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EXISTENCE OF EXTREMAL SOLUTIONS OF PBVP FOR FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider the existence of solutions of periodic boundary value problems (PBVP) for first order differential equations. By using Mönch and Von Harten inequality, we prove the existence of extremal solutions of (PBVP) in any of four groups different conditions in Banach spaces.

1. Introduction and Preliminaries

Let E be a real Banach space with $\|\cdot\|$ and E^* denote the dual space of E. Let α be the Kuratowski's measure of noncompactness on E. The definition of α is as follows:

$$\alpha(B) = \inf\{\varepsilon > 0 : B \text{ is covered by a finite number of sets with diameter} \leq \varepsilon\},$$

where $B \subset E$ bounded. For its properties see [1, 4].

We list the following definitions and lemmas for convenience.

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Definition 1.1. Let K be a subset of E. K is said to be a *cone* if K is a closed convex subset such that $\lambda \cdot K \subset K$ for any $\lambda \geq 0$ and $K \bigcap (-K) = \{\Theta\}$. We denote $K^* = \{\varphi \in E^* : \varphi(u) \geq 0 \text{ for all } u \in K\}$.

Definition 1.2. Let K^0 be the inner part of K and $K^0 \neq \Phi$. By means of K the partial orders $\leq$ and $<$ are defined as

$$x \leq y \text{ iff } y - x \in K,$$

$$x < y \text{ iff } y - x \in K^0.$$

Definition 1.3. A cone K is said to be *normal* if there exists a real number $N > 0$ such that $0 \leq v \leq u$ implies $\| v \| \leq N \| u \|$, where N is independent of u, v.

We shall always assume in this paper that K is a normal cone and K^0 is nonempty.

Lemma 1.1 (see [1]). *Let K be a cone. Then $x \in K$ iff $\varphi(x) \geq 0$ for all $\varphi \in K^*$.*

Lemma 1.2 (see [1]). *Let $f \in C[[a, b], E]$ and there is an at most countable subset Γ of $[a, b]$ such that f'_+ exists and $\| f'_+ \| \leq M$ for all $t \in [a, b] \setminus \Gamma$ (where $M > 0$). Then $\| f(t_1) - f(t_2) \| \leq M | t_1 - t_2 |$ for any $t_1, t_2 \in [a, b]$.*

Lemma 1.3 (see [3]). *Let $\{x_n\}$ be a sequence of continuously differentiable functions from $J = [a, b]$ to E such that there is some $\mu \in L^1(a, b)$ with $\| x_n(t) \| \leq \mu(t)$ and $\| x'_n(t) \| \leq \mu(t)$ on J. Let $\psi(t) = \alpha(\{x_n(t) : n \geq 0\})$. Then ψ is absolutely continuous and*

$$\psi'(t) \leq 2\alpha(\{x'_n(t) : n \geq 0\}) \text{ a.e. on } J.$$

2. Main Results

We shall consider the following periodic boundary value problem

(PBVP) for first order differential equations

$$u' = f(t, u), u(0) = u(2\pi), \tag{2.1}$$

where $f \in C[[0, 2\pi] \times E, E]$.

The functions $\alpha_0, \beta_0 \in C^1[[0, 2\pi], E]$ are said to be a *lower solution* and an *upper solution* of (2.1), respectively, if

$$\alpha_0' \leq f(t, \alpha_0), \alpha_0(0) \leq \alpha_0(2\pi);$$

$$\beta_0' \geq f(t, \beta_0), \beta_0(0) \geq \beta_0(2\pi).$$

For any $\alpha_0, \beta_0 \in C[[0, 2\pi], E]$ such that $\alpha_0(t) \leq \beta_0(t)$ on $[0, 2\pi]$, we define

$$[\alpha_0, \beta_0] = \{\eta \in C[[0, 2\pi], E] : \alpha_0(t) \leq \eta(t) \leq \beta_0(t) \text{ on } [0, 2\pi]\}.$$

In this paper, we shall prove the following theorem.

Theorem 2.1. *Assume that*

(A_0) (i) $v_0, w_0 \in C^1[[0, 2\pi], E], w_0 \leq v_0$ *and*

$$f(t, u) - f(t, \bar{u}) \leq M(u - \bar{u}), \forall u, \bar{u} \in [w_0, v_0], u \geq \bar{u},$$

$$\alpha(f(t, B)) \leq L\alpha(B), \forall B \subset D,$$

where $D = \{u \in E : w_0(t) \leq u \leq v_0(t), t \in [0, 2\pi]\}$ *and* $M > 0, L > 0$.

