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NEW RESULT ON THE ULTIMATE BOUNDEDNESS OF SOLUTIONS FOR CERTAIN FIFTH-ORDER NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATION

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Abstract

This paper studies the boundedness of solution $x(t)$ of the fifth-order nonlinear delay differential equation

$$x^{(5)}(t) + \psi(t, x(t-r), \dot{x}(t-r), \ddot{x}(t-r), \dddot{x}(t-r), x^{(4)}(t-r))\, x^{(4)}(t)$$
$$+ f(\dddot{x}(t-r)) + \alpha_3 \ddot{x}(t) + \alpha_4 \dot{x}(t) + \alpha_5 x(t)$$
$$= p(t, x(t-r), \dot{x}(t-r), \ddot{x}(t-r), \dddot{x}(t-r), x^{(4)}(t-r)).$$

Sufficient conditions for the boundedness of solutions are obtained by constructing a Lyapunov functional.

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1. Introduction

The study of higher-order nonlinear delay differential equations has received considerably much attention, and still receiving such from various researchers. Among the scores of articles on the qualitative theory of differential equations, the number of articles on boundedness of solutions to nonlinear fifth-order differential equations with delay is significantly less than those on differential equations without delay, for example: Abou-El-Ela and Sadek [1], Chukwu [7], Sinha [21], Tunç [23-27] and references quoted therein, which contain the differential equations without delay or with delay. Up to this moment, the investigations concerning the boundedness of solutions of nonlinear equations of fifth-order with delay have not been fully developed.

In particular, in 2009, Ogundare [14] used the Lyapunov's second method to give sufficient criteria for the zero solution to be globally asymptotically stable, as well as the uniform boundedness of all solutions with their derivatives for the nonlinear fifth-order differential equation

$$x^{(5)} + ax^{(4)} + b\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}),$$

where a and b are two positive constants. The functions f, g, h and p are continuous in their respective arguments displayed explicitly.

Later, in 2009, Tunç [26] established sufficient conditions for the boundedness of the solution of a nonlinear delay differential equation of fifth-order

$$\begin{aligned} & x^{(5)} + f_1(t, x(t-r), \dot{x}(t-r), \ddot{x}(t-r), \dddot{x}(t-r), x^{(4)}(t-r))x^{(4)} \\ & + \alpha_2\dddot{x} + \alpha_3\ddot{x} + \alpha_4\dot{x} + f_5(x(t-r)) \\ = {} & p(t, x(t-r), \dot{x}(t-r), \ddot{x}(t-r), \dddot{x}(t-r), x^{(4)}(t-r)), \end{aligned}$$

where f_1, f_5 and p are continuous functions for the arguments displayed explicitly. r is a positive constant delay; α_2, α_3 and α_4 are some positive constants.

Recently, in 2013, Ademola and Arawomo [2] obtained criteria for uniform stability, uniform boundedness and uniform ultimate boundedness of solutions for third-order nonlinear delay differential equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x}) + g(x(t - r(t)), \dot{x}(t - r(t))) + h(x(t - r(t))) = p(t, x, \dot{x}, \ddot{x}),$$

where $0 \leq r(t) \leq \gamma$, $\gamma > 0$ is a constant, the functions f, g, h and p are continuous in their respective arguments.

In this paper, we consider the following non-autonomous fifth-order delay differential equation:

$$\begin{aligned} & x^{(5)}(t) + \psi(t, x(t - r), \dot{x}(t - r), \ddot{x}(t - r), \dddot{x}(t - r), x^{(4)}(t - r))x^{(4)}(t) \\ & + f(\dddot{x}(t - r)) + \alpha_3\ddot{x}(t) + \alpha_4\dot{x}(t) + \alpha_5 x(t) \\ = \; & p(t, x(t - r), \dot{x}(t - r), \ddot{x}(t - r), \dddot{x}(t - r), x^{(4)}(t - r)), \end{aligned} \tag{1.1}$$

which is equivalent to the system

$$\begin{aligned} & \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = u, \\ & \dot{u} = -\psi(t, x(t - r), y(t - r), z(t - r), w(t - r), u(t - r))u - f(w) \\ & \quad - \alpha_3 z - \alpha_4 y - \alpha_5 x + \int_{t-r}^{t} f'(w(s))u(s)ds \\ & \quad + p(t, x(t - r), y(t - r), z(t - r), w(t - r), u(t - r)), \end{aligned} \tag{1.2}$$

where ψ, f and p are continuous functions for the arguments displayed explicitly in (1.1). r is a positive constant delay; α_3, α_4 and α_5 are some positive constants; the derivative $f'(w)$ exists and is continuous for all w, and all solutions considered are assumed to be real-valued.

