1} = \sqrt{gk}\left(1+\frac{1}{2}\kappa\right), \quad \omega_2 = \sqrt{gk_2} - \sqrt{gk}\left(1-\frac{1}{2}\kappa\right) \tag{3.8}$$

implies $\delta = \frac{1}{2}\kappa$ to the first order in wavenumber and frequency perturbations.

It is well known that one simple solution of the nonlinear Schrödinger equation (2.26) depending on time only is

$$A(t) = A_0 \exp\left[-\frac{i}{2}\omega_0 k_0^2 A_0^2 t\right], \tag{3.9}$$

where A_0 is a constant. This solution essentially represents the fundamental components of the Stokes wave. We consider a perturbation of (3.9) and express it in the form

$$a(x,t) = A(t)[1+B(x,t)]. \tag{3.10}$$

Substituting this result into (2.26) yields

$$i(1+B)A_t + iAB_t - \left(\frac{\omega_0}{8k_0^2}\right)AB_{xx}$$
$$= \frac{1}{2}\omega_0 k_0^2 A_0^2[(1+B) + BB^*(1+B) + (B+B^*)B + (B+B^*)]A, \tag{3.11}$$

where B^* is the complex conjugate of B. Ignoring squares of B, we obtain

$$iB_t - \left(\frac{\omega_0}{8k_0^2}\right)B_{xx} = \frac{1}{2}\,\omega_0 k_0^2 A_0^2 (B + B^*). \tag{3.12}$$

We next seek a solution for B in the form

$$B(x,\, t) = B_1 \exp(\Omega t + ilx) + B_2 \exp(\Omega^* t - ilx), \tag{3.13}$$

where B_1 and B_2 are complex constants, l is a real wavenumber, and Ω is a growth rate (possibly complex) to be determined. Substituting the solution for B into (3.12) yields a pair of coupled equations,

$$\left(i\Omega + \frac{\omega_0 l^2}{8k_0^2}\right)B_1 - \frac{1}{2}\,\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0, \tag{3.14}$$

$$\left(i\Omega^* + \frac{\omega_0 l^2}{8k_0^2}\right)B_2 - \frac{1}{2}\,\omega_0 k_0^2 A_0^2 (B_1^* + B_2) = 0. \tag{3.15}$$

Taking the complex conjugate of (3.15) gives

$$\left(-i\Omega + \frac{\omega_0 l^2}{8k_0^2}\right)B_2^* - \frac{1}{2}\,\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0. \tag{3.16}$$

The pair of linear homogeneous equations (3.14) and (3.16) for B_1 and B_2^* admits a nontrivial eigenvalue for Ω, provided

$$\begin{vmatrix} i\Omega + \dfrac{\omega_0 l^2}{8k_0^2} - \dfrac{1}{2}\,\omega_0 k_0^2 A_0^2 & -\dfrac{1}{2}\,\omega_0 k_0^2 A_0^2 \\ -\dfrac{1}{2}\,\omega_0 k_0^2 A_0^2 & -i\Omega + \dfrac{\omega_0 l^2}{8k_0^2} - \dfrac{1}{2}\,\omega_0 k_0^2 A_0^2 \end{vmatrix} = 0, \tag{3.17}$$

which is equivalent to

$$\Omega^2 = \frac{1}{2}\left(\frac{\omega_0 l}{2k_0}\right)^2\left(k_0^2 A_0^2 - \frac{l^2}{8k_0^2}\right). \tag{3.18}$$

The growth rate Ω is real (and positive) and purely imaginary depending on whether $l^2 < 8k_0^2 A_0^2$ on $l^2 > 8k_0^4 A_0^2$. The former case represents a wave solution for B, and the latter case corresponds to the Benjamin-Feir instability with criteria in terms of the nondimensional wavenumbers $\tilde{l} = (l/k_0)$ as

$$\tilde{l}^2 < 8k_0^2 A_0^2. \tag{3.19}$$

Thus, the range of instability is

$$0 < \tilde{l} < \tilde{l}_c = 2\sqrt{2}k_0 A_0. \tag{3.20}$$

Since Ω is a function of $\tilde{l}$, the maximum instability occurs at $\tilde{l} = \tilde{l}_{\max} = 2k_0 A_0$, with the maximum growth rate given by

$$\Re\{\Omega_{\max}\} = \frac{1}{2}\omega_0 K_0^2 A_0^2, \tag{3.21}$$

where $\Re$ implies the real part.