(ii) *if* $u_n \in [w_0, v_0], u_n(0) = u_n(2\pi)(n = 1, 2, ...), \{u_n\}$ *is equicontinuous and uniformly bounded and* u_n *is absolutely continuous for each n, then* $\{u_n(0)\}$ *is relative compact.*

(A_1) (i) $v_0' \leq f(t, v_0)(t \in [0, 2\pi]), v_0(0) \leq v_0(2\pi);$

(ii) $w_0' \geq f(t, w_0)(t \in [0, 2\pi]), w_0(0) \geq w_0(2\pi).$

(A_2) (i) $v_0' \leq f(t, v_0) - M\gamma_1, (t \in [0, 2\pi]), v_0(0) < v_0(2\pi),$

where $M > 0, \gamma_1 = \dfrac{e^{-2M\pi}}{1 - e^{-2M\pi}}(v_0(2\pi) - v_0(0));$

(ii) $w_0' \geq f(t, w_0) + M\gamma_2$, $(t \in [0, 2\pi])$, $w_0(0) > w_0(2\pi)$,

where $M > 0$, $\gamma_2 = \dfrac{e^{-2M\pi}}{1 - e^{-2M\pi}}(w_0(0) - w_0(2\pi))$.

(A_3) (A_1)(i) *and* (A_2)(ii).

(A_4) (A_1)(ii) *and* (A_2)(i).

Then (A_0) *holds and any one of the conditions* $(A_1), (A_2), (A_3), (A_4)$ *implies that there exist monotone sequences* $\{v_n\}, \{w_n\}$ *such that* $v_n \to r$, $w_n \to \rho$ *as* $n \to \infty$ *uniformly and monotonically on* $[0, 2\pi]$ *and that* ρ, r *are minimal and maximal solutions of* (2.1), *respectively*, (*in* $[w_0, v_0]$).

The proof of the theorem will be completed by a series of lemmas.

Lemma 2.1. *Let* $m \in C^1[[0, 2\pi], E]$. *Then*

(i) $m'(t) \leq Mm(t)$ *for* $t \in [0, 2\pi]$, *where* $M > 0$ *and* $m(0) \leq m(2\pi)$, *implies* $m(t) \geq 0$ *on* $[0, 2\pi]$.

(ii) $m'(t) \leq Mm(t) - M\gamma$ *for* $t \in [0, 2\pi]$, *where* $M > 0$ *and* $\gamma = \dfrac{e^{-2M\pi}}{1 - e^{-2M\pi}}(m(2\pi) - m(0))$ *and* $m(0) < m(2\pi)$, *implies* $m(t) \geq 0$ *on* $[0, 2\pi]$.

Proof. If the conclusion is false, then there exist a $\varphi \in K^*$ and a $t_0 \in [0, 2\pi]$ and $\varepsilon > 0$ such that

$$\varphi(m(t_0)) = -\varepsilon, \ \varphi(m(t)) \geq -\varepsilon \ (\forall t \in [0, 2\pi]). \tag{2.2}$$

In fact, if $m(t) \geq 0$ is not true for all $t \in [0, 2\pi]$, then there exists at least a $t_1 \in [0, 2\pi]$ such that $m(t_1) \notin K$. By Lemma 1.1, there is a $\varphi \in K^*$ such that $\varphi(m(t_1)) < 0$. Let $\psi(t) = \varphi(m(t))$ on $[0, 2\pi]$. Then $\psi(t)$ has a minimal value $-\varepsilon(\varepsilon > 0)$, that is, (2.2) holds.

(1) Let (i) hold.

If $t_0 \in [0, 2\pi)$, then $\varphi(m(t)) - \varphi(m(t_0)) \geq 0$ by (2.2). Let $t \to t_0+$, we have $\varphi(m'(t_0)) \geq 0$. By (i),

$$\varphi(m'(t_0)) \leq \varphi(Mm(t_0)) = M\varphi(m(t_0)) = -M\varepsilon < 0,$$

which is a contradiction.

If $t_0 = 2\pi$, then we have $\varphi(m(2\pi)) = -\varepsilon$. Also $m(0) \leq m(2\pi)$, so $\varphi(m(0)) \leq \varphi(m(2\pi)) = -\varepsilon$. By (2.2), $\varphi(m(0)) \geq -\varepsilon$, therefore $\varphi(m(0)) = -\varepsilon$. We lead to the same contradiction if replacing t_0 with 0 in the above proof.