2. Preliminaries and Main Result

In order to reach the main result of this paper, we shall give some basic information to the boundedness criteria; we consider a general non-autonomous delay differential system

$$\dot{x} = f(t, x_t), \quad x_t = x(t+\theta), \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2.1}$$

where $f : [0, \infty) \times C_H \to \mathcal{R}^n$ is a continuous mapping, $f(t, 0) = 0$ and we suppose that f takes closed bounded sets into bounded sets of $\mathcal{R}^n$.

Definition 2.1 [5]. Let $V(t, \phi)$ be a continuous functional defined for $t \ge 0$, $\phi \in C_H$. The derivative of V along solutions of (2.1) will be denoted by $\dot{V}$ and is defined by the following relation:

$$\dot{V}_{(2.1)}(t, \phi) = \limsup_{h \to 0} \frac{V(t+h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ is the solution of (2.1) with $x_{t_0}(t_0, \phi) = \phi$.

The following is the classical theorem on uniform boundedness and uniform ultimate boundedness for the solution of (2.1).

Theorem 2.1 [5]. *Let $V(t, \phi_t) : \mathcal{R} \times C \to \mathcal{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) $W(|\phi(t)|) \le V(t, \phi_t) \le W_1(|\phi(t)|) + W_2\left(\int_{t-r}^{t} W_3(|\phi(s)|) ds\right)$ *and*

(ii) $\dot{V}_{(2.1)}(t, \phi_t) \le -W_3(|\phi(t)|) + M$, *for some* $M > 0$, *where* $W(r)$ *and* $W_i(r)$ $(i = 1, 2, 3)$ *are wedges*;

then the solutions of (2.1) *are uniformly bounded and uniformly ultimately bounded for bound B.*

The following will be our main boundedness result for (1.1).

Theorem 2.2. *In addition to the basic assumptions imposed on the functions* ψ, f *and* p *appearing in* (1.1), *we assume that there are positive constants* $\alpha_1, \ldots, \alpha_5, \varepsilon, \varepsilon_o, \delta, \lambda, \rho, \mu$ *and L such that the following conditions hold*:

(i) $\alpha_1 > 0, \alpha_1\alpha_2 - \alpha_3 > 0, \alpha_5 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0.$

$$\delta_o := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \tag{2.2}$$

$$\Delta_1 = \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) > 2\varepsilon\alpha_2 \ \textit{and} \tag{2.3}$$

$$\Delta_2 = \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5)}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1} > 0. \tag{2.4}$$

(ii) $2\varepsilon_o \le \psi(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1$

$$\le \min\left\{\frac{\varepsilon\alpha_4}{3\delta^2}, \frac{\varepsilon}{4\alpha_1^2}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{3\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}\right\}.$$

(iii) $f(0) = 0, \dfrac{f(w)}{w} \ge \alpha_2;\ w \ne 0, |f'(w(s))| \le L$ *and*

$$\left\{\frac{f(w)}{w} - \alpha_2\right\}^2 \le \min\left\{\frac{\varepsilon^2\alpha_4}{3\delta^2}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{3\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}\right\}.$$

(iv) $|p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))|$

$\le p_1(t) + p_2(t)(|y| + |z| + |w| + |u|),$

where $p_1(t)$ *and* $p_2(t)$ *are continuous functions satisfying*

$$p_1(t) \le p_o,$$

where $0 < p_o < \infty$ *and there exists* $\varepsilon_1 > 0$ *such that*

$$0 \le p_2(t) \le \varepsilon_1.$$

Then the solutions of the system (1.2) *are uniformly bounded and uniformly ultimately bounded provided that the inequality*

$$r < \min\left\{\frac{\varepsilon\alpha_4}{3\delta L}, \frac{\varepsilon}{8\alpha_1 L}, \frac{\varepsilon_o}{2(L + 2\rho)}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{6\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}\right\}$$

holds.