In order to establish the connection with Benjamin-Feir instability described earlier, we need the velocity potential for the fundamental wave mode multiplied by $\exp(ks)$. It turns out that the term proportional to B_1 is the upper sideband, while that proportional to B_2 is the lower sideband. Hence, the equivalent of the Benjamin result $(\kappa = 2\delta)$ is satisfied. The main conclusion of the preceding analysis is that the Stokes water waves are definitely unstable.

In summary, the theoretical prediction of the Benjamin-Feir sideband instability is that of a *breakup* of a nonlinear wave, with the energy eventually spread over a number of small-amplitude waves. With the identification of the soliton, it has been conjectured that the final state of an unstable Stokes wave would be one or more solitons. These predictions were found to be in remarkable agreement with their experimental results, which indicated that the monochromatical gravity waves produced in the experiment transfer energy to a wavenumber adjacent to that of the carrier wave by the nonlinear interactions. Benjamin and Feir observed what was apparently a disintegration of a mechanically produced finite-amplitude deep water wave. They interpreted this disintegration (or breakup) as being due to a sideband instability of the finite-amplitude wave. Based upon the linear stability analysis, they also described the mechanism as a resonant coupling between the primary wave train and a pair of wave modes at sideband frequencies and wavenumber fractionally different from the fundamental frequency and wavenumber. In consequence of coupling, the energy is then transferred from the primary wave to the sidebands at a rate that can increase exponentially in time or distance as the nonlinear interactions develop. The general conclusion is that finite-amplitude water waves are unstable.

4. Higher-order Instability of Wave Trains

Longuet-Higgins [9, 10] reported a new type of instability for finite-amplitude gravity waves. Its discovery came from computations based on Stokes' [15] conformal mapping representation of two-dimensional gravity waves in deep water. Longuet-Higgins showed at small wave steepness ka there is no supercritical of the discrete wavenumbers. However, as ka was increased be found supercritical instability among wavenumbers k_1, k_2 satisfying

$$k_1 + k_2 = 2$$

the fundamental wavenumber and frequency being normalized to unity. The corresponding frequency condition for four-wave resonance is

$$\omega_1 + \omega_2 = 2$$

but this cannot be satisfied as $ka \to 0$ for Longuet-Higgins' chosen wavenumbers. Onset of supercritical instability, as ka increases, is due to nonlinear modification of the frequencies ω_1, ω_2 which causes the resonance condition to be nearly satisfied over a finite range of ka. For large and small ka, the wave is "detuned" and no instability takes place. The domain of supercritical instability of two-dimensional disturbances is shown in Figure 2. The most unstable supercritical instability, at given ka, is two-dimensional.

At large values of ka, not far short of that for the highest wave, Longuet-Higgins encountered subcritical instability of the mode with $k_1 = \frac{3}{2}$. Owing to nonlinearity, this attains the same phase speed as the fundamental (i.e., $2k_1 = 3$, $2\omega_1 = 3$) when $ka \approx 0.41$: this is the degenerate five-wave resonance condition. These results clearly demonstrate the role of nonlinearity in turning and detuning resonance.

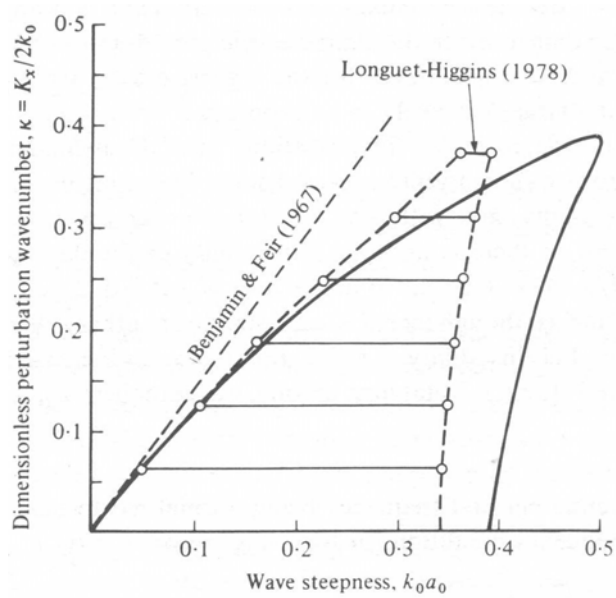

Figure 2. Stability boundary for growth of two-dimensional perturbations of uniform finite-amplitude gravity waves: ∘ results from Longuet-Higgins [10]; the solid line is approximation derived by Zakharov's equation.