(2) Let (ii) hold.

If $t_0 \in [0, 2\pi)$, then $\varphi(m'(t_0)) \geq 0$. Also

$$m'(t_0) \leq Mm(t_0) - M\gamma,$$

so

$$\varphi(m'(t_0)) \leq M\varphi(m(t_0)) - M\varphi(\gamma).$$

Notice that $\gamma > 0$, $\varphi \in K^*$, so $\varphi(\gamma) \geq 0$. This leads to $\varphi(m'(t_0)) < 0$, which is a contradiction.

If $t_0 = 2\pi$, then $\varphi(m(2\pi)) = -\varepsilon$. By (ii),

$$[\varphi(m'(t)) - M\varphi(m(t))]e^{-Mt} \leq -M\varphi(\gamma)e^{-Mt},$$

which, on integration from 0 to 2π, gives

$$\varphi(m(2\pi))e^{-2M\pi} - \varphi(m(0)) \leq \varphi(\gamma)(e^{-2M\pi} - 1).$$

Notice that

$$\varphi(\gamma)(e^{-2M\pi} - 1) = -e^{-2M\pi}(\varphi(m(2\pi)) - \varphi(m(0))),$$

hence

$$\varphi(m(0)) \geq \frac{2e^{-2M\pi}}{1 + e^{-2M\pi}}\varphi(m(2\pi)) = -\frac{2e^{-2M\pi}}{1 + e^{-2M\pi}}\varepsilon > -\varepsilon,$$

which contradicts to $\varphi(m(0)) < \varphi(m(2\pi)) = -\varepsilon$. Therefore, the lemma is true.

Lemma 2.2. *For any* $\eta \in [w_0, v_0]$, *then the linear PBVP*

$$u' = G(t, u), \; u(0) = u(2\pi) \tag{2.3}$$

has a unique solution on $[0, 2\pi]$, *where* $G(t, u) = f(t, \eta(t)) + M(u - \eta(t))$.

Proof. This linear problem can be explicitly solved and a solution of (2.3) is given by

$$u(t) = e^{Mt}\left[u(0) + \int_0^t [f(s, \eta(s)) - M\eta(s)]e^{-Ms}ds\right],$$

where

$$u(0) = u(2\pi) = \frac{1}{e^{-2M\pi} - 1}\int_0^{2\pi} [f(s, \eta(s)) - M\eta(s)]e^{-Ms}ds.$$

We shall show that the solution $u(t)$ of (2.3) is unique. If not, let $u_1(t)$, $u_2(t)$ be two solutions of (2.3). Set $m(t) = u_1(t) - u_2(t)$. Then

$$m'(t) = u_1'(t) - u_2'(t) = M(u_1(t) - u_2(t)) = Mm(t), \; m(0) = m(2\pi).$$

Now using Lemma 2.1(i), it follows that $m(t) \geq 0$, i.e., $u_1(t) - u_2(t) \in K$. By a similar argument we can conclude $u_2(t) - u_1(t) \in K$, i.e., $u_1(t) - u_2(t) \in (-K)$. So $u_1(t) - u_2(t) \in K \cap (-K) = \{\Theta\}$. Therefore, $u_1(t) \equiv u_2(t)$ on $[0, 2\pi]$. The proof is complete.

By Lemma 2.2, we now define a mapping A:

$$A\eta = u, \; \forall \eta \in [w_0, v_0],$$

where u is the unique solution of (2.3). The A possesses the following properties.

Lemma 2.3. *Assume that the conditions of Theorem* 2.1 *are satisfied. Then*

(i) $w_0 \leq Aw_0$, $v_0 \geq Av_0$;

(ii) *A possesses a monotone increasing property on the segment* $[w_0, v_0]$.

Proof. Step 1. Let (A_0), (A_1) hold.

(1) $v_0 \geq Av_0$.

Set $v_1 = Av_0$, $p(t) = v_0(t) - v_1(t)$. Then

$$p'(t) = v_0'(t) - v_1'(t) \leq f(t, v_0(t)) - [f(t, v_0(t)) + M(v_1(t) - v_0(t))]$$
$$= Mp(t),$$

$p(0) = v_0(0) - v_1(0) \leq v_0(2\pi) - v_1(2\pi) = p(2\pi)$. So $p(t) \geq 0$ on $[0, 2\pi]$ by Lemma 2.1(i). This proves $v_0 \geq Av_0$.