Proof. We define the Lyapunov functional $V = V(t, x_t, y_t, z_t, w_t, u_t)$ as

$$2V(t,\, x_t,\, y_t,\, z_t,\, w_t,\, u_t)$$

$$= u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}uz + 2\delta uy + 2\int_0^w f(\xi)d\xi$$

$$+ \left\{\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right\}w^2 + 2\left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}wz$$

$$+ 2\alpha_1\delta wy + 2\alpha_4 wy + 2\alpha_5 wx + \alpha_1\alpha_3 z^2$$

$$+ \left\{\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\right\}z^2 + 2\delta\alpha_2 yz$$

$$+ 2\alpha_1\alpha_4 yz - 2\alpha_5 yz + 2\alpha_1\alpha_5 zx + \frac{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}y^2$$

$$+ (\delta\alpha_3 - \alpha_1\alpha_5)y^2 + \frac{2\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}yx + \delta\alpha_5 x^2$$

$$+ 2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds$$

$$+ 2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds, \tag{2.5}$$

where λ, μ and ρ are some positive constants, which will be determined later and δ is a positive constant satisfying

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon. \tag{2.6}$$

Hence the Lyapunov functional defined in (2.5) can be arranged as the following:

$$2V = \left\{u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y\right\}^2 + \frac{\alpha_4\delta_o}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4}y\right)^2$$

$$+ \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_1\alpha_2 - \alpha_3}\left\{\frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}x + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}y + \alpha_1 z + w\right\}^2$$

$$+ \Delta_2(w + \alpha_1 z)^2 + \alpha_1\alpha_3 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)yz$$

$$+ 2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds$$

$$+ 2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds + \sum_{i=1}^{3} v_i, \tag{2.7}$$

where

$$v_1 := \delta\alpha_5 x^2 - \frac{\alpha_5^2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}x^2,$$

$$v_2 := \left\{\delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_o}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)^2} - \delta^2\right\}y^2,$$

$$v_3 := \frac{\varepsilon}{\alpha_1}w^2 + 2\int_0^w f(\xi)d\xi - \alpha_2 w^2.$$

From (2.6), it is clear that

$$v_1 = \varepsilon\alpha_5 x^2,$$

$$v_2 = \left[\frac{\alpha_5\delta_o}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)} - \varepsilon\left\{\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right\}\right]y^2$$

$$\geq \frac{3\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)}y^2$$

provided that

$$\frac{\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \geq \varepsilon\left\{\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right\}. \tag{I}$$

From (iii), we obtain

$$v_3 = \frac{\varepsilon}{\alpha_1}w^2 + 2\int_0^w\left\{\frac{f(\xi)}{\xi} - \alpha_2\right\}\xi d\xi \geq \frac{\varepsilon}{\alpha_1}w^2.$$

Summing up the three inequalities obtained from v_1, v_2 and v_3 into (2.7), we get

$$2V \geq \left\{u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right\}^2 + \frac{\alpha_4\delta_o}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4} y\right)^2$$

$$+ \Delta_2(w + \alpha_1 z)^2 + \alpha_1\alpha_3 z^2 + \varepsilon\alpha_5 x^2 + \frac{3\alpha_5\delta_o}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2$$

$$+ \frac{\varepsilon}{\alpha_1} w^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$+ 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds + 2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds. \tag{2.8}$$

Clearly, it follows from the first seven terms included in (2.8) that there exist sufficiently small positive constants D_i $(i = 1, ..., 5)$ such that

$$2V \geq D_1 x^2 + 2D_2 y^2 + 2D_3 z^2 + D_4 w^2 + D_5 u^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz$$

$$+ 2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds$$

$$+ 2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds. \tag{2.9}$$

Now, we consider the terms

$$v_4 := D_2 y^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + D_3 z^2,$$

which are contained in (2.9) and by using the inequality $|yz| \leq \frac{1}{2}(y^2 + z^2)$, we obtain

$$v_4 \geq D_2 y^2 + D_3 z^2 - \varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)(y^2 + z^2)$$

$$\geq D_6(y^2 + z^2),$$

for some $D_6 > 0$, $D_6 = \frac{1}{2}\min\{D_2, D_3\}$, if

$$\varepsilon \le \frac{\alpha_1\alpha_4 - \alpha_5}{2(\alpha_3\alpha_4 - \alpha_2\alpha_5)}\min\{D_2, D_3\}. \tag{II}$$

By using the previous inequality, we get from (2.9) that

$$\begin{aligned}2V &\ge D_1x^2 + (D_2 + D_6)y^2 + (D_3 + D_6)z^2 + D_4w^2 + D_5u^2\\ &\quad + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds\\ &\quad + 2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds.\end{aligned} \tag{2.10}$$

As a result, since the integrals

$$2\lambda\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \quad 2\mu\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds \text{ and}$$

$$2\rho\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds$$

are non-negative, it is obvious that there exists a positive constant D_7, which satisfies the following inequality:

$$V(t, x_t, y_t, z_t, w_t, u_t) \ge D_7(x^2 + y^2 + z^2 + w^2 + u^2), \tag{2.11}$$

where $D_7 = \frac{1}{2}\min\{D_1, D_2 + D_6, D_3 + D_6, D_4, D_5\}$, provided that ε is sufficiently small; and (I), (II) hold.