5. Three-dimensional Benjamin-Feir Instability

Following advances in two-dimensional water waves (noted in Sections 2 and 3), interest has been directed to three-dimensional effects, where it might be anticipated that there is an even greater richness of phenomena. The objective here is to establish analytically that three-dimensional progressive waves of finite amplitude (that is, three-dimensional Stokes waves, such as that shown in Figure 3) on water of finite depths are unstable and determine an instability criterion for these types of waves. This proposition, like the Benjamin-Feir instability, implies that in practice, where perturbations from the ideal wave motion are inevitably present, a train of such waves will disintegrate if it travels far enough.

Consider the various simple-harmonic modes present in a slightly disturbed three-dimensional wave motion whose profile is given by

$$\eta = a[\cos\zeta^+ + \cos\zeta^-] + \frac{1}{2}ka^2[\cos 2\zeta^+ + \cos 2\zeta^-], \tag{5.1}$$

where $\zeta^+ = \ell x + my - \omega t,\ \zeta^- = Lx - my - \omega t,\ \ell = k\cos\theta,\ m = k\sin\theta$ and $L = k\sqrt{2-\cos^2\theta}$ (the case $\theta = 0$ is the limit of two-dimensional progressive wave). We have first, for the basic wave train, the fundamental component with amplitude a and argument $\zeta^\pm$, and harmonics with arguments $2\zeta^\pm$, $3\zeta^\pm$, ..., which all advance in the horizontal x- and y-directions with the phase velocity c. For all disturbances, we take a pair of progressive-wave modes which have 'sideband' frequencies and wavenumbers adjacent to $\omega,\ \ell,\ m$ and L, so that their arguments may be expressed as

$$\left.\begin{aligned}
\zeta_1^+ &= \ell(1+\lambda)x + m(1+\mu)y - \omega(1+\delta)t - \gamma_1^+ \\
\zeta_1^- &= L(1+\Lambda)x - m(1+\mu)y - \omega(1+\delta)t - \gamma_1^- \\
\zeta_2^+ &= \ell(1-\lambda)x + m(1-\mu)y - \omega(1-\delta)t - \gamma_2^+ \\
\zeta_2^- &= L(1-\Lambda)x - m(1-\mu)y - \omega(1-\delta)t - \gamma_2^-
\end{aligned}\right\}, \tag{5.2}$$

where λ, μ, δ and Λ are small fractions. The representative amplitudes are denoted by $\varepsilon_1,\ \varepsilon_2$ and are assumed to be much smaller than a. Now, among the products of the nonlinear interactions between these disturbance modes and the basic wave train, there will be components with arguments

$$2\zeta^\pm - \zeta_1^\pm = \zeta_2^\pm + (\gamma_1^\pm + \gamma_2^\pm),$$

$$2\zeta^\pm - \zeta_2^\pm = \zeta_1^\pm + (\gamma_1^\pm + \gamma_2^\pm)$$

and with amplitudes proportional to $a^2\varepsilon_1$ and $a^2\varepsilon_2$, respectively. Thus, if it happens that

$$\phi^{\pm} = \gamma_1^{\pm} + \gamma_2^{\pm} \to \text{constant} \tag{5.3}$$

as the nonlinear processes develop in time, each mode with generate effects that become resonant with the other. Therefore, if $\phi^{\pm} \neq 0, \pi,$ each mode suffers a synchronous forcing action proportional to the amplitude of the other, so that each can grow mutually at an exponential rate.

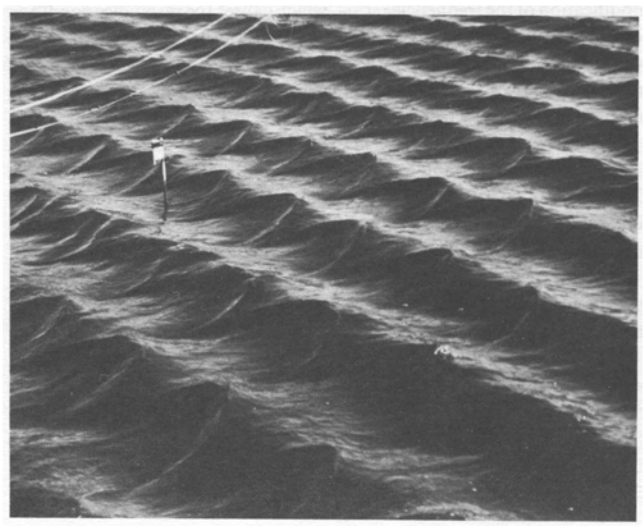

Figure 3. Three-dimensional wave configuration resulting from oblique subcritical instability, for $ak = 0.33$ (from Su et. al. [18]).