(2) $w_0 \leq Aw_0$.

Similarly we can prove it.

(3) Let $\eta_1, \eta_2 \in [w_0, v_0]$ and $\eta_1 \leq \eta_2$. Then $A\eta_1 \leq A\eta_2$.

In fact, setting $u_i = A\eta_i (i = 1, 2)$, $p(t) = u_2(t) - u_1(t)$, we see that, using the conditions $(A_0)(i)$ in Theorem 2.1,

$$p' = [f(t, \eta_2) + M(u_2 - \eta_2)] - [f(t, \eta_1) + M(u_1 - \eta_1)]$$
$$= f(t, \eta_2) - f(t, \eta_1) + M(u_2 - u_1) - M(\eta_2 - \eta_1)$$
$$\leq M(\eta_2 - \eta_1) + M(u_2 - u_1) - M(\eta_2 - \eta_1)$$
$$= Mp,$$

and $p(0) = p(2\pi)$. Consequently Lemma 2.1(i) gives $u_1 \leq u_2$ on $[0, 2\pi]$.

Step 2. Let (A_0), (A_2) hold.

(1) $v_0 \geq Av_0$.

Set $v_1 = Av_0$, $p = v_0 - v_1$. Then

$$p' = v_0' - v_1'$$
$$\leq f(t, v_0) - M\gamma_1 - [f(t, v_0) + M(v_1 - v_0)]$$
$$= Mp - M\gamma_1$$

and

$$p(0) = v_0(0) - v_1(0) < v_0(2\pi) - v_1(2\pi) = p(2\pi),$$

$$p(2\pi) - p(0) = [v_0(2\pi) - v_1(2\pi)] - [v_0(0) - v_1(0)]$$

$$= v_0(2\pi) - v_0(0) (\text{since } v_1(2\pi) = v_1(0)).$$

So

$$\gamma_1 = \frac{e^{-2M\pi}}{1 - e^{-2M\pi}} (p(2\pi) - p(0)).$$

Lemma 2.1(ii) gives $p \geq 0$, i.e., $v_0 \geq v_1$ on $[0, 2\pi]$.

(2) $w_0 \leq Aw_0$.

Similarly we can prove it.

(3) Similarly to Step 1 (3), we can prove that A possesses a monotone increasing property on $[w_0, v_0]$.

Step 3. Let $(A_0), (A_3)$ hold.

By Step 1 (1), (3) and Step 2 (2), the Lemma's conclusion holds.

Step 4. Let $(A_0), (A_4)$ hold.

By Step 1 (2), (3) and Step 2 (1), the Lemma's conclusion holds.

Lemma 2.4. *Assume that the conditions of Theorem* 2.1 *hold*. *We define the sequences*:

$$v_{n+1} = Av_n, \; w_{n+1} = Aw_n, \quad n = 0, 1, 2, 3, \ldots.$$

Then $\{v_n\}, \{w_n\}$ *are monotone, uniformly bounded and equicontinuous and* v_n, w_n *are absolutely continuous for each n*.

Proof. By using Lemma 2.3, we can conclude

$$w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0 \text{ on } [0, 2\pi].$$

Since the cone K is normal and $\| v_0(t) \|, \| w_0(t) \|$ are bounded on $[0, 2\pi]$, it follows that $\{v_n\}, \{w_n\}$ are uniformly bounded on $[0, 2\pi]$.

By assumption (A_0), we have

$$f(t, v_0) - f(t, v_n) \leq M(v_0 - v_n),$$

$$f(t, v_n) - f(t, w_0) \leq M(v_n - w_0).$$

So

$$f(t, v_0) - M(v_0 - v_n) \leq f(t, v_n) \leq f(t, w_0) + M(v_n - w_0).$$

Since K is normal and $\{v_n\}$ is uniformly bounded on $[0, 2\pi]$, we have $\{f(t, v_n)\}$ is uniformly bounded on $[0, 2\pi]$. Notice that

$$v'_{n+1} = f(t, v_n) + M(v_{n+1} - v_n),$$

we conclude that $\{v'_n\}$ is uniformly bounded on $[0, 2\pi]$. Let $\| v'_n(t) \| \leq M_0$, $n = 1, 2, 3, \ldots, t \in [0, 2\pi]$ (where $M_0 > 0$). By using Lemma 1.2, we have

$$\| v_n(t_1) - v_n(t_2) \| \leq M_0 | t_1 - t_2 |, \quad \forall t_1, t_2 \in [0, 2\pi].$$

So $\{v_n(t)\}$ is equicontinuous and absolutely continuous on $[0, 2\pi]$. Similarly we can prove that $\{w_n\}$ possesses similar properties.