Further, since $|f'(w(s))| \le L$ implies that $f(w) \le Lw$, for all w. This inequality and the fact $2pq \le p^2 + q^2$, then the functional V defined in (2.5) yields

$$\begin{aligned}&V(t, x_t, y_t, z_t, w_t, u_t)\\ &\le \eta_1(x^2 + y^2 + z^2 + w^2 + u^2)\\ &\quad + \int_{-r(t)}^{0}\int_{t+s}^{t} \eta_2[x^2(\theta) + y^2(\theta) + z^2(\theta) + w^2(\theta) + u^2(\theta)]d\theta ds,\end{aligned} \tag{2.12}$$

where

$$\eta_1 := \frac{1}{2}\max\Bigg[\alpha_5\left\{1+\alpha_1+\delta+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right\}, \delta(1+\alpha_1+\alpha_2+\alpha_3)$$

$$+\alpha_4\left\{1+\alpha_1+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}+\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right\}-\alpha_5(1-\alpha_1),$$

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}(1+\alpha_1+\alpha_2)+(\alpha_1-1)(\alpha_4+\alpha_5)+\alpha_3(1+\alpha_1)$$

$$-\delta(1+\alpha_1-\alpha_2), \delta(\alpha_1-1)+\alpha_3+\alpha_4+\alpha_5+\frac{L}{2}+\alpha_1+\alpha_1^2$$

$$+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}(\alpha_1-1), 1+\alpha_1+\delta+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\Bigg]$$

and $\eta_2 := \frac{1}{2}\max\{1, \lambda, \mu, \rho\}$.

From estimates (2.11) and (2.12), the condition (i) of Theorem 2.1 is satisfied.

Now, let $(x_t, y_t, z_t, w_t, u_t)$ be a solution of system (1.2). Then a direct computation along this solution shows that

$$\frac{dV}{dt}(t, x_t, y_t, z_t, w_t, u_t)$$

$$= -\{\psi(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\}u^2$$

$$-\left[\alpha_1\frac{f(w)}{w}-\left\{\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right\}\right]w^2$$

$$-\left\{\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-(\delta\alpha_2+\alpha_1\alpha_4-\alpha_5)\right\}z^2$$

$$-\left\{\delta\alpha_4-\frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right\}y^2-\alpha_1\{\psi(t, x, y, z, w, u)-\alpha_1\}wu$$

$$- \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\{\psi(t, x, y, z, w, u) - \alpha_1\}zu$$

$$- \delta\{\psi(t, x, y, z, w, u) - \alpha_1\}yu$$

$$- \delta\left\{\frac{f(w)}{w} - \alpha_2\right\}wy - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\left\{\frac{f(w)}{w} - \alpha_2\right\}wz$$

$$+ u\int_{t-r}^{t} f'(w(s))u(s)ds + \alpha_1 w\int_{t-r}^{t} f'(w(s))u(s)ds$$

$$+ \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z\int_{t-r}^{t} f'(w(s))u(s)ds$$

$$+ \delta y\int_{t-r}^{t} f'(w(s))u(s)ds$$

$$+ \left\{u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right\}$$

$$\times\, p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))$$

$$+ \rho u^2 r + \lambda z^2 r + \mu w^2 r - \rho\int_{t-r}^{t} u^2(s)ds$$

$$- \mu\int_{t-r}^{t} w^2(s)ds - \lambda\int_{t-r}^{t} z^2(s)ds. \qquad (2.13)$$

Making use of the assumptions (i)-(iv) and (2.6), we get

$$\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \geq 2\varepsilon_o,$$

$$\alpha_1\frac{f(w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}$$

$$= \alpha_1\left\{\frac{f(w)}{w} - \alpha_2\right\} - \left\{\alpha_1\alpha_2 - \alpha_3 + \delta - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right\}$$

$$\geq \varepsilon,$$

$$\frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\delta\alpha_2 + \alpha_1\alpha_4 - \alpha_5)$$

$$= \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) - \varepsilon\alpha_2$$

$$\geq 2\varepsilon\alpha_2 - \varepsilon\alpha_2 = \varepsilon\alpha_2$$

and

$$\delta\alpha_4 - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} = \varepsilon\alpha_4,$$

by (i), (ii), (iii) and (2.6).