The crucial task of the analysis is to show that, for a given specification a, ℓ, m, L, ω of the basic wave train, the property (5.3) is possible for some nonzero λ, μ, Λ and δ. This property would be impossible in the absence of the basic wave train, or if the amplitude a was too small for there to be significant nonlinear coupling with the sideband modes, because of the *net* frequencies $\omega_{1,2}^2$ and wavenumbers $k_{1,2}^{\pm}$ (where $k^+ = \sqrt{\ell^2 + m^2}$ and $k^- = \sqrt{L^2 - m^2}$) of these modes would have to obey the dispersion relation

$$\omega_{1,2}^2 = gk_{1,2}^{\pm} \tag{5.4}$$

given by linearized theory (Lamb [6, Section 229]). Letting $k_{1,2}^{\pm} = k^{\pm}(1 \pm \kappa)$, we see that the frequencies $\omega(1 \pm \delta)$ appearing explicitly in (5.2) cannot both satisfy (5.4), and the discrepancy must be accommodated by allowing $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ to be slowly-varying functions of time. If we put $\delta = \frac{1}{2}\kappa$, which means that $\delta\omega/\kappa k^{\pm} = \frac{1}{2}c^{\pm} = \frac{1}{2}(g/k^{\pm})^{1/2}$ is the group velocity for an infinitesimal wave with wavenumbers $k^{\pm}$, then (5.4) is satisfied to a first approximation for small δ even if $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ are constants; but to a second approximation (5.4) requires that

$$\frac{d\phi^{\pm}}{dt} = -\omega\delta^2.$$

Thus the effect of dispersion, in so far as it may be independent of nonlinear effects, is to detune the prospective resonance between second-harmonic components of the basic wave motion and the sideband modes.

Despite the ground in common with previous studies of Benjamin-Feir instability (all for two-dimensional waves), the presentation of a new treatment from first principles is judged desirable. There are several essential results in the present extension to three-dimensional waves that are not readily accessible from existing analyses: in particular, we need to find that the actual asymptotic value of the phase function $\phi^{\pm}$ in order to predict the ultimate rate of amplification of an unstable three-dimensional disturbance on water of infinite depth.

Thus, we need to examine the stability of three-dimensional wave motion, described by (5.1), by first constructing a velocity potential φ. Therefore, we assume

$$\eta = H + \sigma\tilde{\eta}, \quad \varphi = \Phi + \sigma\tilde{\varphi} \tag{5.5}$$

and derive linearized equations for $\tilde{\eta}$ and $\tilde{\varphi}$. In (5.5), σ is small number whose square can be neglected.

On the basis of the ideas explained above, the perturbation is assumed to consist of pair of sideband modes, together with the products of their interaction with the primary wave train. Then by expressing

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2, \quad \tilde{\varphi} = \tilde{\varphi}_1 + \tilde{\varphi}_2,$$

we may construct solutions for $\tilde{\eta}_i$ and $\tilde{\varphi}_i$ $(i = 1, 2)$.

These solutions are then substituted in (5.5) and into the dynamic and kinematic boundary conditions. After linearizing in σ we reduce all terms to simple-harmonic components. Next, by separating the coefficients proportional to $a\varepsilon_i$ we derive pairs of simultaneous, coupled ordinary differential equations for ε_i and ϕ as a function of time. Finally, from the solution obtained for ε_i we derive stability/instability criterion for three-dimensional waves on water of infinite depth.

Ross and Sajjadi [1] considered three-dimensional irrotational motion in an infinitely deep water, where a periodic wave train with fundamental frequency ω moves obliquely, with α representing direction measured with respect to the positive x-axis. They assumed harmonics have phases 2ζ, 3ζ, ..., where $\zeta = lx + my - \omega t$. They then introduced small disturbances in the form of sidebands with frequencies $\omega(1 \pm \delta)$ and, where over time, energy is transferred from the primary wave to the sidebands. Using methods introduced by Benjamin and Feir for two-dimensional motion (described above), they showed that the magnification will become unbounded if the sideband frequencies meet the condition

$$0 < \delta \le \frac{a}{\sqrt{2}}(l \sec\alpha + m \csc\alpha),$$

where a denotes the fundamental amplitude of the perturbed wave train. The limiting cases $\alpha \to 0$ yields the two-dimensional Benjamin-Feir instability. Indeed, if we put $k = \sqrt{l^2 + m^2}$, then this condition for instability may be expressed as

$$0 < \delta \leq \sqrt{2}ak,$$

in formal agreement with the two-dimensional theory. On the other hand, the limit as $\alpha \to \frac{1}{2}\pi,$ where the waves are propagating at an angle $\frac{1}{2}\pi,$ the instability represents standing waves such as those studied by Penny and Price [2].

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