Lemma 2.5. *The sequences* $\{v_n\}, \{w_n\}$ *defined in Lemma* 2.4 *is relatively compact in* $C[[0, 2\pi], E]$.

Proof. By Lemma 2.4, $\{v_n\}$, $\{w_n\}$ are uniformly bounded and equicontinuous. So, by using Ascoli-Arzela theorem, we only prove

$$\alpha(\{v_n(t) : n \geq 0\}) = \alpha(\{w_n(t) : n \geq 0\}) = 0, \quad \forall t \in [0, 2\pi].$$

Setting $\varphi(t) = \alpha(\{v_n(t) : n \geq 0\})$. Obviously $\{v_n(t)\}$ satisfies Lemma 1.3, so

$$\begin{aligned} \varphi'(t) &\leq 2\alpha(\{v'_n(t) : n \geq 1\}) \\ &= 2\alpha(\{f(t, v_{n-1}) + M(v_n(t) - v_{n-1}(t)) : n \geq 1\}) \end{aligned}$$

$$\leq 2\alpha(\{f(t, v_{n-1}(t)) : n \geq 1\}) + 2M \cdot 2\varphi(t)$$

$$\leq 2(L + 2M)\varphi(t).$$

By using assumption (A_0), $\varphi(0) = 0$, so we have

$$\varphi(t) \leq \varphi(0)e^{2(L+2M)t} = 0,$$

i.e., $\varphi(t) = 0$ on $[0, 2\pi]$. Similarly we can prove $\alpha(\{w_n(t) : n \geq 0\}) = 0$ on $[0, 2\pi]$.

Proof of Theorem 2.1. By Lemma 2.4, Lemma 2.5, $\{v_n\}$ and $\{w_n\}$ are convergent sequences. Let

$$\lim_{n\to\infty} v_n = r, \quad \lim_{n\to\infty} w_n = \rho \quad \text{in } C[[0, 2\pi], E].$$

Obviously $w_0 \leq \rho \leq r \leq v_0$ and ρ, r are solutions of (2.1).

We shall show that ρ, r are minimal and maximal periodic solutions of (2.1) in $[w_0, v_0]$.

Let u be any solution of (2.1) and $u \in [w_0, v_0]$. Let us assume that for some integer $k > 0$, $w_{k-1} \leq u \leq v_{k-1}$. Then setting $p = v_k - u$, we get

$$\begin{aligned} p' &= v_k' - u' \\ &= f(t, v_{k-1}) + M(v_k - v_{k-1}) - f(t, u) \\ &\leq M(v_{k-1} - u) + M(v_k - v_{k-1}) \\ &= Mp \end{aligned}$$

and $p(0) = p(2\pi)$. This implies by Lemma 2.1(i) $p(t) \geq 0$ on $[0, 2\pi]$, i.e., $v_k \geq u$ on $[0, 2\pi]$. Using similar argument we get $w_k \leq u$ on $[0, 2\pi]$. It follows by induction that $w_k \leq u \leq v_k$ on $[0, 2\pi]$ for all *n*. Hence, we have $\rho \leq u \leq r$ on $[0, 2\pi]$. Therefore, ρ, r are minimal and maximal periodic solutions of (2.1), respectively.

Corollary. *Let* $E = R^n$, $(\textit{specially } E = R^1)$, *the conditions* $(A_0)(\text{ii})$ *and* $\alpha(f(t, B)) \le L\alpha(B)$ *in* $(A_0)(\text{i})$ *be cancelled*. *Then Theorem* 2.1 *is valid*.

Proof. When E is a special Banach space R^n (or R^1), $(A_0)(\text{ii})$ holds by using Ascoli-Arzela theorem. For any $B \subset R^n$ bounded, $f(t, B)$ is also bounded since f is continuous in $[0, 2\pi] \times R^n$. So $\alpha(f(t, B)) = 0$ and $\alpha(B) = 0$ by using finite covering theorem. Thus, the all conditions in Theorem 2.1 are satisfied.

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