By using the assumption $|f'(w(s))| \leq L$ from (iii), and the inequality $2|ab| \leq a^2 + b^2$, we obtain the following inequalities:

$$u\int_{t-r}^{t} f'(w(s))u(s)ds \leq \frac{L}{2}ru^2(t) + \frac{L}{2}\int_{t-r}^{t} u^2(s)ds,$$

$$\alpha_1 w\int_{t-r}^{t} f'(w(s))u(s)ds \leq \frac{\alpha_1 L}{2}rw^2(t) + \frac{\alpha_1 L}{2}\int_{t-r}^{t} u^2(s)ds,$$

$$\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z\int_{t-r}^{t} f'(w(s))u(s)ds$$

$$\leq \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\frac{L}{2}rz^2(t) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\frac{L}{2}\int_{t-r}^{t} u^2(s)ds$$

and

$$\delta y\int_{t-r}^{t} f'(w(s))u(s)ds \leq \frac{\delta L}{2}ry^2(t) + \frac{\delta L}{2}\int_{t-r}^{t} u^2(s)ds.$$

Making use of these inequalities into (2.12), we obtain

$$\frac{dV}{dt} \leq -\left(\frac{\varepsilon\alpha_4}{3} - \frac{\delta L}{2}r\right)y^2$$

$$
\begin{aligned}
& -\left[\frac{\varepsilon\alpha_2}{3}-\left\{\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}L+\lambda\right\}r\right]z^2 \\
& -\left\{\frac{\varepsilon}{4}-\left(\frac{\alpha_1 L}{2}+\mu\right)r\right\}w^2 \\
& -\left\{\frac{\varepsilon_o}{2}-\left(\frac{L}{2}+\rho\right)r\right\}u^2 \\
& -\left[\rho-\left\{\frac{L}{2}+\frac{\alpha_1 L}{2}+\frac{\delta L}{2}+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}L\right\}\right]\int_{t-r}^{t}u^2(s)ds \\
& -\lambda\int_{t-r}^{t}z^2(s)ds-\mu\int_{t-r}^{t}w^2(s)ds \\
& +\left|u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right| \\
& \times|\,p(t,\, x(t-r),\, y(t-r),\, z(t-r),\, w(t-r),\, u(t-r))\,| \\
& -\sum_{k=5}^{9}v_k, \qquad (2.14)
\end{aligned}
$$

where

$$
\begin{aligned}
v_5 &:= \frac{1}{4}(\psi-\alpha_1)u^2+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}(\psi-\alpha_1)zu+\frac{\varepsilon\alpha_2}{3}z^2, \\
v_6 &:= \frac{1}{4}(\psi-\alpha_1)u^2+\alpha_1(\psi-\alpha_1)wu+\frac{\varepsilon}{4}w^2, \\
v_7 &:= \frac{1}{4}(\psi-\alpha_1)u^2+\delta(\psi-\alpha_1)uy+\frac{\varepsilon\alpha_4}{3}y^2, \\
v_8 &:= \frac{\varepsilon}{4}w^2+\delta\left\{\frac{f(w)}{w}-\alpha_2\right\}wy+\frac{\varepsilon\alpha_4}{3}y^2
\end{aligned}
$$

and

$$
v_9 := \frac{\varepsilon}{4}w^2+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left\{\frac{f(w)}{w}-\alpha_2\right\}wz+\frac{\varepsilon\alpha_2}{3}z^2.
$$

It is clear that the expressions given by $v_5, \ldots, v_9$ represent certain specific quadratic forms, respectively.

Making use of the basic information on the positive semi-definite of a quadratic form, one can easily conclude that $v_5 \geq 0$, $v_6 \geq 0$, $v_7 \geq 0$, $v_8 \geq 0$ and $v_9 \geq 0$ provided that

$$\psi - \alpha_1 \leq \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{3\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \quad \psi - \alpha_1 \leq \frac{\varepsilon}{4\alpha_1^2}, \quad \psi - \alpha_1 \leq \frac{\varepsilon\alpha_4}{3\delta^2},$$

$$\left\{\frac{f(w)}{w} - \alpha_2\right\}^2 \leq \frac{\varepsilon^2\alpha_4}{3\delta^2} \quad \text{and} \quad \left\{\frac{f(w)}{w} - \alpha_2\right\}^2 \leq \frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{3\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2},$$

respectively.

Thus, in view of the above discussion and inequality (2.13), it follows that

$$\begin{aligned}
\frac{dV}{dt} \leq & -\left(\frac{\varepsilon\alpha_4}{3} - \frac{\delta L}{2}r\right)y^2 \\
& - \left[\frac{\varepsilon\alpha_2}{3} - \left\{\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)}L + \lambda\right\}r\right]z^2 \\
& - \left\{\frac{\varepsilon}{4} - \left(\frac{\alpha_1 L}{2} + \mu\right)r\right\}w^2 \\
& - \left\{\frac{\varepsilon_o}{2} - \left(\frac{L}{2} + \rho\right)r\right\}u^2 \\
& - \left[\rho - \left\{\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)}L\right\}\right]\int_{t-r}^{t} u^2(s)ds \\
& - \lambda\int_{t-r}^{t} z^2(s)ds - \mu\int_{t-r}^{t} w^2(s)ds \\
& + \left|u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y\right| \\
& \times |\, p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))\,|. \qquad (2.15)
\end{aligned}$$

So we can choose the constants λ, μ and ρ in the following form:

$$\rho = \frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} L,$$

$$\mu = \frac{\alpha_1 L}{2} \quad \text{and} \quad \lambda = \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} L.$$

Then the inequality in (2.15) becomes

$$\begin{aligned}\frac{dV}{dt} \le &-\left(\frac{\varepsilon\alpha_4}{3} - \frac{\delta L}{2} r\right) y^2 \\ &- \left\{\frac{\varepsilon\alpha_2}{3} - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} Lr\right\} z^2 \\ &- \left(\frac{\varepsilon}{4} - \alpha_1 Lr\right) w^2 \\ &- \left\{\frac{\varepsilon_o}{2} - \left(\frac{L}{2} + \rho\right) r\right\} u^2 \\ &+ \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right| \\ &\times | p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) |. \qquad (2.16)\end{aligned}$$

Thus, one obtains easily that

$$\begin{aligned}\frac{dV}{dt} &\le -\sigma(y^2 + z^2 + w^2 + u^2) + \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right| \\ &\quad \times | p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) | \\ &\le -\sigma(y^2 + z^2 + w^2 + u^2) + \eta_3(| y | + | z | + | w | + | u |) \\ &\quad \times | p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) |,\end{aligned}$$

where $\eta_3 := \max\left\{\delta, \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}, \alpha_1, 1\right\}$, for some constant $\sigma > 0$, provided that

$$r < \min\left\{\frac{\varepsilon\alpha_4}{3\delta L}, \frac{\varepsilon}{8\alpha_1 L}, \frac{\varepsilon_o}{2(L + 2\rho)}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{6\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}\right\}.$$

In view of (iv), we get

$$\frac{dV}{dt} \leq -(\sigma - 4\eta_3\varepsilon_1)(y^2 + z^2 + w^2 + u^2) + \eta_3 p_o(|y| + |z| + |w| + |u|).$$

By choosing $\varepsilon_1 < 4^{-1}\eta_3^{-1}\sigma,$ there exists $\eta_4 = \sigma - 4\eta_3\varepsilon_1 > 0$ and from the fact $(|y| + |z| + |w| + |u|)^2 \leq 4(y^2 + z^2 + w^2 + u^2),$ then

$$\frac{dV}{dt} \leq -\eta_4(y^2 + z^2 + w^2 + u^2) + 2\eta_3 p_o(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}},$$

choose $(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}} \geq 2\eta_3 p_o \eta_4^{-1},$ then we find

$$\frac{dV}{dt} \leq -\eta_4(y^2 + z^2 + w^2 + u^2) + 4\eta_3^2 p_o^2 \eta_4^{-1}, \tag{2.17}$$

for all *x*, *y*, *z*, *w* and *u*. From estimate (2.17), hypothesis (ii) of Theorem 2.1 is satisfied. Also, from estimates (2.11) and (2.12), condition (i) of Theorem 2.1 follows. This completes the proof of Theorem 2.2. □

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